DELIGNE'S THEOREM ON THE SEMISIMPLICITY OF A POLARIZED VHS OVER A QUASIPROJECTIVE BASE

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1. Introduction

The purpose of this expository note is to give Deligne's proof of the semisimplicity of a polarized complex variation of Hodge structures over a quasiprojective base [**Del87**, Proposition 1.13].

Theorem 1.1. *Let H be a polarized complex VHS over a quasiprojective base B. The the underlying flat bundle decomposes as*

$$H = \bigoplus_{i \in I} L_i \otimes W_i, \tag{1.1}$$

where the L_i are pairwise non-isomorphic irreducible flat bundles and the W_i are complex vector spaces.

Furthermore, the L_i and W_i admit polarized complex VHS, unique up to shifting the bigrading, such that equation 1.1 is an equality of complex polarized VHS.

Deligne states this theorem for weight 0 VHS, but this assumption is not used in the proof. Deligne assumes the existence of an underlying locally free $\mathbb{Z}$ local system $H_{\mathbb{Z}}$ such that $(H_{\mathbb{Z}}) \otimes \mathbb{C} = H$, but states that unpublished work of M. Nori can be used to remove this assumption.

For simplicity, we will assume that H is a real polarized pure VHS which has an underlying $\mathbb{Z}$ local system. However, the proof will only produce the structure of a complex VHS on the L_i and W_i, and will not produce underlying $\mathbb{Z}$ local systems for the L_i and W_i .

The key tool in the proof is the following result of Schmid [**Sch73**, Theorem 7.22], which is sometimes called the Theorem of the Fixed Part.

Theorem 1.2. *H be a complex polarized VHS over a quasiprojective base B. Let e be a global flat section of H, and write $e = \sum e_{p,q}$, with $e_{p,q} \in H^{p,q}$. Then each $e_{p,q}$ is again flat.*

This theorem is beyond the scope of this note, but the reader may see [**CMSP03**, Theorem 13.1.8] for an introduction. In fact Schmid states this result with some additional assumptions, but, as Deligne remarks in [**Del87**], they are not required.

M. Nori has communicated to the author that any flat bundle equipped with a harmonic metric, over a sufficiently nice base, is semisimple, owing to the fact that with respect to this metric, the perp of any flat subbundle is again flat. The Hodge metric on a real VHS is an example of a harmonic metric.

2. Definitions

Here we review the basic definitions, which seem to vary slightly according to the source.

Definition 2.1. A *pure complex variation of Hodge structure of weight* w over a complex manifold B is a local system of $\mathbb{C}$ modules H which admits a decomposition into complex C^∞ subbundles

$$H = \bigoplus_{p+q=w} H^{p,q}$$

satisfying *Griffiths' transversality relation*: If ∇ is the natural flat connection on H, and

$$F^p = \sum_{r \geq p} H^{r,w-r}$$

is the *Hodge filtration*, then $\nabla_\xi F^p \subset F^{p-1}$, where ξ is any holomorphic vector field on H.

A pure complex VHS H is said to be *real* if there is an underlying $\mathbb{R}$ local system $H_{\mathbb{R}}$ whose complexification is H, and the *reality constraint* $H^{p,q} = \overline{H^{q,p}}$ is satisfied. Sometimes the existence of an underlying $\mathbb{Z}$ or $\mathbb{Q}$ local system is assumed in the definition of a real VHS. We will assume a the existence of an underlying $\mathbb{Z}$ local system.

Definition 2.2. The weight w real VHS H is *polarized* if it is equipped with a nondegenerate bilinear form

$$b : H \times H \to \mathbb{C},$$

where $\mathbb{C}$ is the trivial $\mathbb{C}$ valued local system, which satisfies:

(1) $b(x, y) = (-1)^w b(y, x)$.
(2) $b(x, y) = 0$ if $x \in H^{p,q}$ and $x \in H^{p',q'}$ with $p \neq q'$.
(3) $i^{p-q} b(x, \overline{x}) > 0$ if $x \in H^{p,q}$ and $x \neq 0$.

The *Hodge inner product* h on H is defined by $h(x, y) = b(Cx, \overline{y})$, where the *Weil operator* C acts on H by multiplication by i^{p-q} on $H^{p,q}$. The conditions above imply that h is indeed a positive definite Hermitian form.

Given two complex polarized VHS H and H', the same structure may naturally be assigned to $H \oplus H'$, $H \otimes H'$, $H \wedge H'$, and $\mathrm{Hom}(H, H')$.

3. Semisimplicity of the Local System

In this section we will show that if the base B is quasiprojective, then the underlying flat bundle is simple, that is, a direct sum of flat bundles which do not contain any nontrivial flat subbundles. An equivalent definition of semisimple is that all flat subbundles are complemented.

We follow Deligne's outline in [**Del87**], which explains how his previous work [**Del71**] and Schmid's Theorem together give the result. It is only in this section that we will need an underlying $\mathbb{Z}$ local system $H_{\mathbb{Z}}$.

The proof considers the action of $\mathbb{C}^*$ on H, where $t \in \mathbb{C}^*$ acts on $H^{p,q}$ by multiplication by t^{p-q}. A subbundle V of H is preserved by the action if and only if $V = \bigoplus_{p,q} V \cap H^{p,q}$. In particular, if V is fixed by $\mathbb{C}^*$ then it is fixed by the Weil operator C, so hence its perp with respect to b is the same as its perp with respect to the Hodge inner product h.

Lemma 3.1. *Suppose that V is a flat rank 1 subbundle of H, and $\pi_1(B)$ acts on V through a finite group of n-th roots of unity. Then tV is again flat, for all $t \in \mathbb{C}^*$.*

Proof. For tV to be flat, it suffices for $(tV)^{\otimes n} \subset H^{\otimes n}$ to be flat. But $(tV)^{\otimes n} = t(V^{\otimes n})$, and $V^{\otimes n}$ has a flat global section e. By Schmid's Theorem, the $e_{p,q}$ are flat, and hence te is flat. Since $(tV)^{\otimes n}$ has a flat global section, it is of course flat. ∎

Now let d be the smallest dimension of a nontrivial flat subbundle of H, and let $W \subset H$ be sum of all d-dimensional flat subbundles of H. By construction, W is semisimple. Fix V a d dimensional flat subbundle of H, and V' a complement for V in W. For $t \in \mathbb{C}^*$, our goal is now to show that tV is also flat.

The bundle W is defined over $\mathbb{Q}$, in that there is $\mathbb{Q}$ local system $W_{\mathbb{Q}}$ whose complexification is W. To see this, consider the monodromy of H. The subbundle W corresponds to the sum of all the d dimensional invariant subspaces of the monodromy representation. The Galois group of $\mathbb{C}$ over $\mathbb{Q}$ acts on the set of (equivalence classes of) irreps of $\pi_1(B)$. Since the monodromy of H may be defined over $\mathbb{Q}$, if a d dimensional irrep occurs in the monodromy representation of H, then so must all its Galois conjugates. In a similar way we see that the set of irreps which occur in the monodromy of W is preserved by Galois conjugation, so W is defined over $\mathbb{Q}$.

Note that the bundle $(\wedge^r W)^{\otimes 2}$ is trivial, where $r = \dim W$. Indeed, this is because that $\wedge^r W = (\wedge^r W_{\mathbb{Z}}) \otimes \mathbb{C}$, where $W_{\mathbb{Z}} = W \cap H_{\mathbb{Z}}$, and because $\pi_1(B)$ may only act on $\wedge^r W_{\mathbb{Z}}$ by ± 1.

Now, the bundle $\wedge^d V \otimes \wedge^{r-d} V' \subset \wedge^d H \otimes \wedge^{r-d} H$ is isomorphic to $\wedge^r W$. Hence, we may apply Lemma 3.1 to conclude that

$$t(\wedge^d V \otimes \wedge^{r-d} V') = (t \wedge^d V) \otimes (t \wedge^{r-d} V')$$

is flat. This implies that $t \wedge^d V = \wedge^d tV$ is flat, and hence that tV is flat.

Since, for any d dimensional flat subbundle V of H, tV is also flat, we can conclude that W is t invariant for all $t \in \mathbb{C}^*$. Taking $t = i$, we find that the semisimple subbundle W is C invariant, and hence that its h-perp is equal to its b-perp. Its b-perp is obviously flat, whereas its h perp is obviously a complement to W. Hence this common perp space is a flat complement to W.

Induction on the dimension of H gives the result that the underlying flat bundle is semisimple. This allows us to write

$$H = \bigoplus_{i \in I} L_i \otimes W_i, \tag{3.1}$$

where the L_i are simple flat bundles and the W_i are complex vector spaces.

To show Theorem 1.1, it remains to assign the L_i and W_i polarized VHS, so that the decomposition of H is an equality of polarized VHS.

4. Polarized VHS on the L_i and W_i

We begin with a decomposition of H as $\bigoplus_{i \in I} L_i \otimes W_i$. The VHS structure on H induces a VHS structure on $\mathrm{End}(H)$. Let us consider the space $\Gamma(\mathrm{End}(H))$ of global flat sections of $\mathrm{End}(H)$. By Schurr's Lemma,

$$\Gamma(\mathrm{End}(H)) = \bigoplus_{i \in I} \Gamma(\mathrm{End}(W_i)).$$

Note that $\mathrm{End}(W_i)$ is a trivial bundle, with fibers equal to the space of endomorphisms of a fiber of the trivial bundle W_i.

Lemma 4.1. *For each i, there is a flat global section p_i of weight $(0,0)$ in $\mathrm{End}(W_i)$ which is a rank 1 projection.*

Proof. We will consider the action of $\mathbb{C}^*$ on $\mathrm{End}(H)$, wherein $t \in \mathbb{C}^*$ acts as t^{p-q} on the (p,q) part of the Hodge decomposition. By Schmid's Theorem, this action preserved the space of flat global sections, and so induces an action on $\bigoplus_{i \in I} \Gamma(\mathrm{End}(W_i))$.

In particular, this action provides a continuous map

$$\mathbb{C}^* \to \mathrm{Aut}\left(\bigoplus_i \Gamma(\mathrm{End}(W_i))\right).$$

By continuity, the image of this map is contained in $\bigoplus_i \mathrm{Aut}(\Gamma(\mathrm{End}(W_i)))$.

By the Skolem-Noether theorem, every automorphism of the full matrix algebra $\Gamma(\mathrm{End}(W_i))$ is given by conjugation. Hence there is a continuous map $\rho : \mathbb{C}^* \to \Gamma(\mathrm{Aut}(W_i))$, so that the action of $t \in \mathbb{C}^*$ on $\mathrm{End}(W_i)$ is given by conjugation by $\rho(t)$.

For each i, decompose $\Gamma(W_i)$ into $\bigoplus \Gamma(W_i)^p$, where $\rho(t)$ acts on $\Gamma(W_i)^p$ by t^p. Pick p so $\Gamma(W_i)^p$ is non-empty, and choose a line $\mathbb{C} \in \Gamma(W_i)^p$. This defines a sub line bundle in W_i, and a projection p_i onto the line bundle. The projection p_i must be a sum of endomorphisms of weight (k, k), since it is fixed by $\mathbb{C}^*$. Since $\mathrm{End}(H)$ is a VHS of weight 0, it follows that p_i must be of pure weight $(0, 0)$. ∎

Now, L_i is isomorphic to $p_i H$, and this expression induces the structure of a VHS on L_i. The VHS structure on W_i is given by the isomorphism $W_i = \mathrm{Hom}(L_i, H)$

We omit the check that the VHS on the L_i and W_i combine to give the VHS on H.

The VHS on L_i and W_i are not unique. Indeed, L_i can be tensored with an arbitrary rank 1 VHS, and W_i with its dual, and the result does not change the VHS on $L_i \otimes W_i$. This tensoring operation amounts to changing the weights (bigrading) of the W_i and S_i. However, the above proof shows that the VHS on W_i is determined up to shift of bigrading by the VHS of $\mathrm{End}(H)$, which is induced from that of H. The projection p_i will always be an isomorphism of VHS from H to L_i tensor a rank 1 subbundle of W_i of pure weight, which shows that the VHS on L_i is well defined up to shifting the bigrading.

References

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