CONVOLUTION OF VOLUME MEASURES

ALEX WRIGHT

Abstract. If M_1 and M_2 are hypersurfaces in $\mathbb{R}^n$ and μ_1 and μ_2 are their volume measures, we provide a formula for the absolutely continuous part h of $\mu_1 * \mu_2$. We prove h is continuous off a compact set of measure zero, and calculate it explicitely if M_1 and M_2 are spheres.

1. Introduction and Main Formula

Every oriented Riemann manifold M is endowed with a natural volume form dA. In oriented local coordinates $x_1, \cdots, x_n$,

$$dA = \sqrt{\det(g_{ij})} dx_1 \cdots dx_n.$$

See [**1**], p.257-262 for this and related facts. Corresponding to dA, there is a natural volume measure μ so that $\int f dA = \int f d\mu$ for continuous functions f.

In this note, M_1 and M_2 will be compact hyper-surfaces in $\mathbb{R}^n$. A transversality argument shows that M_1 and M_2 are orientable ([**4**]); hence these manifolds possess natural volume measures μ_1 and μ_2. As another consequence of orientability, we find that M_1 and M_2 possess ortho-normal vector fields n_1 and n_2.

A differentiable manifold is called *real analytic* if the transition functions are real analytic, that is, locally expressable as real power series. Ragozin proved that if M_1 and M_2 are real analytic, then $\mu_1 * \mu_2$ is absolutely continuous to Lebesgue measure m ([**2**], the proof is short). In this case we say simply that $\mu_1 * \mu_2$ is absolutely continuous and write $\mu_1 * \mu_2 \in L^1$. If M_1 and M_2 are spheres, Ragozin also explicitly computed the Radon-Nikodym derivative of $\mu_1 * \mu_2$.

Our primary result derives a formula for the absolutely continuous part of $\mu_1 * \mu_2$. If M_1 and M_2 are real analytic, this completely describes $\mu_1 * \mu_2$. We also prove that $\mu_1 * \mu_2$ is continuous off a compact set of measure zero, and re-derive Ragozin's formulas regarding spheres.

Define $\theta : M_1 \times M_2 \to [0, \pi)$ as the angle between $n_1(x_1)$ and $n_2(x_2)$, so $\sin \theta(x_1, x_2) = \sqrt{1 - (n_1(x_1) \cdot n_2(x_2))^2}$. Let $p : M_1 \times M_2 \to \mathbb{R}^n$ be the addition map; $p(x, z) = x + z$. Since M_1 and M_2 are compact, the set of critical points C_p of p is compact, and the set of critical values $C_v := p(C_p)$ is also compact. By Sard's Theorem C_v has measure zero.

It is important to note that y is a regular value for p if and only if M_1 and $y - M_2$ are transverse. Hence, for such a y, $N_y = M_1 \cap (y - M_2)$ is an $n-2$ dimensional sub-manifold of $\mathbb{R}^n$. It can be oriented by the following convention: $\partial x_1, \cdots, \partial x_{n-2} \in T_x N_y$ are oriented if $n_1(x), n_2(y-x), \partial x_1, \cdots, \partial x_{n-2}$ are oriented in $\mathbb{R}^n$. Thus N_y possesses a natural volume measure μ_{N_y}.

Theorem 1.1. *If $f \in C_c(\mathbb{R}^n - C_v)$, and*

$$h(y) = \int_{N_y} \frac{d\mu_{S_y}}{sin(\theta(x_1, x_2))},$$

then

$$\int_{\mathbb{R}^n} fd(\mu_1 * \mu_2) = \int_{\mathbb{R}^n} fhd.$$

Thus, h is the absolutely continuous part of $\mu_1 * \mu_2$.

Proof. By the Submersion Theorem ([**1**], p.133), for every $x \in M_1 \times M_2$ there are local coordinates $x_1, \cdots, x_{2n-2}$ near x so that

$$p(x_1, \cdots, x_{2n-2}) = (x_{n-1}, \cdots, x_{2n-2}).$$

Let π_1 and π_2 be the projections of $M_1 \times M_2$ onto M_1 and M_2 respectively. Let $s_1, \cdots, s_{n-1}$ be local coordinates for M_1 near $\pi_1(x)$, and let $t_1, \cdots, t_{n-1}$ be local coordinates for M_2 near $\pi_1(x)$. If $x \notin C_p$, then without loss of generality (reordering the t_i if necessary), we may assume that

$$\text{span}\{\partial p(x)/\partial s_1, \cdots \partial p(x)/\partial s_{n-1}, \partial p(x)/\partial t_1\} = \mathbb{R}^n.$$

In the future, considering how M_1 and M_2 are embedded in $\mathbb{R}^n$, we write, for example, ∂s_i instead of $\partial p/\partial s_i$. Now, the collection

$$\partial s_1, \cdots, \partial s_{n-1}, \partial t_i, \partial x_1, \cdots, \partial x_{n-2}$$

is linearly independent in $T_x(M_1 \times M_2)$. So, if we consider the map $x \mapsto (x_1, \cdots, x_{n-2}, t_1, s_1, \cdots, s_{n-1})$, defined on a neighborhood of x, we see that it has an invertible derivative at x. The inverse function theorem gives that these can serve as local coordinates for some suitably small neighbourhood of x. Thus we can obtain a neighborhood U_x of x with local coordinates $x_1, \cdots, x_{n-2}, t_1, s_1, \cdots, s_{n-1}$ with the following properties: U_x is rectangular in these coordinates; $p(x)$ does not depend on $x_1, \cdots, x_{n-2}$; $\pi_1 \circ s_i = s_i$ for $i = 1, \cdots, n-1$; and $\pi_2(t_1) = t_1$. These local coordinates will be crucial bellow.

Take $f \in \mathcal{C}_c(\mathbb{R}^n - C_v)$, supported on a compact set K disjoint from C_v. Take a finite open subcover $U_1, \cdots, U_m$ of the $U_x, x \in M_1 \times M_2$

for $p^{-1}(K)$. Let $r_1, \cdots, r_m$ be a partition of unity subordinate to this open subcover ([**3**], p.40). Now,

$$\begin{aligned}\int_{\mathbb{R}^n} fd(\mu_1 * \mu_2) &= \int_{M_1\times M_2} f(x+z)d\mu_1(x)\times d\mu_2(z)\\ &= \sum_k \int_{M_1\times M_2} r_k(x,z)f(x+z)d\mu_1(x)\times d\mu_2(z)\end{aligned}$$

At this point, we restrict out attention to a single coordinate patch U_k, and use our good local coordinates $x_1, \cdots, x_{n-2}, t_1, s_1, \cdots, s_{n-1}$. Let g_{M_1}, g_{M_2}, $g_{M_1\times M_2}$, g_{N_y}, $g_{\mathbb{R}^n}$ be the metric tensor for the manifolds indicated in the subscripts. We get

$$\int_{U_k} r_k \cdot (f\circ p)\cdot \sqrt{\det(g_{M_1\times M_2})}dx_1\cdots dx_{n-2}dt_1ds_1\cdots ds_{n-1}.$$

To compute $\sqrt{\det g_{M_1\times M_2}}$ we define a $2n$ by $n-1$ matrix J, whose top block J_1 is the Jacobian of $\pi_1\circ p$ with respect to the local coordinates of M_1 and the standard coordinates of $\mathbb{R}^n$, and whose bottom block J_2 is similarly the Jacobian of $\pi_2\circ p$. We have $g = JJ^t$. If the first $n-1$ columns of J represent the s_i coordinates, then fact that $\pi_1\circ s_i = s_i$ gives that J is block upper triangular. Thus

$$\det g_{M_1\times M_2} = \det JJ^t = \det(J_1J_1^t)\det(J_2J_2^t) = \det g_{M_1}\det g_{M_2}$$

where g_{M_1} is expressed in local coordinates $\partial s_1, \cdots, \partial s_{n-1}$ and g_{M_2} is expressed in the local coordinates $\pi_2 x_1, \cdots, \pi_2 x_{n-2}, t_1$.

So, the integral above becomes

$$\int_{U_k} r_k p\sqrt{\det g_{M_1}}\sqrt{\det g_{M_2}}dx_1\cdots dx_{n-2}ds_1\cdots ds_{n-1}dt_1.$$

We have that $t_1, s_1, \cdots, s_{n-1}$ serve as local coordinates near $p(x)$, and that ([**1**], Lemma 10.38)

$$\sqrt{g_{\mathbb{R}^n}(s_1,\cdots,s_{n-1},t_1)} = \langle n_1(\pi_1(x)), \partial t_1\rangle\sqrt{g_{M_1}}$$

where we write $g_{\mathbb{R}^n}(s_1,\cdots,s_{n-1},t_1)$ to stress the use of the non-standard local coordinates for $\mathbb{R}^n$.

Recall that $N_y = M_1\cap(y-M_2)$ is an $n-2$ dimensional oriented sub-manifold of $\mathbb{R}^n$. It is also an oriented sub-manifold of co-dimensions 1 in $y-M_2$. Thus N_y possesses an orthonormal vector field $n_y(x)$ as a sub-manifold of $y-M_2$. We have

$$\sqrt{g_{M_2}} = \langle n_y(y-\pi_2(x)), \partial t_1\rangle\sqrt{g_{N_y}}.$$

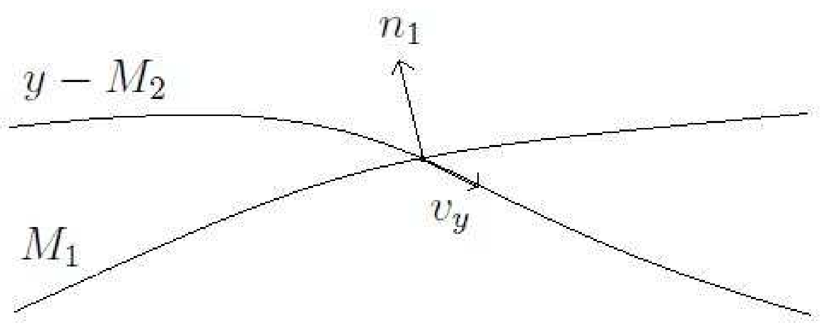
Hence, the integral above is

$$\int_{U_k} r_k \cdot (f \circ p) \cdot \frac{\langle n_y(y - \pi_2(x)), \partial t_1 \rangle}{\langle n_1(\pi_1(x)), \partial t_1 \rangle} \cdot dA_{N_y} dA_{\mathcal{R}^n}.$$

We can write $\partial t_1 = v_y + v_y^{\perp}$, where $v_y \in T_{\pi_2(x)} N_y$, and $\langle v_y^{\perp}, v_y \rangle = 0$. So

$$\begin{aligned} \frac{\langle n_y(y - \pi_2(x)), \partial t_1 \rangle}{\langle n_1(\pi_1(x)), \partial t_1 \rangle} &= \frac{\langle n_y(y - \pi_2(x)), v_y^{\perp} \rangle}{\langle n_1(\pi_1(x)), v_y^{\perp} \rangle} \\ &= \frac{\|v_y^{\perp}\|}{\|v_y^{\perp}\| \cos \varphi} \end{aligned}$$

where φ is the angle between $v_y^{\perp}$ and $n_1(\pi_1(x))$. We now draw a picture and find that $\varphi = \theta(x) + \pi/2$.



Thus we get

$$\int_{U_k} \frac{r_k \cdot (f \circ p)}{sin(\theta(x))} dA_{N_y} dA_{\mathbb{R}^n}.$$

Since we assumed that the U_k are rectangular we can use Fubini's Theorem to integrate first over the N_y coordinates and then over $\mathbb{R}^n$. Then, summing over the k we get the formula as desired.

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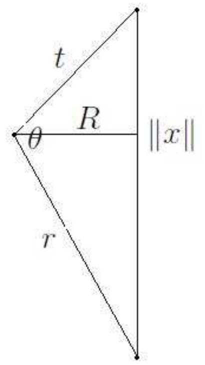
2. Surface measures on Spheres

Let $S_{n-1} \in \mathbb{R}^n$ be the $n-1$ dimensional unit sphere, and let its volume be V_{n-1}. Let μ_r be the volume measure on rS_{n-1} In **[2]**, Ragozin computed h the Radon-Nikodym derivative of $\mu_r * \mu_t$ $(r, t > 0)$. We are able to compute h using our formula and the fact that θ is constand on the N_y for spheres. Our results agree (check constant!!) with Ragozin's up to a constant depending on r and t, which is due to the fact that Ragozin uses measures normalized to have mass 1.

It is clear (from our formula, or more basic facts) that $h(y)$ depends only on $\|y\|$, and that $h(y)$ is zero unless $\|y\| \in (|r-t|, r+t)$. So, given y of appropriate norm, we draw a picture to aid our calculations, where the botton is the origian and the top is y. (Note: $\|x\|$ should read $\|y\|$!!)



Let A be the area of this triangle. Heron's formula yields

$$\begin{aligned} A &= \frac{1}{4}\sqrt{(r+t+\|y\|)(r+t-\|y\|)(\|y\|-r+t)(\|y\|+r-t)} \\ &= \frac{1}{4}\sqrt{((r+t)^2-\|y\|^2)(\|y\|^2-(r-t)^2)}. \end{aligned}$$

Now, $A = \|y\|R/2$ gives

$$R = \frac{\sqrt{((r+t)^2-\|y\|^2)(\|y\|^2-(r-t)^2)}}{2\|y\|}$$

and $A = rt\sin\theta$ gives

$$\frac{1}{\sin\theta} = \frac{2rt}{R\|y\|}.$$

To compute $h(y)$ we integrate the constant $1/\sin\theta$ over an $n-2$ dimensional sphere of radius R. Hence,

$$\begin{aligned} h(y) &= \frac{V_{n-2}R^{n-2}}{\sin\theta} \\ &= \frac{2rtV_{n-2}R^{n-2}}{R\|y\|} \\ &= \frac{2rtV_{n-2}}{\|y\|}\left(\frac{\sqrt{((r+t)^2-\|y\|^2)(\|y\|^2-(r-t)^2)}}{2\|y\|}\right)^{n-3} \\ &= \frac{rtV_{n-2}((r+t)^2-\|y\|^2)^{\frac{n-3}{2}}(\|y\|^2-(r-t)^2)^{\frac{n-3}{2}}}{2^{n-4}\|y\|^{n-2}} \end{aligned}$$

As Ragozin pointed out, we obtain the following as a corroloary.

Corollary 2.1. *If $n \geq 3$ and $r \neq t$, $\mu_r * \mu_t \in C_c(\mathbb{R}^n)$, and $\mu_r^2 \in L_p$ for all $p < n$. Also, for $n = 2$, $\mu_r * \mu_t \in L_p$ for all $p < 2$.*

Ragozin used this result to find examples of singular measures on $\mathbb{R}^n$, $n \geq 3$ whose convolution square is in $C_c(\mathbb{R}^n)$.

Guess: if μ_M^2 is never in $C_c(\mathbb{R}^n)$ if M is a manifold. To be investigated.

3. Continuity Property

We now prove that h always has a certain ammount of continuity.

Theorem 3.1. *h is continuous at each point of $\mathbb{R}^n - C_v$.*

Proof. Take $y \in \mathbb{R}^n - C_v$. For each $x \in N_y$, we have the coordinate patch U_x as above. We can take a finite subcover $U_1, \cdots, U_m$ of N_y, with each U_i centered at a point in N_y. So if V is an open ball contained in $\cap_i p(U_i) \cap (\mathbb{R}^n - C_v)$ and $U = p^{-1}(V)$ we get that that $U \simeq N_y \times V$ through the diffeomorphism $x \mapsto \pi(x), p(x)$, where π comes from the projection in each of the U_i onto the coordinates $x_1, \cdots, x_{n-2}$. The function $\sqrt{g_{M_1 \times M_2}}/\sin\theta(x_1, \cdots, x_{n-2}, y_1)$ is continuous everywhere on $U \simeq N_y \times V$. Now in the $U \simeq N_y$ coordinates, if $y_1 \in V$

$$h(y_1) = \int_{y_1 \times N_y} \frac{\sqrt{g_{M_1 \times M_2}}}{\sin\theta(x_1, \cdots, x_{n-2}, y_1)} dx_1 \cdots dx_{n-2}.$$

As $y_1 \to y$, the inside of this integral converes uniformly to its value at y, since pointwise convergence of continuous functions to a continuous function on a compact set gives uniform convergence. Thus result follows from the fact that integrals can be interchanged with uniform limits of continuous functions. ∎

References

[1] J. Lee, *An Introduction to Smooth Manifolds.* Springer, GTM 218, 2000.
[2] D. Ragozin, *Rotation invariant measure algebras on Euclidean space*, Indiana Univ. Math. J. **23** (1973/74), 1139–1154.
[3] W. Rudin, *Real and Complex Analysis.* McGraw-Hill, 1970.
[4] H. Samelson, *Orientability of Hypersurfaces inR^n*, Proceedings of the American Mathematical Society, 1969.

Pure Math. Dept., U. Waterloo, Waterloo, ON N2L–3G1, CANADA
E-mail address: alexmwright@gmail.com