THE CONJUGACY THEOREMS

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Abstract. We use the theory of differential forms, orientation, and mapping degree to prove that all maximal tori in a compact connected Lie group are conjugate. We also prove that all Lie groups are orientable, and that if G is a compact connected Lie group and T a maximal torus of G, then $\dim G/T$ is even.

1. Introduction

If G is a compact, connected Lie group, we define a maximal torus of G to be a maximal connected abelian subgroup. A dimension argument shows that every such Lie group has at least one maximal torus. If T is a maximal torus of G, then a structure theorem gives that T is isomorphic to a torus $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$. A generator of a torus $\mathbb{T}^n$ is defined to be an element $t \in \mathbb{T}^n$ such that $\{t^j : j = 1, 2, \cdots\}$ is dense in $\mathbb{T}^n$. If $t = [v]$, $v \in \mathbb{R}^k$, then Kronecker's theorem gives that t is a generator if and only if the components of v and 1 are linearly independent over $\mathbb{Q}$.

The rank of G is defined to be $\dim T$, and the Weyl group of T is defined to be $\mathcal{W} = N(T)/T$, where $N(T)$ is the normaliser of T. The Weyl group is proved to be finite in ([2], p. 18). However, maximal tori are not unique, and it is not clear that the rank and the Weyl group do not depend on the choice of maximal torus. This problem motivates us to prove that all maximal tori are conjugate to each other. We proceed by defining and computing a mapping degree, and we eventually conclude that every $g \in G$ is conjugate to some $t \in T$. To do this, we will need to first develop the theory of differential forms, orientation, and integration on manifolds. We assume that a maximal torus T and the Weyl group $\mathcal{W} = N(T)/T$ have been fixed.

2. Differential Forms

Let $x_1, \cdots, x_n$ be standard coordinates on $\mathbb{R}^n$. We define Ω^* to be the algebra over $\mathbb{R}$ generated by $dx_1, \cdots, dx_n$ subject to the relations $(dx_i)^2 = 0$ and $dx_i dx_j = -dx_j dx_i$ if $i \neq j$. As a real vector space, Ω^* has

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a basis $\{1\} \cup \{dx_{i_1}...dx_{i_k} : i_1 < \cdots < i_k, 1 \leq k \leq n\}$. We further define Ω^k to be the subspace of Ω^* generated by $\{dx_{i_1}...dx_{i_k} : i_1 < \cdots < i_k\}$.

The set of **smooth differential k-forms** on $\mathbb{R}^n$ is defined to be

$$\Omega^k(\mathbb{R}^n) = C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} \Omega^k.$$

Thus, if ω is a differential k form, then we can write

$$\omega = \sum f_{i_1,\cdots,i_k} dx_{i_1} \cdots dx_{i_k}$$

in a unique way, where all the coefficient functions are in $C^\infty(\mathbb{R}^n)$. We usually abbreviate this notation to $\omega = \sum f_I dx_I$. Note that, in particular, $\Omega^0(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$.

We define the exterior derivative, $d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$ by

$$\begin{aligned} df &= \sum \frac{\partial f}{\partial x_i} dx_i \ \text{ if } f \in \Omega^0(\mathbb{R}^n) \\ d\omega &= \sum (df_i) dx_I \ \text{ if } \omega = \sum f_I dx_I. \end{aligned}$$

So, for example, in $\Omega^*(\mathbb{R}^2)$, if $f(x,y) = x^2 y$, then

$$\begin{aligned} df &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 2xydx + x^2 dy \\ d(fdx) &= (df)dx = (2xydx + x^2 dy)dx = -x^2 dxdy, \end{aligned}$$

where we have used the relations $(dx)^2 = 0$ and $dxdy = -dydx$.

A smooth map $f : \mathbb{R}^m \to \mathbb{R}^n$ induces a pullback map on smooth functions $f^* : \Omega^0(\mathbb{R}^n) \to \Omega^0(\mathbb{R}^m)$ defined by $f^*(g) = g \circ f$. Let $x_1, \cdots, x_m$ and $y_1, \cdots, y_n$ be the standard coordinates on $\mathbb{R}^m$ and R^n, and let f_i be the i-th coordinate function of f. Then we extend f^* to a pullback on all forms $f^* : \Omega^*(\mathbb{R}^n) \to \Omega^*(\mathbb{R}^m)$ by

$$f^*\left(\sum g_I dy_{i_1} \cdots dy_{i_q}\right) = \sum (g_I \circ f) df_{i_1} \cdots df_{i_q}.$$

Note in particular that if f maps $\mathbb{R}^n$ to $\mathbb{R}^m$, then f^* maps $\Omega^*(\mathbb{R}^m)$ to $\Omega^*(\mathbb{R}^n)$ (the order of n and m are reversed), and that the the pullback of a k-form is again a k form. For example, if $f : \mathbb{R}^3 \to \mathbb{R}^2 : (x_1, x_2, x_3) \mapsto (x_1, x_2 + x_3^2)$, we consider $\mathbb{R}^2$ to have coordinates y_1, y_2 and we have

$$\begin{aligned} f^*(y_1 y_2^2 dy_1 dy_2) &= f^*(y_1 y_2^2)(df_1)(df_2) = x_1^2(x_2 + x_3^2)^2 dx_1 d(x_2 + x_3^2) \\ &= x_1^2(x_2 + x_3^2)^2 dx_1(dx_2 + 2x_3 dx_3) \\ &= x_1^2(x_2 + x_3^2)^2 dx_1 dx_2 + 2x_1^2(x_2 + x_3^2)^2 x_3 dx_1 dx_3. \end{aligned}$$

Proposition 2.1. *Let T be a smooth function from $\mathbb{R}^n$ to $\mathbb{R}^n$ with coordinate functions $T_1, \cdots, T_n$. If $x_1, \cdots, x_n$ are the standard coordinates of $\mathbb{R}^n$, then*

$$T^*(dx_1 \cdots dx_n) = dT_1 \cdots dT_n = J(T) dx_1 \cdots dx_n,$$

where $J(T)$ is the determinant of the Jacobian of T.

Proof. If π is a permutation of $(1, \cdots, n)$, then since $dx_i dx_j = -dx_j dx_i$, $dx_{\pi(1)} \cdots dx_{\pi(n)} = \mathrm{sgn}(\pi) dx_1 \cdots dx_n$. Here $\mathrm{sgn}(\pi)$ the usual sign of a permutation. Combining this with the fact that $(dx_i)^2 = 0$, we get

$$\begin{aligned} dT_1 \cdots dT_n &= (s \sum_j \frac{\partial T_1}{\partial x_j} dx_j) \cdots (s \sum_j \frac{\partial T_n}{\partial x_j} dx_j) \\ &= s \sum_\pi \frac{\partial T_1}{\partial x_{\pi(1)}} \cdots \frac{\partial T_n}{\partial x_{\pi(n)}} dx_{\pi(1)} \cdots dx_{\pi(n)} \\ &= J(T) dx_1 \cdots dx_n \end{aligned}$$

□

Proposition 2.2. *In the notation above, f^* commutes with d.*

Proof. See ([1], p.19). □

Proposition 2.3. *In the notation above, if ω and τ are forms, then $f^*(\omega\tau) = f^*(\omega) f^*(\tau)$.*

Proof. This is apparent from the definition of f^*. □

The idea of differential forms can be extended to smooth manifolds. (We will assume all manifolds are smooth.) Recall that a smooth n-manifold M is a second countable, Hausdorff topological space with a **consistent atlas**. A consistent atlas is an open cover of M by sets $\{U_\alpha\}_{\alpha \in A}$ for which each U_α is homeomorphic to $\mathbb{R}^n$. If these homeomorphisms are $\phi_\alpha : U_\alpha \to \mathbb{R}^n$, we also require that the transitions functions $\phi_\alpha \circ \phi_\beta^{-1}$ are smooth functions where defined. Sometimes we write $\{(U_\alpha, \phi_\alpha)\}$ for an atlas instead of $\{U_\alpha\}$.

A differential k-form ω on a smooth manifold M is a collection of differential forms ω_α on each chart U_α so that, if V_α and V_β are open sets in $\mathbb{R}^n$ with $(\phi_\alpha \phi_\beta^{-1})(V_\beta) = V_\alpha$, then $(\phi_\alpha \phi_\beta^{-1})^* V(\omega_\alpha)$ agrees with ω_β. We denote the set of differential forms on M (resp. k-forms) by $\Omega^*(M)$ (resp. $\Omega^k(M)$). The above theorems imply that d, and the product of forms (usually called **wedge product**), are well defined on $\Omega^*(M)$.

Note that $\phi_\alpha \phi_\beta^{-1}$ is a map from V_β to V_α, not from $\mathbb{R}^n$ to $\mathbb{R}^n$, so $(\phi_\alpha \phi_\beta^{-1})*$ is a map from $\Omega^*(V_\alpha)$ to $\Omega^*(V_\beta)$. This requires a notion of differential forms $\Omega^*(U)$ for a subset U of $\mathbb{R}^n$. However, all the definitions and theory above works for open subsets of $\mathbb{R}^n$ just as easily as for $\mathbb{R}^n$ itself, so this is not a problem.

In summary, a differential form on a manifold can be thought of as a collection of differential forms on the charts that agree on the intersections of charts. There is a rival definition of differential forms, which we describe on $\mathbb{R}^n$. Recall that the tangent space to x in $\mathbb{R}^n$, T_x, has basis $\partial x_1, \cdots, \partial x_n$. Given the form $\omega = dx_{i_1} \cdots dx_{i_k}$ on $\mathbb{R}^n$, and a point $x \in \mathbb{R}^n$, we define $\omega(x)$ to be the unique alternating k-linear functional on T_x such that

$$\begin{aligned} \omega(x)\left(\partial x_{\pi(i_1)}, \cdots, \partial x_{\pi(i_k)}\right) &= \operatorname{sgn}(\pi) \text{ if } \pi \text{ is a permutation} \\ \omega(x)\left(\partial x_{j_1}, \cdots, \partial x_{j_k}\right) &= 0 \text{ if } \{j_1, \cdots, j_k\} \neq \{i_1, \cdots, i_k\} \end{aligned}$$

Note that if $V_1 = \sum_i f_{1i} \partial x_i, \cdots, V_k = \sum_i f_{ki} \partial x_i$ are k smooth vector fields on $\mathbb{R}^n$, then $x \mapsto \omega(x)(V_1(x), \cdots V_k(x))$ is a smooth function from M to $\mathbb{R}$. Since this is true for forms $\omega = dx_{i_1} \cdots dx_{i_k}$, it is in fact true for all k forms $\omega \in \Omega^k(\mathbb{R}^n)$.

Conversely, let ω' any function that assigns to each point $x \in M$ an alternating, multilinear functional $\omega'(x)$ on T_x, such that if $V_1, \cdots V_k$ are smooth vector fields on $\mathbb{R}^n$, then $\omega'(x)(V_1(x), \cdots, V_k(x))$ is smooth. If we assume the V_i are written as above, we get

$$\begin{aligned} & \omega'(x)(V_1(x), \cdots, V_k(x)) \\ = \ & \omega'\Big(\sum_i f_{1i}(x) \partial x_i, \cdots, \sum_i f_{ki}(x) \partial x_i\Big) \\ = \ & \sum_{\{i_1 < \cdots < i_k\}} \sum_{\pi} \operatorname{sgn}(\pi) f_{\pi(i_1)}(x) \cdots f_{\pi(i_k)}(x) \omega'(x)(\partial x_{i_1}, \cdots, \partial x_{i_k}). \end{aligned}$$

So

$$\omega' = \sum_{\{i_1 < \cdots < i_k\}} \omega'(\partial x_{i_1} \cdots, \partial x_{i_k}) dx_{i_1} \cdots dx_{i_k},$$

where ω' on the right hand side is considered to be a smooth function from $\mathbb{R}^n$ to $\mathbb{R}$ that sends x to $\omega'(x)(\partial x_{i_1}, \cdots, \partial x_{i_k})$.

This proves that differential k-forms on $\mathbb{R}^n$ are the same thing as functions that map points to alternating k-linear functionals on tangent spaces, in a way that respects the smoothness of vector fields. This statement is true if we replace $\mathbb{R}^n$ with a smooth n manifold M; it is proved by applying the discussion above to charts.

Recall that if $f : M \to N$ is a smooth map between manifolds and $x \in M$, there is an induced map $f_* : T_x \to T_{f(x)}$ called the push forward. If V is a vector field on M, $f_* V$ is not in general a vector field on N (f may not be onto, for example), but we can nonetheless define $f_* V$.

Proposition 2.4. *If $f : M \to N$ is smooth, $\omega \in \Omega^k(N)$, and $V_1, \cdots, V_k$ are smooth vector fields on M then*

$$f^*(\omega)(V_1, \cdots, V_k) = \omega(f_*(V_1), \cdots, f_*(V_k)).$$

Proof. In order to avoid a description of how to compute push forwards, we omit this proof. The interested reader should consult ([3]) and try to prove the proposition for $M = N = \mathbb{R}^n$. $\square$

3. Volume Forms and Orientation

A **volume form** on an n-manifold M is a nowhere vanishing n-form on M. For example, $dx_1 \cdots dx_n$ is a volume form on $\mathbb{R}^n$, as is $-dx_1 \cdots dx_n$.

Let M be a manifold with atlas $\{(U_\alpha, \phi_\alpha)\}$. This atlas is called **oriented** if all transitions functions $\phi_\alpha \circ \phi_\beta^{-1}$ have positive Jacobian determinants everywhere they are defined. A manifold is called **orientable** if it has an oriented chart.

Proposition 3.1. *An n-manifold M is orientable if and only if it has a volume form.*

Proof. See ([1], p. 29). $\square$

Any two volume forms ω and ω' on a manifold M are related by $\omega = f\omega'$ for some non vanishing function f. If M is connected, then f is either everywhere positive or everywhere negative. So on a connected, orientable manifold, volume forms form two equivalence classes, where ω is equivalent to ω' if $\omega = f\omega'$ and f is everywhere positive. Each equivalence class is called an orientation, and written $[M]$. For example, the standard orientation on $\mathbb{R}^n$ is given by $dx_1 \cdots dx_n$.

In the proof of proposition 3.1, oriented atlases are associated with volume forms and vice versa. A volume form ω defines the orientation on an oriented chart $\{(U_\alpha, \phi_\alpha)\}$, if when ω is restricted to each chart, it is $m(x)dx_1 \cdots dx_n$ for some positive function $m(x)$. In this case, we say that ω is associated with the orientation given by the chart $\{(U_\alpha, \phi_\alpha)\}$, or simply that ω is associated with this orientable chart.

Theorem 3.2. *Lie groups are orientable.*

Proof. Note that if ω is a function that maps points to alternating n-linear functionals on tangent spaces, the corresponding form is non zero if and only if, when $V_1, \cdots, V_n$ are vector fields on M that are linearly independent at each point, then $\omega(V_1, \cdots, V_n)$ is a nowhere vanishing function on M.

Let G be a Lie group and let ω_e be any non zero alternating n-linear functional on the tangent space T_e of the identity e. Let L_g be the left multiplication by g, and let L_g^* be the corresponding pull back on forms. The we can define $\omega(g) = L_{g^{-1}}^*\omega_e$. Using the above description of vanishing forms, we can prove that ω is a no-where vanishing n-form on G, so G must be orientable. $\square$

Using the proposition 2.4, we can prove that $L_g^*(\omega) = \omega$. In fact this proof is in some sense trivial, but requires a nuanced understanding of push forwards, and in that sense is tricky. The experienced reader is encouraged to work out this proof.

The fact $L_g^*(\omega) = \omega$ is known as translation invariance of the volume form. In fact, ω is very closely related to a Haar measure, a translation invariant measure of the group. So we denote ω as dg. We do this because the Haar measure is denoted dg; we do not mean that dg is the exterior derivative of some other form. (It isn't!)

4. Integration on Manifolds

We say that a function f has **compact support** if it is zero outside of some compact set in the domain. Given a function f of compact support, we define

$$\int_{\mathbb{R}^n} f dx_{\pi(1)} \cdots dx_{\pi(n)} = \int_{\mathbb{R}^n} \operatorname{sgn}(\pi) f |dx_1 \cdots dx_n|$$

where the absolute value bars in the second integral are intended to indicate that that integral is a Riemann integral, whereas the first integral is the integral of a form.

Suppose T is a diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$. Suppose further that $y_1, \cdots, y_n$ are the standard coordinates of $\mathbb{R}^n$ and $y_i = x_i \circ T(y_1, \cdots, y_n)$ are coordinates after the diffeomorphism. Then, by proposition 2.1, we have

$$\int T^*(f dx_1 \cdots dx_n) = \int (f \circ T) J(T) dy_1 \cdots dy_n.$$

By the change of variables theorem, this is equal to $\pm \int_{\mathbb{R}^n} f dy_1 \cdots dy_n$ with positive sign if $J(T) > 0$ and negative sign if $J(T) < 0$.

Given a such a diffeomorphism T of $\mathbb{R}^n$, we call T orientation preserving if $J(T)$ is everywhere positive. By the above, integration on $\mathbb{R}^n$ is invariant under orientation preserving diffeomorphisms.

Given an open cover $\{U_\alpha\}$ of a manifold M, **a partition of unity subordinate to** $\{U_\alpha\}$ is defined to be a collection of smooth functions $\rho_\alpha : M \to \mathbb{R}$ such that ρ_α has support in U_α (this means that $\rho_\alpha(x) \neq 0$ implies $x \in U_\alpha$) and $\sum \rho_\alpha$ is the constant function 1 on M. We also require that only finitely many ρ_α are non zero at any point.

An example, note that $\{\sin^2(\theta/2), \cos^2(\theta/2)\}$ is a partition of unity of the unit circle subordinate to the open cover $\{(0, 2\pi), (-\pi, \pi)\}$.

Theorem 4.1. *For every manifold M, and every open cover $\{U_\alpha\}$ of M, there is a partition of unity of M subordinate to $\{U_\alpha\}$.*

Proof. See ([3], p.63). □

Let τ be a compactly supported n-form and $[M]$ be an orientation on M. Given an oriented atlas $\{(U_\alpha, \phi_\alpha)\}$ corresponding to the orientation $[M]$ and a partition of unity ρ_α subordinate to $\{U_\alpha\}$, we define

$$\int_{[M]} \omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha \tau,$$

where $\int_{U_\alpha} \rho_\alpha \tau$ means $\int_{\mathbb{R}^n} (\phi_\alpha^{-1})^*(\rho_\alpha \tau)$. Basically, we integrate over the manifold by using a partition of unity and integrating over charts. Usually, with a fixed orientation of M understood, we leave out the square brackets and write $\int_M \tau$.

Proposition 4.2. *The definition of the integral $\int_M \tau$ is independent of the oriented atlas $\{(U_\alpha, \phi_\alpha)\}$ and partition of unity ρ_α.*

Proof. See ([1], p.30). □

The most important theorem about integrals on manifolds is Stoke's theorem. We state a special case:

Theorem 4.3 (Stoke's Theorem)**.** *Let M be an oriented n dimensional smooth manifold. Let ω be an $n-1$ form on M with compact support. Then $\int d\omega = 0$.*

Proof. See ([1], p.32). □

Forms like $d\omega$ are called **exact forms**, so this case of Stoke's theorem says that the integral of an exact form on an oriented manifold is zero. The more general Stoke theorem discusses the integral of exact forms on manifolds with boundary.

5. Mapping Degree

Let M and N be compact n-manifolds with fixed orientations. As an application of the powerful Poincaré duality theorem ([1], p.44) we get that if ω and ω' are two n forms (necessarily with compact support), and if $\int_M \omega = \int_M \omega'$, then $\omega - \omega' = d\tau$, for some $n-1$ form τ. This fact, together with Stoke's theorem above, allows us to define the **mapping degree** of a smooth function $f : M \to N$. Let $\omega \in \Omega^n(N)$ be a form with integral 1 and define the mapping degree of f to be

$$\deg(f) = \int_M f^*(\omega).$$

If $\omega' \in \Omega^n(N)$ is another n-form with integral 1, we get that $\omega - \omega' = d\tau$ for some $\tau \in \Omega^{n-1}(N)$ and we get

$$\begin{aligned}\int_M f^*(\omega) - \int_M f^*(\omega') &= \int_M f^*(\omega - \omega') \\ &= \int_M f^*(d\tau) = \int_M df^*(\tau) = 0\end{aligned}$$

by Stoke's theorem. So mapping degree is in fact well defined. It can be proven that mapping degree is always an integer, and that homotopic maps have the same mapping degree ([1], p.40, p.35). However, we will not need these facts.

Proposition 5.1. *In the notation above, if f is not onto, then* $\deg f = 0$.

Proof. Pick $y \in M$ not in the range of f. The range of f is compact, so we can find a neighborhood U of y that is disjoint from the range of f. There exists an n-form ω, zero outside of U, with $\int_M \omega = 1$. (We will not prove this, but creating such "bump" functions is common in the theory of calculus of manifolds.) Using either definition of differential forms, it is straightforward to show that $f^*\omega = 0$. So

$$\deg(f) = \int_M f^*(\omega) = 0.$$

□

Recall that if we have an oriented chart of M, say $\{U_\alpha\}$, then a volume form α is associated with this orientation if α restricts to $a(x)dx_1 \cdots dx_n$ on each U_α and $a(x)$ is strictly positive.

Proposition 5.2. *Let $f : M \to N$ be a smooth map between compact oriented n-manifolds, and let α and β be volume forms on M and N associated with the orientations on M and N. If* $\det(f)$ *is defined by*

$$f^*(\beta) = \det(f)\alpha,$$

and $y \in N$ is a point with a finite pre-image $f^{-1}(y) = \{x_1, \cdots, x_n\}$ and $\det(f)(x_i) \neq 0$ *for all i, then*

$$\deg(f) = \sum_i \operatorname{sgn}(\det(f)(x_i)).$$

Proof. First note that $\det(f)$ is indeed a function of $x \in M$, and it is well defined since α is nowhere vanishing. Separate the x_i with disjoint charts U_i such that $f(U_i) \subset V$, where V is a chart for y. So all the U_i and V are homeomorphic to $\mathbb{R}^n$. By choosing the n-form β with to be zero outside of V, we get $\deg(f) = \sum_i \deg(f_i)$, where $f_i : \mathbb{R}^n \to \mathbb{R}^n$ is

f restricted to $U_i \simeq \mathbb{R}^n$. Thus it suffices to prove the theorem for a map $f : \mathbb{R}^n \to \mathbb{R}^n$ and a point y with a single pre-image $f^{-1}(y) = x$. Without loss of generality, x and y are both assumed to be 0.

Let $\alpha = a(x)dx_1 \cdots dx_n$ and $\beta = b(x)dx_1 \cdots dx_n$ be the two volume forms on the domain and range. Since α and β are associated to the orientations on the domain and range, we have that $a(x)$ and $b(x)$ are strictly positive. Then

$$f^*(\beta) = f^*(b(x))f^*(dx_1 \cdots dx_n) = f(b(x))J(f)dx_1 \cdots dx_n,$$

by proposition 2.1, where $J(f)$ is the Jacobian determinant of f. We also have that $f^*(\beta) = \det(f)\alpha$. So, we get that

$$\det(f) = b(f(x)))J(f)/a(x),$$

and in particular, since a and b are strictly positive, $\mathrm{sgn}(\det(f)) = \mathrm{sgn}(J(f))$. Since $\det(f)(0) \neq 0$, f must be a diffeomorphism of a neighbourhood of 0 onto a neighbourhood $V' \subset V$ of 0. By reducing the size of the U_i and V, and by changing β to be zero outside the new, smaller V', we can assume that the $f : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism.

Since f is a diffeomorphism, as discussed in section 4, we have

$$\int_{\mathbb{R}^n} f^*(\beta) = \mathrm{sgn}(J(f)) \int_{\mathbb{R}^n} \beta = \mathrm{sgn}(J(f)) = \mathrm{sgn}(\det(f))$$

and the result is proved. □

6. Maximal Tori

Recall that we have fixed a compact, connected Lie group G and a maximal torus T, and that we will speak of the Weyl group $\mathcal{W} = N(T)/T$, which we assume is finite. Let $k = \dim T$, and $n = \dim G$. By the closed subgroup theorem, a closed subgroup of a Lie group is again a Lie group. We also have the following.

Theorem 6.1. *Let H be a closed subgroup G, and let $\pi : G \to G/H : g \mapsto gH$ be the canonical quotient map. Then the topological space G/H is a smooth manifold of dimension $\dim G - \dim K = n - k$.*

Proof. See ([2], p.14), or for a more detailed discussion, ([4], p.218). □

Proposition 6.2. *If H is connected, G/H is orientable.*

Proof. As in the proof that Lie groups are orientable, let τ_e be an alternating $(n-k)$-linear functional on the tangent space T_{eT} to eT. We define a volume form $\tau(gT) = L^*_{g^{-1}}\tau_e$, where $L_{g^{-1}}$ is the map $hT \mapsto g^{-1}hT$. In general, L_g is not independent of choice of representative in gT, so it is not obvious that τ is well defined. To prove it is well defined, we need to show that if $h \in H$, then

$$L^*_{g^{-1}}\tau_e = L^*_{(gh)^{-1}}\tau_e = (L_{h^{-1}}L_{g^{-1}})^*\tau_e = L^*_{g^{-1}}L^*_{h^{-1}}\tau_e.$$

(It is true in general that $(fg)^* = g^*f^*$ for appropriate smooth functions f and g.) So in particular, it suffices to show that $L^*_{h^{-1}}\tau_e = \tau_e$. It makes sense to speak of this equality since $L_{h^{-1}}$ maps eT to itself.

For each h, $L^*_{h^{-1}}\tau_e$ is alternating $n-k$-linear functional on T_{eT}. Since the later space has dimension $n-k$, it can be shown that up to a scalar multiple, it has a unique alternating $n-k$-linear functional (corresponding to the determinant). So for all h, $L^*_{h^{-1}} = c(h)L^*_e$, and, in particular, $c(h)$ is a homomorphism from H to the multiplicative group $\mathbb{R}^*$. Since H is compact and connected, $c(h)$ must be identically 1. Thus, we get that $L^*_{h^{-1}}\tau_e = \tau_e$ for all $h \in H$. $\square$

We denote τ as $d(gH)$ since again we get that $L^*_g(d(gH)) = d(gH)$; $d(gH)$ is translation invariant.

Let $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{t}$ the sub-algebra of the torus. Pick an inner product on $\mathfrak{g}$ witch is invariant under the action of the adjoint representation $Ad(g)$ for all $g \in G$. Let $\mathfrak{g}/\mathfrak{t}$ be the orthogonal complement to $\mathfrak{t}$ with respect to this invariant inner product. The adjoint map restricted to the torus $Ad|_T$ acts trivially on every vector in $\mathfrak{t}$, and non trivially on every nonzero vector in $\mathfrak{g}/\mathfrak{t}$. By definition, Ad acting trivially on $v \in \mathfrak{g}/\mathfrak{t}$ means $Ad(t)(v) = v$ for all $t \in T$. If this is the case, pick $v(s)$, a one parameter subgroup in G with derivative v at the identity. Then $tv(s)t^{-1}$ is a one parameter subgroup in G with derivative v at the origin, since $Ad(t)(v) = v$. By the uniqueness of one parameter subgroups, it must be that $tv(s)t^{-1} = v(s)$ for all $t \in \mathbb{R}$. Thus, $v(\mathbb{R})$ commutes with T and $v(\mathbb{R})T$ is a torus containing T. Since T is maximal, this means $v(\mathbb{R}) \subset T$, so $v \in \mathfrak{t}$. Thus $v \in \mathfrak{t} \cap \mathfrak{g}/\mathfrak{t}$ so $v = 0$.

Since $\mathfrak{g}/\mathfrak{t}$ is an invariant subspace of Ad, we can define an induced action of T on $\mathfrak{g}/\mathfrak{t}$, denoted $Ad_{G/T} : T \to \mathrm{Aut}(\mathfrak{g}/\mathfrak{t}) : t \to Ad(t)|_{\mathfrak{g}/\mathfrak{t}}$. By proposition 6.1, since T is closed, G/T is a manifold of dimension $n - k$, where $n = \dim G$ and $k = \dim T$. The canonical projection $\pi : G \to G/T$ is smooth and hence induces a map π_* from $\mathfrak{g} = \mathfrak{g}/\mathfrak{t} \oplus \mathfrak{t}$ to the tangent space to G/T at the point eT. We claim that π_* maps $\mathfrak{g}/\mathfrak{t}$ isomophically onto this tangent space; this is apparent from a careful study of one of the proofs referenced in proposition 6.1. So, from now on, we will identify the tangent space to eT in G/T with $\mathfrak{g}/\mathfrak{t}$ via this map π_*.

T is a Lie group, so we can find a translation invariant volume form dt on T. So we get a translation invariant form $d(gT)dt$ on $G/T \times T$. Both $(d(gT)dt)(e)$ and $dg(e)$ are non zero alternating n-linear functionals on $\mathfrak{g}/\mathfrak{t} \oplus \mathfrak{t}$. Since, up to multiplication by a constant, there is a unique alternating multilinear n-linear functional on $\mathfrak{g}/\mathfrak{t} \oplus \mathfrak{t}$ (corresponding to determinant), we get that $(d(gT)dt)(e) = c \cdot dg(e)$ for some non-zero constant c. By replacing τ with $-\tau$ if necessary, we get that $c > 0$.

Now, if we give $G/T \times T$ and G the orientations associated to $d(gT)dt$ and dg, we can state the main lemma of this essay.

Lemma 6.3 (Main Lemma)**.** *Let G be a compact connected Lie group and T a maximal torus in G. Then the map*

$$q : G/T \times T \to G,\ (gT, t) \mapsto gtg^{-1}$$

has mapping degree $\deg(q) = |\mathcal{W}|$*, where $|\mathcal{W}|$ is the order of the Weyl group associated to T.*

Given this lemma, we easily get the conjugation theorems.

Theorem 6.4 (Conjugation Theorem 1)**.** *In a compact, connected Lie group G, every element is conjugate to an element in any fixed maximal torus.*

As an example, consider $SU(n)$, the special unitary matrices of size n. It can be shown by elementary means that a maximal torus is the diagonal matrices in some basis. So, the statement that every element is conjugate to an element in this maximal torus just says that every special unitary matrix can be diagonalised by a matrix in $SU(n)$!

Proof. By an proposition 5.1, since the mapping degree of the map q is non zero, q must be surjective. So for every $g \in G$ there are $hT \in G/T, t \in T$ so that $q(hT, t) = g$. Since $q(hT, t) = hth^{-1}$, we are done. □

Theorem 6.5 (Conjugation Theorem 2)**.** *Any two maximal tori in a compact, connected Lie group are conjugate.*

Proof. Let T and T' be two maximal tori. Pick a generator t of T, and, by the previous theorem, find $g \in G$ so that $gtg^{-1} \in T'$. Then, for all $n > 0$, $gt^n g^{-1} = (gtg^{-1})^n \in T'$. Thus $gTg^{-1} = \mathrm{cl}\{gt^n g^{-1} : n > 0\} \subset T'$, where cl denotes closure. Since T is maximal, we get that $T' = gTg^{-1}$ as desired. □

We now proceed to the proof of the main lemma.

Proof of Main Lemma. As in proposition 5.2, the determinant of the conjugation map q is defined by the equation $q^* dg = det(q)(d(gH)dt)$.

Lemma (Helper Lemma 1)**.**

$$\mathrm{sgn}(\det(q)(gT,t)) = \mathrm{sgn}(\det(Ad_{G/T}(t^{-1} - id_{G/T}))),$$

where $id_{G/T}$ is the identity on $\mathfrak{g}/\mathfrak{t}$.

Proof. This proof relies on proposition 2.4, and also required a good understanding of push forwards. Let $V_1, \cdots, V_n$ be smooth vector fields on $G/T \times T$. Then

$$\begin{aligned}
& q^* dg(V_1(gT,t), \cdots, V_n(gT,t)) \\
= \;& dg(q_*(V_1(gT,t)), \cdots q_*(V_n(gT,t))) \\
= \;& L^*_{q(gT,t)^{-1}} dg(q_*(V_1(gT,t)), \cdots q_*(V_n(gT,t))) \\
= \;& dg((L_{q(gT,t)^{-1}})_* q_*(V_1(gT,t)), \cdots, (L_{q(gT,t)^{-1}})_* q_*(V_1(gT,t))) \\
= \;& dg(e)((L_{q(gT,t)^{-1}} \circ q)_*(V_1(gT,t)), \cdots, (L_{q(gT,t)^{-1}} \circ q)_*(V_n(gT,t))),
\end{aligned}$$

where we have used the translation invariance of dg and proposition 2.4. Note in particular that in this last expression we have written $dg(e)$ to emphasize that all arguments of dg lie in the tangent space $\mathfrak{g}/\mathfrak{t} \oplus \mathfrak{t}$ to the identity. Also, we have

$$\begin{aligned}
& (dg(H)dt)(V_1(gT,t), \cdots, V_n(gT,t)) = \\
& (dg(H)dt)(e)((L_{(gT,t)^{-1}})_* V_1(gT,t), \cdots, (L_{(gT,t)^{-1}})_* V_n(gT,t))
\end{aligned}$$

where we have used the translation invariance of $dg(H)dt$. Now, we use the fact that $(dg(H)dt)(e) = c \cdot dg(e)$, and write

$$W_i(gT,t) = (L_{(gT,t)^{-1}})_* V_i(gT,t),$$

to get

$$(dg(H)dt)(V_1(gT,t), \cdots, V_n(gT,t)) = c \cdot dg(e)(W_1, \cdots, W_n)$$

and

$$\begin{aligned}
& q^* dg \quad (V_1(gT,t), \cdots, V_n(gT,t)) \\
= \;& dg(e) \quad ((L_{q(gT,t)^{-1}} \circ q \circ L_{(gT,t)})_* W_1(gT,t), \cdots, \\
& \qquad\quad (L_{q(gT,t)^{-1}} \circ q \circ L_{(gT,t)})_* W_n(gT,t)).
\end{aligned}$$

Since each $W_i(gT,t)$ is just a vector in $\mathfrak{g}/\mathfrak{t} \oplus \mathfrak{t}$, as in proposition 2.1, we get that this last expression is equal to

$$J(L_{q(gT,t)^{-1}} \circ q \circ L_{(gT,t)}) dg(e)(W_1, \cdots, W_n).$$

Thus

$$q^* dg = J(L_{q(gT,t)^{-1}} \circ q \circ L_{(gT,t)}) dg(e)(W_1, \cdots, W_n),$$

where J is the Jacobian determinant. Hence

$$\det(q) = J(L_{q(gT,t)^{-1}} \circ q \circ L_{(gT,t)})/c.$$

Note that this is just the determinant of $(L_{q(gT,t)^{-1}} \circ q \circ L_{(gT,t)})_*$ over c.

Now, we compute

$$(L_{q(gT,t)^{-1}} \circ q \circ L_{(gT,t)})(hT, s) = c(g)(c(t^{-1})(h)sh^{-1}),$$

where $c(a)(b) = aba^{-1}$ is conjugation. Let f be the function $(hT, s) \mapsto c(t^{-1})(h)sh^{-1}$. So by the chain rule

$$(L_{q(gT,t)^{-1}} \circ q \circ L_{(gT,t)})_* = Ad(g)f_*$$

and

$$\det((L_{q(gT,t)^{-1}} \circ q \circ L_{(gT,t)})_*) = \det(Ad(g)) \det(f_*).$$

Now, $Ad(g)$ is a unitary real matrix, so it must have determinant ± 1. Since G is connected and $Ad(e)$ has determinant 1, we get that $Ad(g)$ always has determinant 1. So $\deg(g) = \det(f_*)$.

To compute f_*, we only need to determine its action on differentiable curves through (eT, e). Say $v \in \mathfrak{g}/\mathfrak{t}$ and $v(r) \in G, r \in \mathbb{R}$ is a one parameter subgroup with derivative v at e. Then $(v(r)T, e)$ is a curve through (eT, e) with the corresponding derivative, and we can compute the action of $f*$ on this curve.

$$\begin{aligned} f_*(v) &= \frac{d}{dr} f((v(r)T, e) \\ &= \frac{d}{dr} c(t^{-1})(v(r))v(-r)) = \frac{d}{dr} t^{-1}v(r)tv(-r) \end{aligned}$$

We can express $t^{-1}v(r)tv(-r)$ as the composition $r \mapsto (r, r)$ and $(r, q) \mapsto t^{-1}v(r)tv(-q)$ and apply the chain rule to get that $f_*(v) = Ad(t^{-1})(v) - v$. Since $v \in \mathfrak{g}/\mathfrak{t}$, we write that f^* is $Ad_{G/H}(t^{-1}) - id_{G/H}$ on $\mathfrak{g}/\mathfrak{t}$.

Now suppose $v \in \mathfrak{t}$ and $v(r) \in G$ is a one parameter subgroup with derivative v at e. Then

$$\begin{aligned} f_*(v) &= \frac{d}{dr} f((eT, v(r)) \\ &= \frac{d}{dr} c(t^{-1})(e)v(r)) = \frac{d}{dr} v(r) = v. \end{aligned}$$

So f^* is the identity on $\mathfrak{t}$. Thus, in a matrix corresponding to the decomposition $\mathfrak{g}/\mathfrak{t} \oplus \mathfrak{t}$, we can write f_* as

$$\begin{pmatrix} Ad_{G/T}(t^{-1}) - id_{G/T} & 0 \\ 0 & id_T \end{pmatrix}$$

The result follows.

□

Lemma (Helper Lemma 2)**.** *There is a generator t of T such that*

(i) $|q^{-1}(t)| = |\mathcal{W}|$.

(ii) $Ad_{G/T}(t^{-1})$ *has no real eigenvalues. Consequently,* $\dim G/T$ *is even.*
(iii) $\det(q) > 0$ *at each of these points.*

Proof. We will in fact see that we can pick t to be the square of any generator of T.

(i) Let t be any generator of T. Note $q^{-1}(t) = \{(gT, s) \in G/T \times T : gsg^{-1} = t\}$. Of course, $g^{-1}tg$ depends only gT. Also, if $g^{-1}tg \in T$, then

$$\mathrm{cl}(\{g^{-1}tg = (g^{-1}tg)^n : n > 0\}) = g^{-1}Tg \subset T,$$

where cl is closure, so $g \in N(T)$. For every $gT \in N(T)/T$ we can find a unique s (which must be $g^{-1}tg$) with $(gT, s) \in q^{-1}(t)$.

(ii) Since $Ad_{G/T}(t^{-1})$ is an orthogonal linear transformation on $\mathfrak{g}/\mathfrak{t}$, its real eigenvalues (if any) must be ± 1. By Kronecker's theorem, we see that t^2 is again a generator of T, so by replacing t with t^2 we get that $Ad_{G/T}(t^{-1})$, for this new value of t, can only have real eigenvalues 1. If $Ad_{G/T}(t^{-1})$ had eigenvalue 1 with eigenvector $v \in \mathfrak{g}/\mathfrak{t}$, then we'd have $Ad_{G/T}(t^{-n})v = v$ for all n. However, $\{t^{-n} : n > 1\}$ is dense in T, so we'd get that $Ad_{G/T}(s)v = v$ for all $s \in T$. As discussed above, $Ad_{G/T}$ acts non trivially on each non zero vector in $\mathfrak{g}/\mathfrak{t}$, so this is impossible.

(iii) Basic linear algebra shows that $Ad_{G/T} - id_{G/T}$ has no real eigenvalues if $Ad_{G/T}$ has no real eigenvalues. So part (iii) follows from the following claim, and Helper Lemma 1. An real matrix with no real eigen values has positive determinant. Proof: Consider the characteristic polynomial of the matrix. It is monic, and its constant term is the determinant. (In general, the constant term is $(-1)^d$ times the determinant, where d is the dimension of the matrix. Since our matrix has no real eigenvalues, it must be of even dimension.) So, by the intermediate value theorem, if the determinant is not positive, the characteristic polynomial has a real root. This implies that the matrix has a real eigen value, a contradiction.

□

At this point, we can finally prove this main lemma. Pick t as in helper lemma 2. Then since $q^{-1}(t)$ consists of exactly $|\mathcal{W}|$ points and $\det(q) > 0$ at each of these points, proposition 5.2 gives that $\deg(q) = |\mathcal{W}|$. □

7. Corollaries

For completeness, we state without proof some of the corollaries of the conjugation theorems. As mentioned, the first two corollaries are that rank and Weyl group are well defined. We also have:

Corollary 7.1. *The exponential map of a compact connected Lie group is surjective.*

Corollary 7.2. *Let G be a compact connected Lie group, and T a maximal torus of G, and let $Z(H)$ denote the center of a subgroup H.*

(i) *$Z(T) = T$,*
(ii) *If $S \subset T$ is a connected abelian subgrouo, then $Z(S)$ is the union of the (maximal) tori containing S.*
(iii) *The center of G is the intersection of all maximal tori in G. In particular, $Z(G)$ is contained in any maximal torus.*

Corollary 7.3. *The Weyl group (of a compact, connected Lie group) acts effectively on the maximal torus. Thus, we may interpret the Weyl group as a group of automorphisms of T.*

Corollary 7.4. *Two elements of a maximal torus are conjugate (in a compact connected Lie group G) if and only if they lie in the same orbit under the action of the Weyl group.*

Some of these corollaries are tough to prove, but all make use of the conjugacy theorems. There are also consequences of the conjugacy theorem regarding class functions that are important in representation theory, but we omit those results here.

8. Final Remarks

The conjugacy theorems are deep theorems with important consequences. Their proof here is based on ([2]) and ([5]), which in turn references exposé number 26 in ([7]) by Serre as the original source. These are not the only proofs of the conjugacy theorems that use the notion of mapping degree; ([6]) offers a proof by computing the mapping degree of a different map and showing that the equation $g^k = h$ always has a solution for g (given $k > 0, h \in G$) in connected compact Lie groups. Also, [8] offers two proofs of the conjugacy theorems, one algebraic, and one using the Lefschetz fixed point theorem.

Our approach has several advantages. Firstly, it relates to the theory of differential forms and de Rham cohomology, which is a very important theory. Secondly, with only a little extra work, it is possible to prove the Weyl integration formula from the results presented here.

Thirdly, the discussion of of volume forms can be extended to define Haar measure on Lie groups. And finally, we get $\dim G/T$ is even. The fact that we can approach all these theorems in a unified way illustrates the power of topological ideas and the depth of Lie group theory.

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