BEURLING'S THEOREM

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Abstract. Hardy spaces are defined, and a proof of Beurling's theorem, describing the invariant subsapces of the unilateral shift, is given.

Recall that on $\ell^2(\mathbb{Z})$ and $\ell^2(\mathbb{N})$ respectively we have the bilateral and unilateral shifts, W and U.

$$W((\cdots, a_{-1}, a_0, a_1, a_2, \cdots)) = (\cdots, a_{-2}, a_{-1}, a_0, a_1, \cdots)$$
$$W^*((\cdots, a_{-1}, a_0, a_1, a_2, \cdots)) = (\cdots, a_{-0}, a_1, a_2, a_3, \cdots)$$
$$U((a_0, a_1, a_2, \cdots)) = (0, a_0, a_2, \cdots)$$
$$U^*((a_0, a_1, a_2, \cdots)) = (a_1, a_2, a_3, \cdots)$$

A closed subspace $\mathcal{M}$ of an operator $T : \mathcal{H} \to \mathcal{H}$ is said to be invariant if $T\mathcal{M} \subset \mathcal{M}$ and reducing if in addition $T(\mathcal{M}^\perp) \subset \mathcal{M}^\perp$. The unilateral shift has many invariant subspaces; for example $\overline{\text{span}\{e_n : n \geq N\}}$, where e_n is the standard basis. The purpose of this note is to describe all the invariant subspaces of U. Along the way we will also describe the invariant and reducing subspaces of W.

It is hard to describe all the invariant subspaces of U on ℓ^2, so we move to $L^2(\mathbb{T})$ (where $\mathbb{T}$ is the unit circle in $\mathbb{C}$). In this context every $f \in L^2(\mathbb{T})$ corresponds to a Fourier series $\sum_{n=-\infty}^{\infty} a_n z^n$ and

$$W(f) = W(\sum_{n=-\infty}^{\infty} a_n z^n) = \sum_{n=-\infty}^{\infty} a_n z^{n+1} = M_z f.$$

That is, the bilateral shift is just M_z, multiplication by z!

To discuss the unilateral shift in this context, we need the Hardy-Hilbert space, defined as $\mathcal{H}^2 = \{f = \sum_{n=0}^{\infty} a_n z^n\} \subset L^2(\mathbb{T})$. Since the Fourier coefficients of $f \in \mathcal{H}^2$ are in l^2, the Fourier series of $f \in \mathcal{H}^2$ converges uniformly to an analytic function on any compact subset of the open unit disk, $\mathbb{D}$, and we see that f is analytic on $\mathbb{D}$. Now if

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$f = \sum_{n=0}^{\infty} a_n z^n$, then

$$Uf = \sum_{n=0}^{\infty} a_n z^{n+1} = M_z f,$$

so we get that the unilateral shift is also just multiplication by z.

Proposition. *If T is an operator on $\mathcal{H}$, $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, and $\mathcal{P}_{\mathcal{M}}$ is the projection onto $\mathcal{M}$, then $\mathcal{M}$ is invariant for T if and only if $\mathcal{P}_{\mathcal{M}}T\mathcal{P}_{\mathcal{M}} = T\mathcal{P}_{\mathcal{M}}$, if and only if $\mathcal{M}^{\perp}$ is an invariant subspace of T^*. Further, M is reducing for T if and only if $\mathcal{P}_{\mathcal{M}}T = T\mathcal{P}_{\mathcal{M}}$, if and only if $\mathcal{M}$ is an invariant subspace for both T and T^*.*

Proof. If $\mathcal{M}$ is invariant for T, then for $f \in \mathcal{H}$ we have $T\mathcal{P}_{\mathcal{M}}f \in \mathcal{M}$ and hence $\mathcal{P}_{\mathcal{M}}T\mathcal{P}_{\mathcal{M}}f = T\mathcal{P}_{\mathcal{M}}f$; thus $\mathcal{P}_{\mathcal{M}}T\mathcal{P}_{\mathcal{M}} = T\mathcal{P}_{\mathcal{M}}$. Conversely, if $\mathcal{P}_{\mathcal{M}}T\mathcal{P}_{\mathcal{M}}f = T\mathcal{P}_{\mathcal{M}}$, then for f in $\mathcal{M}$ we have $Tf = T\mathcal{P}_{\mathcal{M}}f = \mathcal{P}_{\mathcal{M}}T\mathcal{P}_{\mathcal{M}}f = \mathcal{P}_{\mathcal{M}}Tf$, and hence $Tf \in M$. Therefore, $T\mathcal{M} \subset \mathcal{M}$ and $\mathcal{M}$ is invariant for T. Further, since $I - \mathcal{P}_{\mathcal{M}}$ is the projection onto $\mathcal{M}^{\perp}$ and the identity

$$T^*(I - \mathcal{P}_{\mathcal{M}}) = (I - \mathcal{P}_{\mathcal{M}})T^*(I - \mathcal{P}_{\mathcal{M}})$$

is equivalent to $\mathcal{P}_{\mathcal{M}}T^* = \mathcal{P}_{\mathcal{M}}T^*\mathcal{P}_{\mathcal{M}}$, we see that $\mathcal{M}^{\perp}$ is invariant for T^* iff and only if $\mathcal{M}$ is invariant for T. Finally, if $\mathcal{M}$ reduces T, then $T\mathcal{P}_{\mathcal{M}} = \mathcal{P}_{\mathcal{M}}T\mathcal{P}_{\mathcal{M}} = (\mathcal{P}_{\mathcal{M}}T^*\mathcal{P}_{\mathcal{M}})^* = (T^*\mathcal{P}_{\mathcal{M}})^* = \mathcal{P}_{\mathcal{M}}T$, using the facts that we have just proved. □

It is now easy to see that U has no non-trivial reducing subspaces. If $\mathcal{M} \subset \ell^2(\mathbb{N})$ is a non-zero reducing subspace for U, and $0 \neq (a_n)_{n=0}^{\infty} \in \mathcal{M}$, then without loss of generality assume that $a_0 \neq 0$. (Otherwise shift $(a_n)_{n=0}^{\infty}$ to the left some number of times using U^*). Then $(a_n)_{n=0}^{\infty} - UU^*((a_n)_{n=0}^{\infty}) = (a_0, 0, 0, \cdots) \in M$, and so we get $e_0 = (1/a_0)(a_0, 0, 0, \cdots) \in \mathcal{M}$. Thus $U^n(e_0) = e_n \in \mathcal{M}$ for all $n \geq 0$, so $M = \ell^2(\mathbb{N})$.

In turns out however, that the bilateral shift on $L^2(\mathbb{T})$ has a great many reducing subspaces. To prove this, we will need the following lemmas.

Lemma (1)**.** *$\mathcal{M} = \{M_\phi : \phi \in L^\infty(\mathbb{T})\}$ is a maximal abelian subalgebra of $\mathcal{B}(L^2(\mathbb{T}))$, the bounded linear operators on Hilbert space..*

Proof. Suppose that $T \in \mathcal{B}(L^2(\mathbb{T}))$ commutes with $\mathcal{M}$. Set $\Phi = T1$, 1 being the function on $\mathbb{T}$ that is constantly 1. Now if $\phi \in L^\infty(\mathbb{T})$, then

$$T\phi = T(\phi \cdot 1) = TM_\phi 1 = M_\phi T1 = M_\phi \Phi = \phi\Phi.$$

So $T = M_\Phi$, and we need simply to check that $\Phi \in L^\infty(\mathbb{T})$. Set $E_n = \{x \in \mathbb{T} : |\Phi(x)| \geq \|T\| + \frac{1}{n}\}$. Then

$$\begin{aligned}\|T\|\sqrt{\mu(E_n)} &= \|T\|\|\chi_{E_n}\|_2 \geq \|T\chi_{E_n}\|_2 = \|\Phi\chi_{E_n}\|_2 = \left(\int_{E_n} \Phi^2 d\mu\right)^{\frac{1}{2}} \\ &\geq \left(\|T\| + \frac{1}{n}\right)\sqrt{\mu(E_n)}.\end{aligned}$$

Thus $\mu(E_n) = 0$ so $\mu(\{x : |\Phi(x)| > \|T\|\}) = \mu(\cup_n E_n) \leq \sum_n \mu(E_n) = 0$. We conclude that $\|\Phi\|_\infty \leq \|T\|$ and $\Phi \in L^\infty(\mathbb{T})$. □

Lemma (2)**.** *The commutant $\{S \in \mathcal{B}(\mathcal{H}) : SM_z = M_z S\}$ of M_z is $L^\infty(\mathbb{T})$.*

Proof. If $S \in \mathcal{B}(\mathcal{H})$ commutes with M_z, then it commutes with $M_{p(z)}$ whenever $p(z)$ is a polynomial. Now if $\phi \in L^\infty(\mathbb{T})$, then we can pick polynomials $p_n(z)$ such that $p_n(z) \to \phi$ in the L^2 norm. It then follows that $M_{p_n(z)}(f) \to \phi f$ for all $f \in L^2(\mathbb{T})$. Now

$$M_\phi S f = \lim_{n\to\infty} M_{p_n(z)} S f = \lim_{n\to\infty} S M_{p_n(z)} f = S M_\phi f.$$

Thus S commutes with $L^\infty(\mathbb{T})$, so by the preceding lemma, $f \in L^\infty(\mathbb{T})$. □

Corollary. *The reducing subspaces of the bilateral shift are $L^2(E) = \{f \in L^2(\mathbb{T}) : f(x) = 0$ if $x \notin E\}$ for $E \subset \mathbb{T}$ measurable.*

Proof. It is easy to check that those subspaces are invariant under multiplication by $W = M_z$ and $W^* = M_{z^{-1}}$, hence they are reducible. Conversely, if $\mathcal{M}$ is a reducing subspace of W, then $\mathcal{P}_\mathcal{M} M_z = M_z \mathcal{P}_\mathcal{M}$, so by Lemma 2, $\mathcal{P}_\mathcal{M} = M_\phi$ for some $\phi \in L^\infty(\mathbb{T})$. Since $\mathcal{P}_\mathcal{M} = \mathcal{P}_\mathcal{M}^2$, we get $\phi^2 = \phi$, so ϕ is zero or one almost everywhere. Now $\mathcal{M} = \mathcal{P}_\mathcal{M} L^2(\mathbb{T}) = \phi L^2(\mathbb{T}) = L^2(E)$, where $E = \{x \in \mathbb{T} : \phi(x) = 1\}$. □

Theorem. *The non-reducing invariant subspaces of the bilateral shift are of the form $\phi\mathcal{H}^2$, for $|\phi| = 1$ a.e.*

Proof. If $|\phi| = 1$ a.e. then M_ϕ is an isometry, so $M_\phi(\mathcal{H}^2) = \phi\mathcal{H}^2$ is indeed a closed subspace. Furthermore, $W = M_z$ commutes with M_ϕ, so $WM_\phi\mathcal{H}^2 = M_\phi W\mathcal{H}^2 \subset M_\phi\mathcal{H}^2$; thus $\phi\mathcal{H}^2$ is indeed is invariant. Note $\phi \in \phi\mathcal{H}^2$, but $W^*\phi = z^{-1}\phi \notin \phi\mathcal{H}^2$ since $z^{-1} \notin \mathcal{H}^2$; thus $\phi\mathcal{H}^2$ is not reducing.

Conversely, suppose that $\mathcal{M}$ is an invariant, non-reducing subspace of W. Since W is non reducing, $W^{-1}\mathcal{M} = W^*\mathcal{M}$ cannot be a subspace of $\mathcal{M}$, so we get that $W\mathcal{M} \subsetneq \mathcal{M}$. Choose $\phi \in \mathcal{M} \ominus W\mathcal{M}$, with $\|\phi\| = 1$.

W is unitary, so $\phi \perp W^n\phi$ for all $n \geq 1$. Thus for $n \geq 1$,

$$0 = \frac{1}{2\pi}\int_{\mathbb{T}} \phi(z)\overline{\phi(z)z^n}dz = \frac{1}{2\pi}\int_{\mathbb{T}} |\phi(z)|^2 z^{-n}dz.$$

Taking conjugates gives this identity for n negative as well. Thus, $|\phi|$ must be a constant, and since $\|\phi\| = 1$, in fact $|\phi| = 1$ a.e..

We claim that $\mathcal{M} \ominus W\mathcal{M} = \operatorname{span}\phi$. Indeed, the above shows that every function in $\mathcal{M} \ominus W\mathcal{M}$ is of constant norm. So if $\psi \in \mathcal{M} \ominus W\mathcal{M}$ and $\lambda \in \mathbb{C}$, then $\phi - \lambda\psi$ has constant norm. It is easy to pick λ so that $\phi - \lambda\psi$ will be very small on a set of positive measure, and hence $\phi - \lambda\psi$, having constant norm, will always be very small. Since ψ is arbitrarily close to $\operatorname{span}\phi$, we in fact get $\psi \in \operatorname{span}\phi$.

W is unitary, so $W^n(\operatorname{span}\phi) = W^n(\mathcal{M} \ominus W\mathcal{M}) = W^n\mathcal{M} \ominus W^{n+1}\mathcal{M}$. Thus, from the decomposition $\mathcal{M} = (\operatorname{span}\phi) \oplus W\mathcal{M}$, we get the decomposition $\mathcal{M} = (\operatorname{span}\phi) \oplus (\operatorname{span} W\phi) \oplus \cdots \oplus (\operatorname{span} W^n\phi) \oplus W^n\mathcal{M}$. Thus since $W^n\phi = z^n\phi$, we get that $\mathcal{M} = \phi\mathcal{H}^2 \oplus \cap_{n\geq 0} W^n\mathcal{M}$. Since $\cap_{n\geq 0} W^n\mathcal{M} = \{0\}$ (this is true for any subset $\mathcal{M}$ of Hilbert space), we get that $\mathcal{M} = \phi\mathcal{H}^2$. □

Note that this also shows that every non-reducing invariant subspace of W is cyclic: It is generated by the vector ϕ, that is, the subspace is $\overline{\operatorname{span}} \cup_{n\geq 0} W^n$.

Proposition. *$\phi\mathcal{H}^2 = \psi\mathcal{H}^2$ if and only if there is a constant c of modulus 1 such that $\phi = c\psi$.*

Proof. Say $\phi\mathcal{H}^2 = \psi\mathcal{H}^2$. Then $\phi = f_1\psi$, $f_2\phi = \psi$, for some $f_1, f_2 \in \mathcal{H}^2$. It is easily seen that $f_1 = \overline{f_2}$. Since the only analytic functions with analytic conjugate are constants, we get that f_1 and f_2 are constants. □

We define an inner function to be a $\phi \in \mathcal{H}^2$ with $|\phi| = 1$ a.e., and we immediately get Beurling's theorem.

Theorem (Beurling)**.** *The non-zero invariant subspaces of the unilateral shift on $\mathcal{H}^2$ are just $\phi\mathcal{H}^2$, where ϕ is an inner function.*

In particular, note that the invariant subspaces of U on ℓ^2 observed earlier, $\{(a_n)_{n=0}^{\infty} : a_n = 0 \text{ for } n < N\}$ correspond to $\phi = z^N$.

The structure of inner functions is know in detail. In particular, they are all know to have a certain rather complicated form, and in terms of this form is is know when $\phi\mathcal{H}^2 \subset \psi\mathcal{H}^2$. That is, the whole lattice structure of the invariant subspaces of the unilateral shift is known! The interested reader can find a detailed treatment of this material in Martinez-Avendano and Rosenthal's book, [3].

References

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