VISUALIZING THE UNIT BALL OF THE AGY NORM

ALEX WRIGHT

1. Abstract

Avila-Gouëzel-Yoccoz defined a norm on the relative cohomology $H^1(X, \Sigma)$ of a translation surface (X, ω), in [**AGY06**, Section 2] and also [**AG13**, Section 5]. (The relative cohomology $H^1(X, \Sigma)$ is an easy to understand vector space that is constructed in an elementary way from the surface.) This norm plays a major role in the work of the Fields medalists Avila and Yoccoz, and is also related to the work of the Fields medalists McMullen and Mirzakhani. Since its definition only recently in 2013, we still do not have a detailed understanding of this norm. For example, it is unknown if the norm on the vector space $H^1(X, \Sigma)$ contains enough information to reconstruct the translation surface (X, ω), as is famously the case in a related situation (the Teichmüller norm). The goal of this project is to write a computer program to draw pictures of two-dimensional slices of the unit ball. Such a picture will be a convex set in the plane. It probably won't be a round circle in this case, but it may look like some sort of distorted circle. It may or may not have corners, and it may or may not have flat edges. This project will empirically resolve these mysteries and lead directly to ongoing research.

A translation surface can be thought of as a collection of triangles glued together, and $H^1(X, \Sigma)$ can be thought of as the vector space generated by the edges, modulo the relations imposed by requiring that the three edges of any triangle sum to zero. The project will naturally consist of two main modules (as well as a final module that combines everything together). The first module will find a list of saddle connections on the surface, which are straight lines joining the corner of one triangle to the corner of another triangle (and possibly passing through many triangles in between). The second module will take that list as an input, and via an elementary algorithm produce the picture of the norm ball.

Mathematical prerequisites: You should be truly comfortable with the abstract vector spaces, subspaces, quotient spaces, and dual spaces. It isn't required to know about cohomology. You should also be truly comfortable with norms on vector spaces, and know at least one example that doesn't come from an inner product; reading the first bit of the wikipedia page on norms will fulfill this requirement. Additionally you should have at least one of the following two bonus prerequisites. (1) You should be comfortable with surfaces; for example, you should know that an octagon with opposite sides identified is a genus two surface. (2) You should have some experience programming.

Course prerequisites: One of Math 493 (Honors Algebra I) or Math 395 (Honors Analysis I) is required. Exceptions may be provided for students with a very strong performance in Math 296 (Honors Mathematics II) and an enthusiastic recommendation from a faculty member.

2. Norms

A norm is a function $\|\cdot\| : V \to \mathbb{R}$ from a vector space to the real numbers satisfying the following axioms.

(1) $\|v\| \geq 0$, with equality if and only if $v = 0$.
(2) $\|cv\| = |c|\|v\|$.
(3) $\|v + w\| \leq \|v\| + \|w\|$.

Given a norm, the unit ball is defined as $B_1 = \{v : \|v\| \leq 1\}$. It satisfies the following properties.

(1) It is bounded, closed, and contains 0 in its interior.
(2) It is convex, i.e. it contains the straight line between any two of its points.
(3) It is centrally symmetric, i.e. if it contains v then it also contains $-v$.

Any set B_1 satisfying these three properties is the unit ball for some norm. So there is truly a zoo of possibilities.

If v_1, v_2 are linearly independent vectors, we can plot in $\mathbb{R}^2$ the set of (x, y) such that $xv_1 + yv_2$ has norm 1 (or norm at most 1). This allows us to visualize two dimensional slices of the unit ball. A computer program performing this visualization would take as input the two vectors v_1 and v_2, and output a picture of a centrally symmetric convex blob in the plane. This blob might look like a round circle, or a square, or a diamond, or something vastly more complicated.

3. A family of norms on V^*, and the second module

Let V^* denote the dual space to V, i.e. the space of linear functionals. Suppose $\mathcal{S}$ is a set of pairs (v, l_v), where $v \in V$ and $l_v \geq 0$. We think of l_v as some sort of length of the vector v, but this notion of length doesn't have to be defined for all vectors in V, and in particular doesn't have to come from a norm.

We then attempt to define a norm $\|\cdot\|_{\mathcal{S}}$ on V^* as follows. An element $\phi \in V^*$ is a linear functional $\phi : V \to \mathbb{R}$. We define

$$\|\phi\|_{\mathcal{S}} = \sup_{(v,l_v)\in\mathcal{S}} \frac{|\phi(v)|}{l_v}.$$

Sometimes this gives a norm; for now, let's just assume we've picked $\mathcal{S}$ in a sufficiently intelligent way that this does indeed give a norm.

Example 3.1. Suppose $V = \mathbb{R}^2$ and

$$\mathcal{S} = \{((1,0),1),((0,1),1)\}.$$

Let $\phi(x,y) = ax + by$. Then

$$\begin{aligned} \|\phi\|_{\mathcal{S}} &= \sup\left\{\frac{|\phi((1,0))|}{1}, \frac{|\phi((0,1))|}{1}\right\} \\ &= \sup\{a,b\}. \end{aligned}$$

So in this case $\|\cdot\|_{\mathcal{S}}$ defines a norm called the sup-norm or ℓ^∞-norm.

Example 3.2. Suppose $V = \mathbb{R}^2$ and

$$\mathcal{S} = \{((x,y),\sqrt{x^2+y^2}) : x,y \in \mathbb{R}\}.$$

Let $\phi(x,y) = ax + by$. Then

$$\begin{aligned} \|\phi\|_{\mathcal{S}} &= \sup \frac{|\phi(x,y)|}{\sqrt{x^2+y^2}} \\ &= \sup \frac{|ax+by|}{\sqrt{x^2+y^2}} \\ &= \sqrt{a^2+b^2} \end{aligned}$$

by the Cauchy-Schwarz inequality. So in this case $\|\cdot\|_{\mathcal{S}}$ defines the usual Euclidian norm, also called the ℓ^2-norm.

The second module of this project will take as input a finite family $\mathcal{S}$. It will suffice to assume $V = \mathbb{R}^2$. The module will consist of writing a computer program to draw the unit ball of $\|\cdot\|_{\mathcal{S}}$. A good algorithm to do this would be to evenly sample n points on the usual circle, namely

$$\left\{\left(\cos 2\pi\frac{k}{n}, \sin 2\pi\frac{k}{n}\right) : k = 0\ldots,n-1\right\}.$$

For each of these points v, compute its norm, and plot the point

$$\frac{(\cos\frac{k2\pi}{n}, \sin\frac{k2\pi}{n})}{\|(\cos\frac{k2\pi}{n}, \sin\frac{k2\pi}{n})\|_{\mathcal{S}}}.$$

4. Translation surfaces, and the first module

The surfaces we consider will be defined by gluing together triangles in $\mathbb{R}^2$. We will only be allowed to glue triangles along parallel edges of the same length, in a way so that there is one triangle on each side of each edge.

Suppose we fix an orientation on each edge. Concretely, that means that each edge is now a vector, so we can specify that it has a tip and a tail. We will do this in such a way that when we glue triangles along an edge, the orientations agree.

Let V_e denote the abstract vector space generated by the edges. An element of V_e looks like $\sum c_i v_i$, where v_i are edges and c_i are real numbers. Let V_t denote the subspace by the following elements: For each triangle, order the three edges e_1, e_2, e_3 in such a way that $e_1+e_2 = e_3$ as vectors in $\mathbb{R}^2$. Then V_t is spanned by the

$$e_1 + e_2 - e_3.$$

Note that this object is not a vector in $\mathbb{R}^2$. It is a vector in V_e.

We define $H_1(X, \Sigma)$ to be V_e/V_t. (In case you are wondering, Σ is defined to be the subset of the surface arising from the vertices.)

Exercise 4.1. Consider any spanning subtree of the graph of the triangulation of the surface. This gives rise to a subspace of V_e which is isomorphic to V_e/V_t.

We define $H^1(X, \Sigma)$ to be the dual space $H_1(X, \Sigma)^*$. Note that we also have $H_1(X, \Sigma) = H^1(X, \Sigma)^*$.

A saddle connection on the surface is defined to be a straight line joining one corner of a triangle to another, possibly going through many triangles in between, but not passing through any other corners of triangles.

There are infinitely many saddle connections on each translation surface, but only finitely many of length at most L. Each saddle connection γ gives rise to an element $[\gamma]$ of $H_1(X, \Sigma)$, and each saddle connection has a length l_γ. Define

$$\mathcal{S} = \{([\gamma], l_\gamma) : \gamma \text{ a saddle connection}\}$$

and

$$\mathcal{S}_L = \{([\gamma], l_\gamma) : \gamma \text{ a saddle connection of length at most } L\}.$$

The first module of the project will take L and a surface as its input, and compute $\mathcal{S}_L$. There is already some open source software that does some related things, and we can either try to modify it or try to imitate some of its algorithms. This software is available on the webpage of Ronen Mukamel. There is also software written by Vincent Delecroix and Pat Hooper available at `https://github.com/videlec/sage-flatsurf`.

5. The definition of the AGY norm, and the final module

The AGY norm on $H^1(X, \Sigma)$ is defined as $\| \cdot \|_{\mathcal{S}}$, where $\mathcal{S}$ is as above. Since this $\mathcal{S}$ is infinite, we can't compute it, but the second module computes $\mathcal{S}_L$ for each $L > 0$.

The final module should only be started once the first two modules are done or almost done. It will accept the input from the user, and interface between the first two modules. It will accept as input a surface, two linearly independent vectors ϕ_1, ϕ_2 of $H^1(X, \Sigma)$. There will also be parameters $L > 0$ and $n > 0$ that determine the accuracy to which we want to draw the unit ball. (We will consider saddle connections of length at most L, and plot the unit ball by sampling n evenly spaced points on the circle.) It will call the second module to compute $\mathcal{S}_L$. It will then translate $\mathcal{S}_L$ into the context of $\mathbb{R}^2$, allowing us to call the first module and draw the unit ball.

References

[AG13] Artur Avila and Sébastien Gouëzel, *Small eigenvalues of the Laplacian for algebraic measures in moduli space, and mixing properties of the Teichmüller flow*, Ann. of Math. (2) **178** (2013), no. 2, 385–442. MR 3071503

[AGY06] Artur Avila, Sébastien Gouëzel, and Jean-Christophe Yoccoz, *Exponential mixing for the Teichmüller flow*, Publ. Math. Inst. Hautes Études Sci. (2006), no. 104, 143–211.