HODGE-TEICHMÜLLER PLANES AND FINITENESS RESULTS FOR TEICHMÜLLER CURVES

CARLOS MATHEUS AND ALEX WRIGHT

ABSTRACT. We prove that there are only finitely many algebraically primitive Teichmüller curves in the minimal stratum in each prime genus at least 3. The proof is based on the study of certain special planes in the first cohomology of a translation surface which we call Hodge-Teichmüller planes.

We also show that algebraically primitive Teichmüller curves are not dense in any connected component of any stratum in genus at least 3; the closure of the union of all such curves (in a fixed stratum) is equal to a finite union of affine invariant submanifolds with unlikely properties. Results of this type hold even without the assumption of algebraic primitivity.

Combined with work of Nguyen and the second author, a corollary of our results is that there are at most finitely many non-arithmetic Teichmüller curves in $\mathcal{H}(4)^{hyp}$.

1. INTRODUCTION

Context. There is an analogy between the dynamics of the natural $SL(2, \mathbb{R})$–action on the moduli space of unit area translation surfaces and homogenous space dynamics. This analogy has been very fruitful in the study of rational billiards, interval exchange transformations, and translation flows, as well as related problems in physics. From this point of view, a key goal is to obtain a classification of $SL(2, \mathbb{R})$–orbit closures, in partial analogy with Ratner's Theorem for homogeneous spaces.

Quite a lot of progress has occurred on this problem. McMullen has classified orbit closures in genus 2 [**McM07**], and Eskin-Mirzakhani-Mohammadi have proved that all orbit closures are *affine invariant submanifolds* [**EM, EMM**]. Progress on understanding affine invariant submanifolds is ongoing [**Wria, Wrib**]. Despite all this, a classification even just of closed orbits is not available in any genus greater than 2.

As is common practice, we will call closed $SL(2, \mathbb{R})$–orbits *Teichmüller curves*. (The projections of these orbits to the moduli space of Riemann surfaces are also called Teichmüller curves; these projections are in fact complex algebraic curves which are isometrically immersed with respect to the Teichmüller metric.) Teichmüller curves yield examples of translation surfaces and billiards with optimal dynamical properties [**Vee89**], as well as families of algebraic curves with extraordinary algebro-geometric properties [**Möl06b, Möl06a, McM09**].

As is the case with lattices in Lie groups, Teichmüller curves come in commensurability classes, and are either arithmetic or not. In the non-arithmetic case, every commensurability class contains a unique minimal representative, which is called primitive [**Möl06a**]. Teichmüller curves that are primitive for algebraic reasons are called algebraically primitive, and it is precisely these curves which are "as non-arithmetic as possible."

There are infinitely many algebraically primitive Teichmüller curves in $\mathcal{H}(2)$, which were constructed independently by Calta and McMullen [**Cal04, McM03a**] and classified by McMullen [**McM05**]. McMullen showed that there is only one algebraically primitive Teichmüller curve in $\mathcal{H}(1, 1)$ [**McM06b**]. Finiteness of algebraically primitive Teichmüller curves is known in $\mathcal{H}(g-1, g-1)^{hyp}$ by work of Möller [**Möl08**], and in $\mathcal{H}(3, 1)$ by work of Bainbridge-Möller [**BM12**]. Bainbridge and Möller have informed the second author that together with Habegger they have recently established new finiteness results that are complementary to those in this paper. Their methods are very different from ours.

The assumption of *algebraic* primitivity is made primarily for two reasons. Firstly, the key tools include Möller's algebro-geometric results [**Möl06b, Möl06a**], and these results are most restrictive in the algebraically primitive case. Secondly, in some situations finiteness is not true for primitive but not algebraically primitive curves. For example, McMullen [**McM06a**] has constructed infinitely many primitive Teichmüller curves in the minimal stratum in genus 3 and 4 (none of which are algebraically primitive).

Statement of results. Our main result is

Theorem 1.1. *There are only finitely many algebraically primitive Teichmüller curves in the minimal stratum in prime genus at least 3. Furthermore, algebraically primitive Teichmüller curves are not dense in any connected component of any stratum in genus at least 3.*

A purely algebro-geometric reformulation of the first statement of Theorem 1.1 is possible: the locus of all eigenforms in the projectivized

minimal stratum in prime genus at least 3 is a countable set together with at most a finite number of algebraic curves.[1]

A key new idea in the proof of Theorem 1.1 is the study of Hodge-Teichmüller planes. A *Hodge-Teichmüller plane* at a translation surface M is a plane $P \subset H^1(M, \mathbb{R})$ such that for any $h \in SL(2, \mathbb{R})$, the plane[2] $hP \subset H^1(hM, \mathbb{R})$ satisfies $\dim_{\mathbb{C}}(hP)^{1,0} = 1$. It is then automatic that $\dim_{\mathbb{C}}(hP)^{0,1} = 1$. Here

$$(hP)^{1,0} = H^{1,0}(hM) \cap (hP \otimes \mathbb{C}) \quad \text{and} \quad (hP)^{0,1} = H^{0,1}(hM) \cap (hP \otimes \mathbb{C}).$$

Hodge-Teichmüller planes are thusly named because they respect the Hodge decomposition along the entire Teichmüller disk of a translation surface. (The Teichmüller disk of a translation surface is the projection of the $SL(2, \mathbb{R})$–orbit to the moduli space of Riemann surfaces.) The consideration of these planes is motivated by Möller's description of the variation of Hodge structure over a Teichmüller curve [**Möl06b**].

Theorem 1.2. *Suppose $\mathcal{M}$ is an affine invariant submanifold that contains a dense set of Teichmüller curves whose trace fields have degree at least k. Then every translation surface in $\mathcal{M}$ has k orthogonal Hodge-Teichmüller planes.*

In particular, suppose $\mathcal{M}$ is an affine invariant submanifold of genus g translation surfaces, and algebraically primitive Teichmüller curves are dense in $\mathcal{M}$. Then every translation surface in $\mathcal{M}$ has g orthogonal Hodge-Teichmüller planes.

The orthogonality is either with respect to the Hodge inner product, or the symplectic form: for a Hodge-Teichmüller plane, the orthogonal complement is the same for both. When we say that a translation surface "has k orthogonal Hodge-Teichmüller planes" more precisely we mean that the translation surface has a set of k pairwise orthogonal Hodge-Teichmüller planes (there may also be more).

If $M = (X, \omega)$, then the tautological plane $P = \mathrm{span}_{\mathbb{R}}(\mathrm{Re}(\omega), \mathrm{Im}(\omega))$ is Hodge-Teichmüller. In genus two, the orthogonal complement of this plane always gives a second Hodge-Teichmüller plane. However,

---

[1]Teichmüller curves can be characterized by the torsion and real multiplication conditions discovered by Möller [**Möl06b, Möl06a**]. In particular, any curve in the projectivized minimal stratum, all of whose points $(X, \mathbb{C} \cdot \omega)$ have the property that $\mathrm{Jac}(X)$ has real multiplication with ω as an eigenform, must be the projectivization of a Teichmüller curve. Added in proof: A generalization of this unpublished observation of the second author has been very recently used by Filip, in his work showing that all $SL(2, \mathbb{R})$ orbit closures are varieties [**Fila, Filb**].

[2]See Section 2.3 for details.

Theorem 1.3. *In any connected component of any stratum of unit area translation surfaces in genus $g \geq 3$, there is a translation surface which does not have $g - 1$ orthogonal Hodge-Teichmüller planes.*

The final ingredients in the proof of Theorem 1.1 are

Theorem 1.4 (Eskin-Mirzakhani-Mohammadi [**EMM**]). *Any closed $SL(2, \mathbb{R})$–invariant subset of a stratum is a finite union of affine invariant submanifolds.*

Theorem 1.5 (Wright [**Wrib**]). *Let $m \geq 2$ be prime. Any affine invariant submanifold of the minimal stratum in genus m which properly contains an algebraically primitive Teichmüller curve must be equal to a connected component of the stratum.*

Proof of Theorem 1.1 using Theorems 1.2, 1.3, 1.4 and 1.5. Let $\mathcal{H}$ be a connected component of a stratum of genus $g \geq 3$ unit area translation surfaces. By Theorem 1.4, the closure of the union of all algebraically primitive Teichmüller curves in $\mathcal{H}$ is a finite union of affine invariant submanifolds $\mathcal{M}_i, i = 1, \ldots, \ell$.

By Theorem 1.2, every translation surface in each $\mathcal{M}_i$ must have g orthogonal Hodge-Teichmüller planes. Thus Theorem 1.3 gives that no $\mathcal{M}_i$ can be equal to $\mathcal{H}$. In particular, algebraically primitive Teichmüller curves are not dense in $\mathcal{H}$.

Now suppose $\mathcal{H}$ is a connected component of the minimal stratum in prime genus. Then Theorem 1.5 gives that each $\mathcal{M}_i$ must be a Teichmüller curve. In particular, there are only finitely many algebraically primitive Teichmüller curves in $\mathcal{H}$. $\qquad\square$

Additional results. In recent work, Nguyen and the second author have shown that there are no affine invariant submanifolds in $\mathcal{H}(4)^{hyp}$, except Teichmüller curves and $\mathcal{H}(4)^{hyp}$ itself [**NW**]. From this we get

Theorem 1.6 (Matheus-Nguyen-Wright). *There are at most finitely many non-arithmetic Teichmüller curves in $\mathcal{H}(4)^{hyp}$.*

Theorem 1.6 is false in $\mathcal{H}(4)^{\mathrm{odd}}$ by work of McMullen [**McM06a**]. There are at present two known non-arithmetic Teichmüller curves in $\mathcal{H}(4)^{\mathrm{hyp}}$, corresponding to the regular 7-gon and 12-gon. These examples are due to Veech [**Vee89**]. The first is algebraically primitive, and the second is not. Bainbridge and Möller have conjectured that the regular 7-gon is the only algebraically primitive Teichmüller curve in $\mathcal{H}(4)^{\mathrm{hyp}}$ [**BM12**, Ex. 14.4].

Proof using Theorems 1.2, 1.3, and 1.4. By Theorem 1.4, the closure of the union of all non-arithmetic Teichmüller curves in $\mathcal{H}(4)^{hyp}$

is a finite union of affine invariant submanifolds $\mathcal{M}_i, i = 1, \ldots, \ell$. As above, it suffices to rule out the case that some $\mathcal{M}_i$ properly contains a Teichmüller curve.

If this were the case, the work of Nguyen-Wright [**NW**] would give that $\mathcal{M}_i = \mathcal{H}(4)^{hyp}$. Theorem 1.2 would thus give that each translation surface in $\mathcal{H}(4)^{hyp}$ has 2 orthogonal Hodge-Teichmüller planes, contradicting Theorem 1.3. □

Added in proof: recently a similar argument has been used to give results in the other component of $\mathcal{H}(4)$ [**ANW**].

More on the proofs. We will see that the work of Möller [**Möl06b**] shows that any translation surface on a Teichmüller curve has at least k orthogonal Hodge-Teichmüller planes, where k is the degree of the trace field. Theorem 1.2 will then follow from the observation that the limit of Hodge-Teichmüller planes is a Hodge-Teichmüller plane.

The proof of Theorem 1.3 will be an induction on the genus and number of zeros. The base case is

Theorem 1.7. *In both connected components $\mathcal{H}(4)^{hyp}$ and $\mathcal{H}(4)^{odd}$ of the minimal stratum in genus 3, there are translation surfaces that do not have 2 orthogonal Hodge-Teichmüller planes.*

Given a translation surface M, we will consider the action of its affine group on $H^1(M, \mathbb{R})$. We will show that the Zariski closure of the image of this action preserves the set of Hodge-Teichmüller planes. Theorem 1.7 will then be established by exhibiting specific square-tiled surfaces M in $\mathcal{H}(4)^{hyp}$ and $\mathcal{H}(4)^{odd}$ for which the Zariski closure is large enough to imply that there are no Hodge-Teichmüller planes besides the tautological plane.

The inductive step of the proof of Theorem 1.3 is

Theorem 1.8. *Let $\mathcal{H}$ be a connected component of a stratum of genus $g \geq 3$ translation surfaces. Suppose that every translation surface in $\mathcal{H}$ has k orthogonal Hodge-Teichmüller planes.*

Then there is a connected component $\mathcal{H}'$ of the minimal stratum of genus g translation surfaces in which every translation surface has at least k orthogonal Hodge-Teichmüller planes.

If $\mathcal{H}$ is already a connected component of the minimal stratum $\mathcal{H}(2g - 2)$, then there is a connected component $\mathcal{H}'$ of $\mathcal{H}(2g - 4)$ in which every translation surface has at least $k - 1$ orthogonal Hodge-Teichmüller planes.

The proof of Theorem 1.8 uses results developed by Kontsevich-Zorich for the classification of connected components of strata [**KZ03**].

Specifically, we will consider degenerations of translation surfaces $M_n \to M$ that arise from opening up a zero on M or bubbling off a handle on M, and we will show that the limit of Hodge-Teichmüller planes on M_n is a Hodge-Teichmüller plane on M.

Proof of Theorem 1.3 using Theorems 1.7 and 1.8. Suppose, in order to find a contradiction, that there is a connected component of a stratum of genus $g \geq 3$ translation surfaces in which every translation surface has $g - 1$ orthogonal Hodge-Teichmüller planes. By applying Theorem 1.8 a number of times, we eventually see that a connected component of $\mathcal{H}(4)$ must consist entirely of translation surface with 2 orthogonal Hodge-Teichmüller planes. However, this contradicts Theorem 1.7. □

References. For a flat geometry perspective on translation surfaces, see for example [**GJ00**], [**HS06**], and [**SW10**].

There are only very few examples of primitive Teichmüller curves known. They are the Prym curves in genus 2, 3 and 4 [**McM03a, McM06b**] (see also [**Cal04**] for the genus 2 case); the Veech-Ward-Bouw-Möller curves [**BM10**]; and two sporadic examples, one due to Vorobets in $\mathcal{H}(6)$, and another due to Kenyon-Smillie in $\mathcal{H}(1,3)$ [**HS01, KS00**]. These sporadic examples correspond to billiards in the $(\pi/5, \pi/3, 7\pi/15)$ and $(2\pi/9, \pi/3, 4\pi/9)$ triangles respectively, and both Teichmüller curves are algebraically primitive.

See [**Wri13**] for an introduction to the Veech-Ward-Bouw-Möller curves and more background on Teichmüller curves, and Hooper for a flat geometry perspective [**Hoo13**].

The (primitive) Teichmüller curves in $\mathcal{H}(2)$ have been extensively studied by McMullen [**McM05**], Bainbridge [**Bai07**], and Mukamel [**Mukb, Muka**]. The Prym curves are not quite as well understood but have also been extensively studied by Möller [**Möl**] and Lanneau-Nguyen [**LN**].

McMullen has shown that Teichmüller curves in moduli space are rigid [**McM09**].

Organization. In Section 2, we recall some basic material on translation surfaces and their moduli spaces. In particular, the notions of Teichmüller curves, affine invariant submanifolds and the Hodge inner product are quickly reviewed. In this section we give the precise definition of Hodge-Teichmüller planes.

In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorem 1.7. In Section 5 we prove Theorem 1.8 modulo some technical results, which are proved in Section 6.

We conclude the paper in Section 7 by listing a few open problems that may now be accessible.

2. Definitions

In this section we define all the terms used in the introduction, and provide some basic background. Additional background material will be recalled as necessary in the subsequent sections.

2.1. **Abelian differentials and translation surfaces.** Let X be a Riemann surface of genus $g \geq 1$. A (non-trivial) Abelian differential ω on X is simply a (non-zero) holomorphic 1-form. Given an Abelian differential ω on a Riemann surface X, denote by Σ the finite set of zeroes of ω. By locally integrating ω outside Σ, we obtain a translation atlas on $X - \Sigma$, i.e., an atlas whose change of coordinates are translations of the plane $\mathbb{R}^2$. From the point of view of the translation atlas, a zero p of ω is a conical singularity with total angle $2\pi(k + 1)$ where k is the order of zero p. Conversely, given a translation atlas with conical singularities as above, we recover an Abelian differential by pulling back the form $dz = dx + idy$ on $\mathbb{R}^2$ via the charts of the translation atlas. In other words, there is a one-to-one correspondence between Abelian differentials and translation surfaces.

Note that $SL(2, \mathbb{R})$ acts on the set of translation surfaces by postcomposition with the charts of translation atlas. That is, $SL(2, \mathbb{R})$ acts naturally on Abelian differentials. Moreover, this $SL(2, \mathbb{R})$-action

on Abelian differential preserves the total area function $\text{Area}(\omega) :=$ $\frac{i}{2} \int_X \omega \wedge \overline{\omega}$.

We will typically denote a translation surface by $M = (X, \omega)$.

2.2. **Moduli spaces of Abelian differentials.** Let $\mathcal{H}_g$ denote the moduli space of Abelian differentials ω with unit total area (i.e., $\text{Area}(\omega) = 1$) on Riemann surfaces X of genus $g \geq 1$. It is well known that $\mathcal{H}_g$ is naturally stratified, where the strata $\mathcal{H}(k_1, \ldots, k_\sigma)$ are obtained by collecting together Abelian differentials $(X, \omega) \in \mathcal{H}_g$ whose orders of zeros is prescribed by the list $(k_1, \ldots, k_\sigma)$. The action of $SL(2, \mathbb{R})$ on $\mathcal{H}_g$ preserves each stratum $\mathcal{H}(k_1, \ldots, k_\sigma)$. For later use (in this section only), let us denote by

$$\mathbb{R}\mathcal{H}(k_1, \ldots, k_\sigma) = \{(M, t\omega) : t \in \mathbb{R}, (M, \omega) \in \mathcal{H}(k_1, \ldots, k_\sigma)\}.$$

Kontsevich and Zorich have classified the connected components of strata [**KZ03**]. In particular, each stratum has at most 3 connected components.

Work of Masur and Veech shows that the $SL(2, \mathbb{R})$–action on any connected component of a stratum is ergodic with respect to certain fully supported invariant probabilities called Masur-Veech measures [**Mas82, Vee82**]. In particular, typical translation surfaces (with respect to Masur-Veech measures) have dense orbits.

2.3. **Hodge-Teichmüller planes.** We now give a precise definition of a Hodge-Teichmüller plane.

Let $\mathcal{T}_g$ denote the Teichmüller space of genus g Riemann surfaces. A point in $\mathcal{T}_g$ is an isomorphism class of Riemann surface X together with a marking $f : S \to X$, which is an isotopy class of homeomorphisms from a fixed topological surface S to X. Let $H^{1,0}$ denote the bundle of Abelian differentials over $\mathcal{T}_g$. Its fiber over a Riemann surface X is the space of Abelian differentials $H^{1,0}(X)$. There is a map $H^{1,0}$ to $\mathcal{H}_g$ which simply forgets the marking on the Riemann surface. Put differently, $\mathcal{H}_g$ is the quotient of $H^{1,0}$ by the mapping class group.

Let $H^1_{\mathbb{R}}$ be the bundle over $H^{1,0}$ whose fiber over a marked Riemann surface X with Abelian differential ω is $H^1(X, \mathbb{R})$. Since $H^1(X, \mathbb{R})$ is canonically identified with $H^1(S, \mathbb{R})$ via the marking, this bundle is a trivial bundle.

There is a natural $SL(2, \mathbb{R})$ action on the bundle $H^{1,0}$ over Teichmüller space, which projects to the $SL(2, \mathbb{R})$ action on $\mathcal{H}_g$. There is also an action of $SL(2, \mathbb{R})$ on the bundle $H^1_{\mathbb{R}}$, which is typically used to define the Kontzevich-Zorich cocycle (see the next remark). This action is defined to act trivially on the fibers. In other words, given a cohomology class η on a marked (X, ω), and given $h \in SL(2, \mathbb{R})$,

the cohomology class $h(\eta) \in H^1(h(X, \omega), \mathbb{R})$ is defined to be the image of η under the isomorphism of cohomology groups provided by affine map induced by h. (Since the marking on $h(X, \omega)$ is defined by post-composition with this affine map, $h(\eta)$ is identified to the same cohomology class in $H^1(S, \mathbb{R})$ as η, which is why the action is said to be trivial on fibers.)

Consider a plane $P \subset H^1(X, \mathbb{R})$, where (X, ω) is an (unmarked) translation surface. If we pick a marking $S \to X$, we may view P as a plane in a fiber of the bundle $H^1_{\mathbb{R}}$, and hence act on it by $SL(2, \mathbb{R})$. Given a plane $P \subset H^1(X, \mathbb{R})$, we denote by $P^{1,0} = H^{1,0}(X) \cap P_{\mathbb{C}}$ and $P^{0,1} = H^{0,1}(X) \cap P_{\mathbb{C}}$ where $P_{\mathbb{C}} = P \otimes \mathbb{C} \subset H^1(X, \mathbb{C})$ is the complexification of P.

We define P to be Hodge-Teichmüller if for any (equivalently, all) choices of marking $S \to X$, and any $h \in SL(2, \mathbb{R})$, we have that $(hP)^{1,0}$ and $(hP)^{0,1}$ are both one complex dimensional. (In fact the second condition is guaranteed by the first: If $(hP)^{1,0}$ is one dimensional, then so is the complex conjugate subspace $(hP)^{0,1}$.)

The tautological plane at $M = (X, \omega)$ is defined to be

$$\operatorname{span}_{\mathbb{R}}(\operatorname{Re}(\omega), \operatorname{Im}(\omega)) \subset H^1(M, \mathbb{R}).$$

It is a Hodge-Teichmüller plane.

Remark 2.1. The action of $SL(2, \mathbb{R})$ on $H^1_{\mathbb{R}}$ commutes with the action of the mapping class group, so induces an action on the quotient of $H^1_{\mathbb{R}}$ by the mapping class group. This action is sometimes referred to as the Kontsevich-Zorich cocycle, although strictly speaking it is not a cocycle in the usual dynamical terms (unless it is considered to be only measurably defined, or the definitions are modified slightly), since the quotient of $H^1_{\mathbb{R}}$ by the mapping class group is not strictly speaking a vector bundle in the non-orbifold sense; indeed, over points (X, ω) where $\operatorname{Aut}(X, \omega)$ is non-trivial, the fiber is $H^1(X, \mathbb{R})/\operatorname{Aut}(X, \omega)$, and this is not always a vector space. See 4.1 for a definition of $\operatorname{Aut}(X, \omega)$.

Remark 2.2. It would also be possible to define Hodge-Teichmüller planes using the orbifold bundle over moduli space obtained as the quotient of the bundle $H^1_{\mathbb{R}}$ over Teichmüller space by the action of the mapping class group. Because of the issues indicated in previous remark we found it simpler to phrase the definition in Teichmüller space.

Instead of using Teichmüller space, we could use the moduli spaces of translation surfaces with a level structure; these moduli spaces are fine moduli spaces and are smooth manifolds instead of orbifolds. Over

them, the bundle $H^1_{\mathbb{R}}$ is a typical (non-orbifold) bundle with an action of $SL(2,\mathbb{R})$.

These alternate approaches are not substantively different from the approach we have followed.

2.4. **Teichmüller curves.** We will call closed (as subsets of the stratum) $SL(2,\mathbb{R})$–orbits *Teichmüller curves*. The projections of these orbits to the moduli space of Riemann surfaces are also called Teichmüller curves; these projections are in fact algebraic curves which are isometrically immersed with respect to the Teichmüller metric.

Given such a curve $\mathcal{C}$ in moduli space, at each $X \in \mathcal{C}$ consider the initial quadratic differential q of any Teichmüller geodesic segment in $\mathcal{C}$ starting at X. If $q = \omega^2$ is the square of an Abelian differential, then in fact the $SL(2,\mathbb{R})$–orbit of (X,ω) is closed, and its projection to moduli space is exactly $\mathcal{C}$. If q is not the square of an Abelian differential, a double cover construction nonetheless yields a corresponding closed orbit in a moduli space of higher genus translation surfaces. Thus the two perspectives, closed orbits and curves in moduli space, are equivalent.

Denote by $SL(X,\omega) \subset SL(2,\mathbb{R})$ the stabilizer of the translation surface (X,ω). If the $SL(2,\mathbb{R})$–orbit of $M = (X,\omega)$ is closed, then Smillie's Theorem asserts that $SL(X,\omega)$ is a lattice in $SL(2,\mathbb{R})$ [**Vee95, SW04**]. In this case, the $SL(2,\mathbb{R})$-orbit of (X,ω) is isomorphic to $SL(2,\mathbb{R})/SL(X,\omega)$, that is, the unit cotangent bundle of a complete non-compact hyperbolic surface $\mathbb{H}/SL(X,\omega)$ immersed in moduli space.

The field $\mathbb{Q}[\mathrm{tr}(h) : h \in SL(X,\omega)]$ is called the trace field of (X,ω). If (X,ω) lies on a Teichmüller curve, this field is also called the trace field of this Teichmüller curve. It is well known that the trace field of a genus g translation surface is a number field of degree at most g [**KS00, McM03a**].

If a Teichmüller curve has trace field equal to $\mathbb{Q}$, then it is generated by a square-tiled surface [**McM03b, GJ00**]. In this case we say the Teichmüller curve is *arithmetic*. A square-tiled surface is by definition a translation covering of the torus branched over one point (see below).

2.5. **Commensurability and primitivity.** A *translation covering* from a translation surface (X,ω) to another translation surface (X',ω') is defined to be a branched covering of Riemann surfaces $f : X \to X'$ such that $f^*(\omega') = \omega$. A translation surface is called *primitive* if it does not translation cover anything of smaller genus.

The following result was established by Möller [**Möl06a**], and is also explained in [**McM06a**].

Theorem 2.3 (Möller). *Every translation surface M is a translation covering of a primitive translation surface M_{prim}. If M_{prim} has genus greater than 1, then it is unique. If M lies on a Teichmüller curve, then so does M_{prim}, and in this case $SL(M)$ and $SL(M_{prim})$ are commensurable lattices in $SL(2, \mathbb{R})$.*

When a Teichmüller curve consists of primitive translation surfaces, we say this Teichmüller curve is primitive. When two Teichmüller curves $\mathcal{C}_1$ and $\mathcal{C}_2$ have a pair of translation surfaces $M_1 \in \mathcal{C}_1$ and $M_2 \in \mathcal{C}_2$ covering the same primitive translation surface, then we say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are commensurable. Theorem 2.3 gives that any commensurability class of non-arithmetic Teichmüller curves contains a unique primitive Teichmüller curve.

Recall that the trace field of a genus g translation surface is a number field of degree at most g. Recall also that the trace field is a commensurability invariant [**KS00**]. It follows that if the degree of the trace field of a Teichmüller curve of genus g translation surfaces is g, then this Teichmüller curve is primitive. In this case we say that the Teichmüller curve is *algebraically primitive.*

We think of the degree of the trace field as a measure of how non-arithmetic a Teichmüller curve is, and thus algebraically primitive curves are as far as possible from being arithmetic.

2.6. **Affine invariant submanifolds.** Let $\mathcal{H}$ be a finite cover of a stratum $\mathcal{H}(k_1, \ldots, k_\sigma)$ which is a a manifold instead of an orbifold.[3] For example, $\mathcal{H}$ may be taken to be the moduli space of translation surfaces in some given stratum equipped with a level 3 structure, see Remark 2.2 above. For the reasons expressed in Remark 2.1, it is easier to work in $\mathcal{H}$ than in the stratum. Let us denote by $\mathbb{R}\mathcal{H}$ the corresponding finite cover of $\mathbb{R}\mathcal{H}(k_1, \ldots, k_\sigma)$.

Given a translation surface (X, ω), let $\Sigma \subset X$ denote the set of zeros of ω. Pick any basis $\{\xi_1, \ldots, \xi_n\}$ for the relative homology group $H_1(X, \Sigma; \mathbb{Z})$. The map $\Phi : \mathbb{R}\mathcal{H} \to \mathbb{C}^n$ defined by

$$\Phi(X, \omega) = \left(\int_{\xi_1} \omega, \ldots, \int_{\xi_n} \omega \right)$$

defines local *period coordinates* on a neighborhood $(X, \omega) \in \mathbb{R}\mathcal{H}$.

---

[3]This guarantees that there is a family of Riemann surfaces over $\mathcal{H}$, such that fiber over a point $\mathcal{H}$ is a Riemann surface isomorphic to this point. Over such $\mathcal{H}$ there is a well defined (non-orbifold) flat vector bundle whose fiber over a point is its first real cohomology. $\mathcal{H}$ may be taken to be the quotient of Teichmüller space by any torsion free finite index subgroup of the mapping class group.

Let H^1 denote the flat bundle over $\mathbb{R}\mathcal{H}$ whose fiber over (X, ω) is $H^1(X; \mathbb{C})$, and let H^1_{rel} denote the flat bundle whose fiber over (X, ω) is $H^1(X, \Sigma; \mathbb{C})$, where Σ is the set of singularities of (X, ω). Let $p : H^1_{rel} \to H^1$ denote the natural map from relative to absolute cohomology. Also let $H^1_{\mathbb{R}}$ denote the flat bundle whose fiber over (X, ω) is $H^1(X; \mathbb{R})$.

From the description of the local period coordinates, it is clear that the tangent bundle of $\mathbb{R}\mathcal{H}$ is naturally isomorphic to H^1_{rel}: this follows from the fact that $H^1(X, \Sigma; \mathbb{C}) \simeq \mathrm{Hom}(H_1(X, \Sigma; \mathbb{Z}), \mathbb{C})$.

Period coordinates provide $\mathbb{R}\mathcal{H}$ with a system of coordinate charts with values in $\mathbb{C}^n$ and transition maps in $GL(n, \mathbb{Z})$. An *affine invariant submanifold* of $\mathcal{H}$ is an immersed manifold $\mathcal{M} \hookrightarrow \mathcal{H}$ such that each point of $\mathcal{M}$ has a neighborhood whose image is equal to the set of unit area surfaces satisfying a set of real linear equations in period coordinates. For notational simplicity, generally we will treat affine invariant submanifold as subsets of strata, referring to the image of the immersion.

An affine invariant submanifold of a stratum (rather than of a finite cover) is the projection of an affine invariant submanifold via the finite cover map from $\mathcal{H}$ as above to the stratum.

2.7. Hodge and symplectic inner products.
The complex bundle H^1 has a natural pseudo-Hermitian intersection form given by

$$(\alpha, \beta) := \frac{i}{2} \int_X \alpha \wedge \overline{\beta}$$

for α and β in a fiber $H^1(X; \mathbb{C})$ of H^1. This form has signature (g, g) because it is positive-definite on the subbundle $H^{1,0} \subset H^1$ whose fibers are $H^{1,0}(X)$ and it is negative-definite on the subbundle $H^{0,1} \subset H^1$ whose fibers are $H^{0,1}(X)$.

By Hodge's Representation Theorem, any $c \in H^1(X; \mathbb{R})$ is the real part of a unique $h(c) \in H^{1,0}$. In the literature, the *imaginary part* of $h(c)$ is called the Hodge-star $*c$ of c and the operator $c \mapsto *c$ is the *Hodge-star operator*. This operator gives an isomorphism $H^{1,0} \simeq H^1_{\mathbb{R}}$ and thus the restriction of the pseudo-Hermitian form $(\cdot, \cdot)$ to $H^{1,0}$ induces an inner product on $H^1_{\mathbb{R}}$ called the *Hodge inner product*.

Given $V \subset H^1(X; \mathbb{R})$, we denote by $V^{\perp}$, resp. $V^{\dagger}$, the orthogonal complement of V with respect to Hodge inner product, resp. symplectic intersection form. For later use, we note that $P^{\perp} = P^{\dagger}$ whenever P is a Hodge-Teichmüller plane. Indeed, this holds because, from the definition, a Hodge-Teichmüller plane P is Hodge-star invariant and hence the desired fact is a consequence of the following lemma in [**FMZ**]:

Lemma 2.4 (Lemma 3.4 in [**FMZ**]). *For $V \subset H^1(X; \mathbb{R})$, one has $V^\perp = V^\dagger$ if and only if V is Hodge-star invariant, and, in this case, the subspace $V^\perp = V^\dagger$ is Hodge-star invariant.*

3. LIMITS OF HODGE-TEICHMÜLLER PLANES

In this section, we prove Theorem 1.2. The proof is naturally divided into two steps in the following two propositions.

Proposition 3.1. *Let M be a translation surface on a Teichmüller curve. Then M has at least k orthogonal Hodge-Teichmüller planes, where k is the degree of the trace field.*

Proposition 3.2. *Suppose that $M_n \to M$ are translation surfaces, and that P_n is a Hodge-Teichmüller plane at M_n. Let P be a limit of the P_n. Then P is a Hodge-Teichmüller plane at M.*

We emphasize that convergence $M_n \to M$ is in the moduli space of Abelian differentials, not in any specific stratum. Convergence $M_n \to M$ is equivalent to the existence of maps (diffeomorphisms) $f_n : M_n \to M$ such that $(f_n)_*(\omega_n) \to \omega$. The maps f_n may not be uniquely defined, even only up to homotopy, because M may have affine self maps with identity derivative. We say that P is a limit of the P_n if, for some choice of maps f_n, the planes $(f_n)_*(P_n)$ converge to P in the Grassmannian of planes of $H^1(X, \mathbb{R})$.

Proof of Theorem 1.2 using Propositions 3.1 and 3.2. Let $M \in \mathcal{M}$ be a translation surface. By the assumption of density, there are translation surfaces M_n so that $M_n \to M$ and each M_n lies on a Teichmüller curve whose trace field has degree at least k. By Proposition 3.1, each M_n has at least k orthogonal Hodge-Teichmüller planes, say $P_n^{(1)}, \ldots, P_n^{(k)}$. After passing to a subsequence, we may assume that for each $i = 1, \ldots, k$, the planes $P_n^{(i)}$ converge to a plane $P^{(i)}$ at M. By continuity of the Hodge inner product, the $P^{(i)}, i = 1, \ldots, k$ are Hodge orthogonal, and by Proposition 3.2, these planes are Hodge-Teichmüller. Since $M \in \mathcal{M}$ was arbitrary this proves the result. $\square$

3.1. **Möller's description of the VHS over a Teichmüller curve.** Recall that over any Teichmüller curve $\mathcal{M}$ the bundle $H^1_{\mathbb{R}}$ in fact has the structure of a real variation of Hodge structures (VHS). (For a concrete introduction to VHS in the context of Teichmüller curves, see for example [**Wri12**].)

Theorem 3.3 (Möller [**Möl06b**]). *Suppose that $\mathcal{M}$ is a lift of a Teichmüller curve to a finite cover $\mathcal{H}$ of a stratum as above, and suppose*

the trace field has degree k. Then there is a decomposition of VHS over
$\mathcal{M}$

$$H^1_{\mathbb{R}} = \bigoplus_{i=1}^{k} \mathbb{L}_i \bigoplus \mathbb{W},$$

*where $\mathbb{L}_1$ is the tautological bundle and the $\mathbb{L}_i, i = 2, \ldots, k$ are Galois
conjugate to $\mathbb{L}_i$. The $\mathbb{L}_i$'s are symplectically and Hodge orthogonal.*

The statement that this is a decomposition of VHS means precisely
that each of the $k + 1$ subbundles in this decomposition is a flat sub-
bundle of $H^1_{\mathbb{R}}$ which is the sum of of it's $(1, 0)$ and $(0, 1)$ parts. In
particular, each fiber of each $\mathbb{L}_i$ contains a holomorphic one form.

Proof of Proposition 3.1. The fiber of $\mathbb{L}_i$ over a point $(X, \omega) \in \mathcal{M}$
is a plane P_i in $H^1(X, \mathbb{R})$. Any parallel transport of this plane to
any other translation surface in $\mathcal{M}$ again gives a fiber P' of $\mathbb{L}_i$. The
complexification of this fiber always contains a holomorphic one form.
So we conclude that each fiber of each $\mathbb{L}_i$ is a Hodge-Teichmüller plane.
$\square$

Remark 3.4. Using the techniques of the next section, it is possi-
ble to show that for a Teichmüller curve the $\mathbb{L}_i$ are the only Hodge-
Teichmüller planes in $\bigoplus_{i=1}^{k} \mathbb{L}_i$. In particular, an algebraically primitive
Teichmüller curve has no Hodge-Teichmüller planes except those given
by Proposition 3.1. We will not need this result, so its proof is omitted.

3.2. **Limits of Hodge-Teichmüller planes.**

Lemma 3.5. *If $M_n \to M$, and P_n a plane at M_n so that $(P_n)_{\mathbb{C}} \cap
H^{1,0}(M_n) \neq \{0\}$, and P is a limit of the P_n at M, then $P_{\mathbb{C}} \cap H^{1,0}(M) \neq
\{0\}$.*

Proof. This is true because the space of Abelian differentials inside of
first cohomology varies continuously with the complex structure on the
surface. $\square$

Lemma 3.6. *If $M_n \to M$ are marked translation surfaces, and P_n are
planes at M_n and P is a limit of the P_n at M, then for any $h \in SL(2, \mathbb{R})$
$h(P)$ is a limit of the $h(P_n)$.*

The definition of convergence of planes in the cohomology of marked
translation surfaces is identical to the unmarked cases discussed near
the beginning of this section, except that the maps f_n should be re-
quired to commute with the markings.

Proof. By definition there are maps $f_n : (X_n, \Sigma_n) \to (X, \Sigma)$ so that
$(f_n)_*[\omega_n] \to [\omega]$. Fix h. For any (marked) translation surface M, h

induces an affine map $\phi_h : M \to hM$ (and a marking on hM). One checks that the maps $\phi_h \circ f_n \circ \phi_h^{-1} : hM_n \to hM$ induce maps on cohomology which send hP_n to a sequence of planes at hM converging to hP. $\qquad\square$

Proof of Proposition 3.2. Fix h. It suffices to show that $(hP)_{\mathbb{C}}$ intersects $H^{1,0}(hM)$. This follows directly from the previous two lemmas. By Lemma 3.6, hP is a limit of the hP_n. The complexifications of the hP_n each intersect $H^{1,0}(hM_n)$, so Lemma 3.5 may be applied. $\qquad\square$

4. Square-tiled surfaces with no non-trivial good planes

In this section, we prove Theorem 1.7.

4.1. **Definitions.** Recall that the Veech group $SL(M)$ of $M = (X, \omega)$ is the stabilizer of M in $SL(2, \mathbb{R})$. Denote by $\mathrm{Aff}(M)$ the set of affine diffeomorphisms of M, i.e., the set of homeomorphisms of M preserving the set of zeroes and have the form of an affine transformation in the charts of the translation atlas. The Veech group $SL(M)$ is the set of linear parts (derivatives) of the elements of $\mathrm{Aff}(M)$. There is the following exact sequence:

$$1 \to \mathrm{Aut}(M) \to \mathrm{Aff}(M) \to SL(M) \to 1.$$

The affine group $\mathrm{Aff}(M)$ acts naturally (i.e., preserving the natural symplectic intersection form) on the homology (resp. cohomology) group $H_1(M, \mathbb{R})$ (resp. $H^1(M, \mathbb{R})$) of M, inducing natural homomorphisms

$$\mathrm{Aff}(M) \to \mathrm{Sp}(H_1(M, \mathbb{R})) \quad (\text{resp.}, \mathrm{Aff}(M) \to \mathrm{Sp}(H^1(M, \mathbb{R})).$$

Given a translation surface $M = (X, \omega)$, denote by $\Gamma(M)$ the image of $\mathrm{Aff}(M)$ under the natural map $\mathrm{Aff}(M) \to \mathrm{Sp}(H^1(M, \mathbb{R}))$. Let $\overline{\Gamma}(M)$ denote the Zariski closure of $\Gamma(M)$.

Let $H^1_\perp(M, \mathbb{R})$ denote $\mathrm{span}_{\mathbb{R}}(\mathrm{Re}(\omega), \mathrm{Im}(\omega))^\perp$ in $H^1(M, \mathbb{R})$. The perp may be taken with either the symplectic form or the Hodge inner product, and the result is the same (Lemma 2.4). Let $\Gamma_\perp(M)$ be the restriction of $\Gamma(M)$ to $H^1_\perp(M, \mathbb{R})$, and let $\overline{\Gamma}_\perp(M)$ be the Zariski closure of $\Gamma_\perp(M)$.

Remark 4.1. Define $H_1^\perp(M, \mathbb{R}) \subset H_1(M, \mathbb{R})$ to be the annihilator of the tautological plane $\mathrm{span}_{\mathbb{R}}(\mathrm{Re}(\omega), \mathrm{Im}(\omega)) \subset H^1(M, \mathbb{R})$. Define the (homology) tautological plane $H_1^{st}(M, \mathbb{R}) \subset H_1(M, \mathbb{R})$ to be the the orthogonal complement of $H_1^\perp(M, \mathbb{R})$ [**MY10**]. The affine group $\mathrm{Aff}(M)$ respects Poincaré duality: The representations of $\mathrm{Aff}(M)$ in homology and cohomology are in a precise sense dual. If $A \in \mathrm{Aff}(M)$ acts through

a symplectic matrix $M(A)$ in a given symplectic basis of $H_1(M, \mathbb{R})$, then $A \in \text{Aff}(M)$ acts through the matrix $(M(A)^{-1})^t$ in the dual basis in $H^1(M, \mathbb{R})$. It follows that we can compute the groups $\Gamma(M)$, $\Gamma_\perp(M)$ and their Zariski closures by considering the action of $\text{Aff}(M)$ on $H_1(M, \mathbb{R})$ and $H_1^\perp(M, \mathbb{R})$ (i.e., we can use homology instead of cohomology). We will use this fact later, in the proof of Proposition 4.3 below.

4.2. **Outline and strategy.**

Proposition 4.2. *Let M be a translation surface. The set of Hodge-Teichmüller planes at M is invariant under $\overline{\Gamma}(M)$.*

Proposition 4.3. *In each of $\mathcal{H}(4)^{hyp}$ and $\mathcal{H}(4)^{odd}$, there is a translation surface M with $\overline{\Gamma}_\perp(M) = Sp(4, \mathbb{R})$.*

Proof of Theorem 1.7 using Propositions 4.2 and 4.3. Let M be either of the translation surfaces given by Proposition 4.3. Our goal is to show that the only Hodge-Teichmüller plane at M is the tautological plane.

Suppose, in order to find a contradiction, that there is a Hodge-Teichmüller plane P_0 at M which is not the tautological plane. Consider the projection map $\rho : H^1(M, \mathbb{R}) \to H_\perp^1(M, \mathbb{R})$. Set $P = \rho(P_0)$. Since the kernel of ρ is exactly the tautological plane, we see that the subspace P is at least one dimensional. Since the complexification of ρ preserves $H^{1,0}(M)$ and $H^{0,1}(M)$ we see that the complexification of P intersects at least one of $H^{1,0}(M)$ or $H^{0,1}(M)$. Since P is a real subspace, and $H^{1,0}(M)$ and $H^{0,1}(M)$ are complex conjugate, it follows that $P_\mathbb{C} = P \otimes \mathbb{C}$ intersects both of $H^{1,0}(M)$ and $H^{0,1}(M)$. In particular, P is two dimensional.

For all $h \in SL(2, \mathbb{R})$, there is the projection map $\rho : H^1(hM, \mathbb{R}) \to H_\perp^1(hM, \mathbb{R})$ to the orthogonal complement of the tautological plane.

Define $hP \subset H^1(hM, \mathbb{R})$ as in 3.2. Since P_0 is a Hodge-Teichmüller plane, hP_0 intersects both $H^{1,0}(hM)$ and $H^{0,1}(hM)$. Since ρ preserves $H^{1,0}(hM)$ and $H^{0,1}(hM)$, as above we see that $hP = \rho(hP_0)$ also intersects both $H^{1,0}(hM)$ and $H^{0,1}(hM)$. Hence P is a Hodge-Teichmüller plane.

By Proposition 4.2, it follows that $\gamma(P)$ is a Hodge-Teichmüller plane for all γ in the full symplectic group $Sp(H_\perp^1(X, \mathbb{R}))$ on $H_\perp^1(X, \mathbb{R})$.

However, the full symplectic group acts transitively on the set of symplectic planes, and not all symplectic planes in $H_\perp^1(X, \mathbb{R})$ are Hodge-Teichmüller. This is a contradiction.

To see that not all symplectic planes in $H_\perp^1(X, \mathbb{R})$ are Hodge-Teichmüller, simply note that the complexification of most symplectic planes do not

intersect $H^{1,0}$ or $H^{0,1}$. The set of symplectic planes is open in the set of planes, whereas the set of planes whose complexification intersects $H^{1,0}$ has positive codimension. $\square$

4.3. The variety of Hodge-Teichmüller planes.
Here we prove Proposition 4.2. The proof also shows that the set of Hodge-Teichmüller planes at any translation surface is a projective variety.

Proof of Proposition 4.2. Fix a marking on M. For each $h \in SL(2,\mathbb{R})$, define $H_h^{1,0} \subset H^1(M,\mathbb{C})$ by $H_h^{1,0} = h^{-1}(H^{1,0}(hM))$. Here we consider $H^{1,0}(hM)$ to be a subset of $H^1(hM,\mathbb{C}) = H^1(hM,\mathbb{R}) \otimes \mathbb{C}$, and h^{-1} acts on this subset via the usual action discussed in Section 2.

A plane $P \subset H^1(M,\mathbb{R})$ is Hodge-Teichmüller if and only if $P_\mathbb{C} = P \otimes \mathbb{C}$ intersects $H_h^{1,0}$ non-trivially for all $h \in SL(2,\mathbb{R})$.

For each h, the condition that $P_\mathbb{C}$ intersects $H_h^{1,0}$ non-trivially is a polynomial condition on the Grassmanian of planes P in $H^1(X,\mathbb{R})$. (List a basis for $H_h^{1,0}$ next to a basis of P. This matrix should have rank at most $g+1$, which is equivalent to the determinant of every $g+2$ by $g+2$ minor being 0.)

Thus the set $\mathcal{P}$ of Hodge-Teichmüller planes at M is defined by polynomial equations, finitely many for each h. (Thus in total, there are uncountably many polynomial equations, but this is not relevant.) In other words, $\mathcal{P}$ is a subvariety of the Grassmanian of planes in $H^1(M,\mathbb{R})$. It's stabilizer contains $\Gamma(M)$, and hence the Zariski closure $\overline{\Gamma}(M)$. By definition, anything in the stabilizer of $\mathcal{P}$ in $\mathrm{Sp}(H^1(X,\mathbb{R}))$ preserves the set $\mathcal{P}$ of Hodge-Teichmüller planes. $\square$

4.4. Square-tiled surfaces.
A translation surface $M = (X,\omega)$ is called a *square-tiled surface* if there is a ramified finite cover $\pi : X \to \mathbb{R}^2/\mathbb{Z}^2$ unramified outside $0 \in \mathbb{R}^2/\mathbb{Z}^2$ such that $\omega = \pi^*(dz)$. Equivalently, (X,ω) is a square-tiled surface if the relative periods of ω (i.e., the integrals of ω along paths joining two of its zeroes) belong to $\mathbb{Z}\oplus i\mathbb{Z}$.

Combinatorially, a square-tiled surface (X,ω) associated to a covering $\pi : X \to \mathbb{R}^2/\mathbb{Z}^2$ of degree N is determined by a pair of permutations $h, v \in S_N$ acting transitively on $\{1,\ldots,N\}$. Given a square-tiled surface, we may fix a numbering of the N squares, i.e., a bijection from the set of squares to $\{1,\ldots,N\}$. We may then define a pair of permutations $h, v \in S_N$ by declaring that $h(i)$, resp. $v(i)$ is the number of the square to the right, resp. on the top, of the i-th square.

Given $M = (X,\omega)$ a square-tiled surface associated to $\pi : M \to \mathbb{R}^2/\mathbb{Z}^2$, it is a simple task to identify the subspace $H_1^\perp(M,\mathbb{R})$. Indeed, as it is shown in [**MY10**], $H_1^\perp(M,\mathbb{R})$ consists of all absolute homology classes on M projecting to 0 under the covering map $\pi : M \to \mathbb{R}^2/\mathbb{Z}^2$.

We will use this elementary fact to compute the action of the affine group $\mathrm{Aff}(M)$ on $H_1^\perp(M, \mathbb{R})$ for certain square-tiled surfaces of genus 3.

4.5. **Connected components of $\mathcal{H}(4)$.** Using a combinatorial object (extended Rauzy classes), Veech [**Vee90**] showed that the minimal stratum $\mathcal{H}(4)$ of genus 3 translation surfaces has two connected components. More recently, Kontsevich and Zorich [**KZ03**] classified all connected components of all strata of genus $g \geq 2$ translation surfaces. For our purposes, it suffices to know that the two (*hyperelliptic* and *odd*) connected components $\mathcal{H}(4)^{\mathrm{hyp}}$ and $\mathcal{H}(4)^{\mathrm{odd}}$ can be distinguished via the parity of the spin structure.

Given a translation surface $M = (X, \omega)$ of genus $g \geq 1$, we select a canonical symplectic basis $\{\alpha_i, \beta_i : i = 1, \ldots, g\}$ and we compute the Arf invariant

$$\Phi(M, \{\alpha_i, \beta_i : i = 1, \ldots, g\})$$
$$= \sum_{i=1}^{g} (\mathrm{ind}_\omega(\alpha_i) + 1)(\mathrm{ind}_\omega(\beta_i) + 1) \, (\mathrm{mod}\ 2),$$

where $\mathrm{ind}_\omega(\gamma)$ is the degree of the Gauss map associated to the tangents of a curve γ not passing through the zeroes of ω. It can be shown that the Arf invariant is independent of the choice of canonical basis of homology and thus it defines a quantity $\Phi(M) \in \mathbb{Z}/(2\mathbb{Z})$ that is called the parity of the spin structure of M.

In our setting, $M \in \mathcal{H}(4)^{\mathrm{hyp}}$, resp. $M \in \mathcal{H}(4)^{\mathrm{odd}}$, if and only if $\Phi(M) = 0$, resp. $\Phi(M) = 1$.

Remark 4.4. Alternatively, one can show that $M \in \mathcal{H}(4)^{\mathrm{hyp}}$ if and only if M admits an involution (i.e., an element of its affine group $\mathrm{Aff}(M)$ with derivative $-Id$) having exactly 8 fixed points.

4.6. **Monodromy of a square-tiled surface in $\mathcal{H}(4)^{\mathrm{odd}}$.** Let $M_* \in \mathcal{H}(4)^{\mathrm{odd}}$ be the square-tiled surface associated to the pair of permutations $h_* = (1)(2,3)(4,5,6)$ and $v_* = (1,4,2)(3,5)(6)$ (see Figure 4.1).

Lemma 4.5. *Working in a suitable symplectic basis, $\Gamma_\perp(M_*) \subset Sp(4, \mathbb{Z})$ contains the transposes of the inverses of the matrices*

$$A_* = \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 \\ -2 & -4 & 0 & 1 \end{pmatrix}, \quad C_* = \begin{pmatrix} 2 & 2 & 1 & 2 \\ -1 & -1 & -1 & -2 \\ -1 & -2 & 0 & -2 \\ 1 & 2 & 1 & 3 \end{pmatrix}.$$

Proof. Note that M_* decomposes into three horizontal, resp. vertical, cylinders whose waist curves σ_0, σ_1 and σ_2, resp. ζ_0, ζ_1 and ζ_2, have
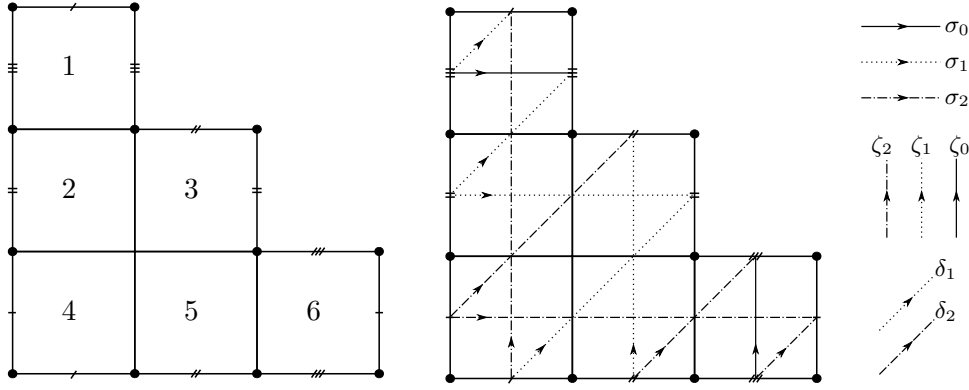
FIGURE 4.1. The square-tiled surface M_* (left), redrawn (center) with the curves referred to below, and index (right).

lengths 1, 2 and 3. Also, M_* decomposes into two cylinders in the slope 1 direction whose waist curves δ_1 and δ_2 are defined by the property that δ_1 intersects σ_0 and δ_2 intersects ζ_2. By direct inspection, it is not hard to check that $\delta_1 = \sigma_1 + \sigma_0 + \zeta_2$ and $\delta_2 = \sigma_2 + \zeta_1 + \zeta_0$.

Denote by $A, B, C \in \mathrm{Aff}(M_*)$ be the affine diffeomorphisms of M_* with linear parts

$$dA = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}, \quad dB = \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}, \quad dC = \begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix}.$$

Geometrically, A, B, C correspond to certain Dehn multitwists in the horizontal, vertical and slope 1 directions.

The actions of A, B, C on homology $H_1(M_*, \mathbb{R})$ are:

$A(\sigma_i) = \sigma_i$ for $i = 1, 2, 3$,

$A(\zeta_0) = \zeta_0 + 2\sigma_2$, $\quad A(\zeta_1) = \zeta_1 + 3\sigma_1 + 2\sigma_2$, $\quad A(\zeta_2) = \zeta_2 + 3\sigma_1 + 2\sigma_2 + 6\sigma_0$,

$B(\sigma_0) = \sigma_0 + 2\zeta_2$, $\quad B(\sigma_1) = \sigma_1 + 3\zeta_1 + 2\zeta_2$, $\quad B(\sigma_2) = \sigma_2 + 3\zeta_1 + 2\zeta_2 + 6\zeta_0$,

$B(\zeta_i) = \zeta_i$ for $i = 1, 2, 3$,

$C(\sigma_0) = \sigma_0 - \delta_1$, $\quad C(\sigma_1) = \sigma_1 - \delta_1 - \delta_2$, $\quad C(\sigma_2) = \sigma_2 - \delta_1 - 2\delta_2$,

$C(\zeta_0) = \zeta_0 + \delta_2$, $\quad C(\zeta_1) = \zeta_1 + \delta_1 + \delta_2$, $\quad C(\zeta_2) = \zeta_2 + 2\delta_1 + \delta_2$.

Note that $\overline{\sigma}_1 := \sigma_1 - 2\sigma_0$, $\overline{\sigma}_2 := \sigma_2 - 3\sigma_0$, $\overline{\zeta}_1 := \zeta_1 - 2\zeta_0$ and $\overline{\zeta}_2 := \zeta_2 - 3\zeta_0$ form a basis of $H_1^{\perp}(M_*, \mathbb{R})$, the symplectic orthogonal of the tautological subbundle $H_1^{st}(M_*, \mathbb{R})$ spanned by $\sigma = \sigma_0 + \sigma_1 + \sigma_2$ and $\zeta = \zeta_0 + \zeta_1 + \zeta_2$. In this basis, the actions of A, B, C on $H_1^{\perp}(M_*, \mathbb{R})$ are given by the symplectic matrices above. By Poincaré duality (see Remark 4.1 above), it follows that $(A_*^{-1})^t, (B_*^{-1})^t, (C_*^{-1})^t \in \Gamma_{\perp}(M_*)$. $\qquad \square$

4.7. Monodromy of a square-tiled surface in $\mathcal{H}(4)^{\mathbf{hyp}}$.
Let $M_{**} \in \mathcal{H}(4)^{\mathrm{hyp}}$ be the square-tiled surface associated to the pair of permutations $h_{**} = (1)(2,3)(4,5,6)$ and $v_{**} = (1,2)(3,4)(5)(6)$ (see Figure 4.2).
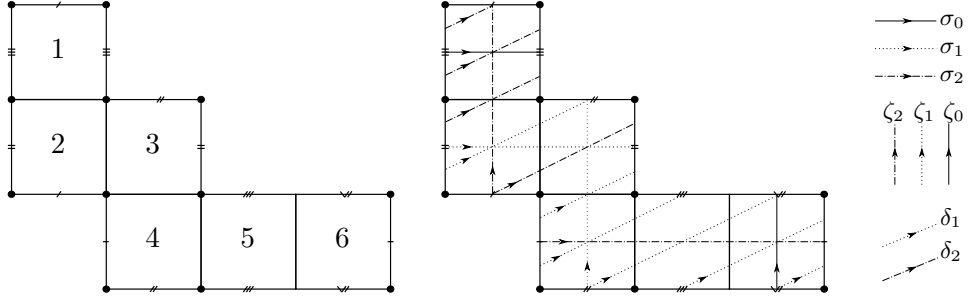

FIGURE 4.2. The square-tiled surface M_{**} (left), re-drawn (center) with the curves referred to below, and index (right).

Lemma 4.6. *Working in a suitable symplectic basis, $\Gamma_{\perp}(M_{**}) \subset Sp(4,\mathbb{Z})$ contains the transposes of the inverses of the matrices*

$$
D_* = \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad
E_* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{pmatrix}, \quad
F_* = \begin{pmatrix} 2 & 3 & 1 & 3 \\ -2 & -5 & -2 & -6 \\ -1 & -3 & 0 & -3 \\ 2 & 6 & 2 & 7 \end{pmatrix}.
$$

Proof. Note that M_{**} decomposes into three horizontal, resp. vertical, cylinders whose waist curves σ_0, σ_1 and σ_2, resp. ζ_0, ζ_1 and ζ_2, have lengths 1, 2 and 3, resp. 1, 2 and 2. Also, M_{**} decomposes into two cylinders in the slope $1/2$ direction whose waist curves δ_1 and δ_2 are defined by the property that δ_1 intersects σ_0 and δ_2 intersects ζ_0. By direct inspection, it is not hard to check that $\delta_1 = \sigma_1 + 2\sigma_0 + \zeta_2$ and $\delta_2 = \sigma_1 + 2\sigma_2 + \zeta_1 + 2\zeta_0$.

Denote by $D, E, F \in \mathrm{Aff}(M_{**})$ be the affine diffeomorphisms of M_{**} with linear parts

$$
dD = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}, \quad
dE = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad
dF = \begin{pmatrix} -7 & 16 \\ -4 & 9 \end{pmatrix}.
$$

Geometrically, D, E, F correspond to certain Dehn multitwists in the horizontal, vertical and slope $1/2$ directions.

The actions of D, E, F on homology $H_1(M_{**}, \mathbb{R})$ are:

$$
\begin{aligned}
&D(\sigma_i) = \sigma_i && \text{for } i = 1, 2, 3, \\
&D(\zeta_0) = \zeta_0 + 2\sigma_2, && D(\zeta_1) = \zeta_1 + 3\sigma_1 + 2\sigma_2, && D(\zeta_2) = \zeta_2 + 3\sigma_1 + 6\sigma_0, \\
&E(\sigma_0) = \sigma_0 + \zeta_2, && E(\sigma_1) = \sigma_1 + \zeta_1 + \zeta_2, && E(\sigma_2) = \sigma_2 + \zeta_1 + 4\zeta_0, \\
&E(\zeta_i) = \zeta_i && \text{for } i = 1, 2, 3, \\
&F(\sigma_0) = \sigma_0 - 2\delta_1, && F(\sigma_1) = \sigma_1 - 2\delta_1 - \delta_2, && F(\sigma_2) = \sigma_2 - 3\delta_2, \\
&F(\zeta_0) = \zeta_0 + 2\delta_2, && F(\zeta_1) = \zeta_1 + 2\delta_1 + 3\delta_2, && F(\zeta_2) = \zeta_2 + 6\delta_1 + \delta_2
\end{aligned}
$$

Note that $\overline{\sigma}_1 := \sigma_1 - 2\sigma_0$, $\overline{\sigma}_2 := \sigma_2 - 3\sigma_0$, $\overline{\zeta}_1 := \zeta_1 - 2\zeta_0$ and $\overline{\zeta}_2 := \zeta_2 - 2\zeta_0$ form a basis of $H_1^{\perp}(M_{**}, \mathbb{R})$, the symplectic orthogonal of the tautological subbundle $H_1^{st}(M_{**}, \mathbb{R})$ spanned by $\sigma = \sigma_0 + \sigma_1 + \sigma_2$ and $\zeta = \zeta_0 + \zeta_1 + \zeta_2$. In this basis, the actions of D, E, F on $H_1^{\perp}(M_{**}, \mathbb{R})$ are given by the symplectic matrices above. By Poincaré duality (see Remark 4.1 above), it follows that $(D_*^{-1})^t, (E_*^{-1})^t, (F_*^{-1})^t \in \Gamma_{\perp}(M_{**})$. $\square$

4.8. **Zariski closure.** Here we complete the proof of Proposition 4.3, by computing the Zariski closures of $\Gamma_{\perp}(M_*)$ and $\Gamma_{\perp}(M_{**})$.

Lemma 4.7. *The Zariski closures of the groups $\langle A_*, B_*, C_* \rangle$ and $\langle D_*, E_*, F_* \rangle$ are both $Sp(4, \mathbb{R})$, where the matrices $A_*, B_*, C_*, D_*, E_*, F_*$ are given in Lemmas 4.5 and 4.6.*

Proof. Denote by $G = \overline{\langle A_*, B_*, C_* \rangle}^{Zariski} \subset Sp(4, \mathbb{R})$ the Zariski closure of the subgroup generated by A_*, B_*, C_*. We claim that $G = Sp(4, \mathbb{R})$. Indeed, denote by $\mathfrak{g}$ the Lie algebra of G and note that

$$
\log A_* = \begin{pmatrix} 0 & 0 & 3 & 3 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}.
$$

Consider the conjugates of $\log A_*$ by

$$
B_*, B_*^2, A_*B_*, A_*^2B_*, B_*A_*B_*, C_*, C_*^2, A_*C_*, B_*C_*.
$$

These nine matrices are, respectively,

$$
\begin{pmatrix} -3 & 3 & 3 & 3 \\ -2 & -10 & -2 & -4 \\ -15 & -21 & 3 & -3 \\ -14 & 34 & 2 & 10 \end{pmatrix},
\begin{pmatrix} -6 & 6 & 3 & 3 \\ -4 & -20 & -2 & -4 \\ -60 & -84 & 6 & -6 \\ 56 & 136 & 4 & 20 \end{pmatrix},
\begin{pmatrix} -6 & 42 & 120 & 210 \\ -28 & -104 & -140 & -370 \\ -15 & -21 & 6 & -42 \\ 14 & 34 & 28 & 104 \end{pmatrix},
$$

$$
\begin{pmatrix} -9 & 81 & 411 & 747 \\ -54 & -198 & -498 & -1332 \\ -15 & -21 & 9 & -81 \\ 14 & 34 & 54 & 198 \end{pmatrix},
\begin{pmatrix} 54 & 522 & 120 & 210 \\ -348 & -1164 & -140 & -370 \\ -999 & -2133 & -54 & -522 \\ 1422 & 3978 & 348 & 1164 \end{pmatrix},
\begin{pmatrix} 4 & 8 & 6 & 6 \\ -2 & -4 & -3 & -3 \\ -4 & -8 & -3 & -3 \\ 4 & 8 & 3 & 3 \end{pmatrix},
$$

$$
\begin{pmatrix} 16 & 32 & 17 & 25 \\ -12 & -24 & -12 & -18 \\ -16 & -32 & -14 & -22 \\ 16 & 32 & 14 & 22 \end{pmatrix},
\begin{pmatrix} 4 & 8 & 10 & 26 \\ -10 & -20 & -19 & -59 \\ -4 & -8 & -7 & -23 \\ 4 & 8 & 7 & 23 \end{pmatrix},
\begin{pmatrix} -2 & 14 & 6 & 6 \\ 1 & -7 & -3 & -3 \\ -4 & 10 & 6 & 6 \\ 1 & 11 & 3 & 3 \end{pmatrix}.
$$

All of these matrices must be in $\mathfrak{g}$. Furthermore, direct computation shows $\log A_*$ and these nine matrices are linearly independent. Since $Sp(4,\mathbb{R})$ has dimension 10, we conclude that $G = Sp(4,\mathbb{R})$.

Next denote by $H = \overline{\langle D_*, E_*, F_* \rangle}^{Zariski} \subset Sp(4,\mathbb{R})$ the Zariski closure of the subgroup generated by D_*, E_*, F_*. We claim that $H = Sp(4,\mathbb{R})$. Indeed, denote by $\mathfrak{h}$ the Lie algebra of H and note that

$$
\log D_* = \begin{pmatrix} 0 & 0 & 3 & 3 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{h}.
$$

Consider the conjugates of $\log D_*$ by

$$
E_*, E_*^2, D_* E_*, D_* E_*^2, E_* D_* E_*, F_*, F_*^2, D_* F_*, E_* F_*.
$$

These nine matrices are, respectively,

$$
\begin{pmatrix} 0 & 6 & 3 & 3 \\ -2 & -10 & -2 & -4 \\ -2 & -4 & 1 & -1 \\ 6 & 24 & 3 & 9 \end{pmatrix},
\begin{pmatrix} 0 & 12 & 3 & 3 \\ -4 & -20 & -2 & -4 \\ -8 & -16 & 2 & -2 \\ 24 & 96 & 6 & 18 \end{pmatrix},
\begin{pmatrix} 12 & 66 & 111 & 255 \\ -22 & -98 & -146 & -364 \\ -2 & -4 & -1 & -11 \\ 6 & 24 & 33 & 87 \end{pmatrix},
$$

$$
\begin{pmatrix} 48 & 252 & 387 & 915 \\ -84 & -372 & -522 & -1308 \\ -8 & -16 & -6 & -42 \\ 24 & 96 & 126 & 330 \end{pmatrix},
\begin{pmatrix} 156 & 720 & 111 & 255 \\ -240 & -1044 & -146 & -364 \\ -96 & -360 & -36 & -120 \\ 624 & 2664 & 360 & 924 \end{pmatrix},
\begin{pmatrix} 12 & 36 & 12 & 30 \\ -24 & -72 & -20 & -58 \\ -15 & -45 & -12 & -36 \\ 30 & 90 & 24 & 72 \end{pmatrix},
$$

$$
\begin{pmatrix} 54 & 162 & 51 & 147 \\ -108 & -324 & -98 & -292 \\ -60 & -180 & -54 & -162 \\ 120 & 360 & 108 & 324 \end{pmatrix},
\begin{pmatrix} 57 & 171 & 219 & 651 \\ -114 & -342 & -434 & -1300 \\ -15 & -45 & -57 & -171 \\ 30 & 90 & 114 & 342 \end{pmatrix},
\begin{pmatrix} 30 & 114 & 12 & 30 \\ -62 & -226 & -20 & -58 \\ -71 & -253 & -20 & -64 \\ 234 & 846 & 72 & 216 \end{pmatrix}.
$$

All of these matrices must be in $\mathfrak{h}$. Furthermore, direct computation shows $\log D_*$ and these nine matrices are linearly independent. Since $Sp(4,\mathbb{R})$ has dimension 10, we conclude that $H = Sp(4,\mathbb{R})$. $\square$

Proof of Proposition 4.3. We consider the square-tiled surfaces $M_* \in \mathcal{H}(4)^{\mathrm{odd}}$ and $M_{**} \in \mathcal{H}(4)^{\mathrm{hyp}}$.

Lemmas 4.5 and 4.6 show that $\Gamma_\perp(M_*) \supset \langle (A_*^{-1})^t, (B_*^{-1})^t, (C_*^{-1})^t \rangle$ and $\Gamma_\perp(M_{**}) \supset \langle (D_*^{-1})^t, (E_*^{-1})^t, (F_*^{-1})^t \rangle$, and Lemma 4.7 proves that both $\langle A_*, B_*, C_* \rangle$ and $\langle D_*, E_*, F_* \rangle$ have Zariski closure equal to $Sp(4, \mathbb{R})$. Since both $\overline{\Gamma}_\perp(M_*)$ and $\overline{\Gamma}_\perp(M_{**})$ are subgroups of $Sp(4, \mathbb{R})$, it follows that they are both equal to $Sp(4, \mathbb{R})$. □

Remark 4.8. The computations in this section were first performed by the first author by hand, using Mathematica only for matrix arithmetic and to verify linear independence of sets of matrices. The computations were then redone independently by the second author by computer, using a short Maple program which computes the action of Dehn multitwists on homology.

5. INDUCTIVE STEP

In this section, we prove Theorem 1.8. We begin by proving the first statement, since the proof is technically easier in this case.

Proposition 5.1. *Let $\mathcal{H}$ be a connected component of a stratum $\mathcal{H}(k_1, \ldots, k_s)$ of genus g translation surfaces, where $s > 1$. Suppose that every translation surface in $\mathcal{H}$ has k Hodge-Teichmüller planes. Then there is a connected component $\mathcal{H}'$ of $\mathcal{H}(2g-2)$ in which every translation surface has k orthogonal Hodge-Teichmüller planes.*

Proposition 5.2 (Kontsevich-Zorich). *Let $\mathcal{H}$ be a connected component of a stratum of genus g translation surfaces with more than one zero. Then there is a connected component $\mathcal{H}'$ of $\mathcal{H}(2g - 2)$ which is contained in the boundary of $\mathcal{H}$.*

Proposition 5.2 is proven in [**KZ03**] (in particular, this statement is included in the proof of [**KZ03**, Prop. 4]).

Proof of Proposition 5.1. Let $\mathcal{H}'$ be given by Proposition 5.2, and let $M \in \mathcal{H}'$ be arbitrary. By Proposition 5.2, we see there is a sequence $M_n \in \mathcal{H}$ with $M_n \to M$. We have assumed that each M_n has k orthogonal Hodge-Teichmüller planes. By Lemma 3.2, we see that M has k Hodge-Teichmüller planes also. □

We now proceed to the second and final statement of Theorem 1.8, whose proof is similar but more involved.

Proposition 5.3. *Let $\mathcal{H}$ be a connected component of the stratum $\mathcal{H}(2g - 2)$. Suppose that every translation surface in $\mathcal{H}$ has k Hodge-Teichmüller planes. Then there is a connected component $\mathcal{H}'$ of $\mathcal{H}(2g-$*

4) *in which every translation surface has at least $k-1$ Hodge-Teichmüller planes.*

Propositions 5.1 and 5.3 together prove Theorem 1.8. To prove Proposition 5.3, we will need to study certain degenerations of translation surfaces to lower genus.

5.1. **Bubbling a handle.** In their classification of connected components of strata, Kontsevich-Zorich introduced a local surgery of Abelian differentials that increases genus by 1. They call their surgery, which involves splitting a zero and then gluing in a torus, "bubbling a handle."

Splitting a zero. Eskin-Masur-Zorich [**EMZ03**] introduced a local surgery of a translation surface known as splitting (or opening up) a zero. A cut and paste operation is performed in a sufficiently small neighborhood of a zero of order m on a translation surface, producing a new translation surface which is flat isometric to the old one outside of this neighborhood, but inside the neighborhood has a pair of zeros of orders m' and m'' with sum $m' + m'' = m$, joined by a short saddle connection with holonomy $v \in \mathbb{R}^2$. We will say that the direction of v in $\mathbb{R}^2$ is the direction of the splitting. (The degenerate case $m'' = 0$ is usually allowed but we will not need it.) For more details, we highly recommend consulting [**KZ03**, Section 4.2] or [**EMZ03**, Section 8.1].

Gluing in the torus. The process of bubbling a handle begins by splitting a zero. The new short saddle connection is then cut, producing a slit. A cylinder is glued into this slit, one end to each half of the slit. This identifies the two endpoints of the slit. The torus is called a handle, which has been "bubbled off" from the original translation surface. The result is a translation surface with the same number of zeros, but the order of one of the zeros (the one which was split) goes up by one, and the genus increases by one. Even after the zero has been split open, there are a number of ways of gluing in a handle, parameterized by the width of the cylinder, and the twist applied when gluing. We will usually glue in a *square handle*, that is, we will take a square with a side parallel to the slit, glue the two sides parallel to the slit to the two sides of the slit, and glue the two remaining sides to each other. See figure 5.3.

Adjacency of strata. The following is a key step in the Kontsevich-Zorich classification of strata.

Proposition 5.4. *For each connected component $\mathcal{H}$ of a minimal stratum, there is a connected component $\mathcal{H}'$ of the minimal stratum in genus*
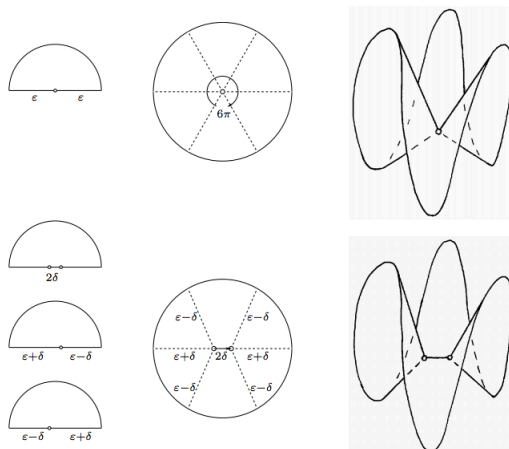
FIGURE 5.1. Splitting a zero is a local surgery which replaces a single zero with a pair of zeros joined by a short saddle connection. In this picture $m = 2, m' = m'' = 1$, and the holonomy v of the newly created short saddle connection is horizontal and has length 2δ. Figure from [**EMZ03, KZ03**].
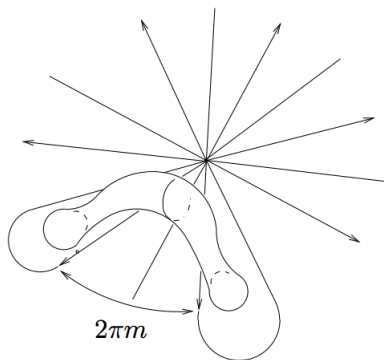

FIGURE 5.2. After opening up a zero, a cylinder or handle is glued in. The process of splitting the zero and then gluing in a cylinder is called bubbling a handle. Figure from [**KZ03**].

one less, so that from every translation surface in $\mathcal{H}'$ it is possible to bubble a square handle so that the resulting translation surface is in $\mathcal{H}$.

Proof. Lemma 14 in [**KZ03**] asserts that given such an $\mathcal{H}$, there is a connected component $\mathcal{H}'$ of the minimal stratum in genus one less, and a translation surface in $\mathcal{H}'$ from which it is possible to bubble a handle and obtain a translation surface in $\mathcal{H}$. The continuity of the
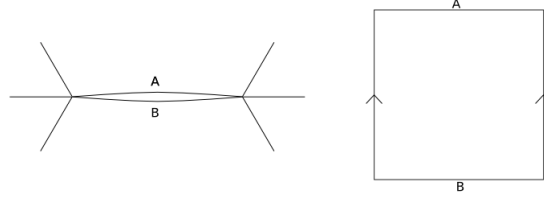
FIGURE 5.3. Another picture of bubbling a square torus
(in a horizontal direction). After the two zeros have been
separated, the short saddle connection joining them is
cut and a square cylinder is glued in. As a result of
gluing in the cylinder the two endpoints of the slit get
identified. (The order of zero in this picture is less than
in the previous picture.)

bubbling a handle operation (Lemma 10 in [**KZ03**]) then gives that
the translation surface in $\mathcal{H}'$ can be changed arbitrarily along paths,
and the cylinder which is bubbled off can be made square.   □

5.2. **The limit of a Hodge-Teichmüller plane as the handle
shrinks.** Before we discuss convergence of Hodge-Teichmüller planes,
we need to discuss convergence of translation surfaces.

Proposition 5.5. *Let (X_1, ω_1) be the result of bubbling a square han-
dle from (Y, ω), and let (X_n, ω_n) be a sequence of translation surfaces
obtained from the same bubbling construction but letting the size of the
square handle go to zero. Then X_n converges in the Deligne-Mumford
compactification to a stable curve X_∞ given by Y joined at a node to a
torus, and ω_n converges to the Abelian differential which is equal to ω
on Y and is zero on the torus. The node on Y is at the zero of ω.*

*Fix $h \in SL(2, \mathbb{R})$, and set $(X_n^h, \omega_n^h) = h(X_n, \omega_n)$, and let $(Y^h, \omega^h) =
h(Y, \omega)$. Then X_n^h converges in the Deligne-Mumford compactification
to Y^h joined at a node to a torus, and ω_n^h converges to the Abelian
differential which is equal to ω^h on Y^h and is zero on the torus.*

One of the key technical points is that the torus is smooth, i.e.,
no additional curve on the torus gets pinched. The proof is deferred
to section 6. We now proceed to limits of Hodge-Teichmüller planes,
proving a version of Proposition 3.2.

Proposition 5.6. *We use the notation and set up of the previous
proposition. Let $\iota : Y \to X_\infty$ denote the natural inclusion. Suppose P_n
is a Hodge-Teichmüller plane at (X_n, ω_n), and P_n converges to a plane
$P \subset H^1(X_\infty, \mathbb{R})$. Then $\iota^*(P)$ is a Hodge-Teichmüller plane at (Y, ω),
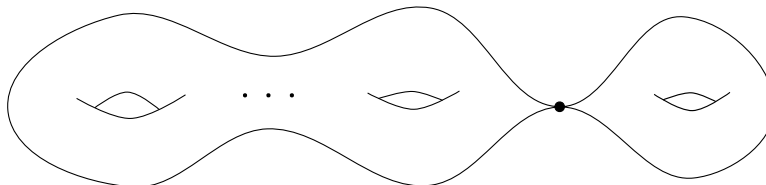unless $\iota^*(P)$ is the zero subspace.*

FIGURE 5.4. The limit X_∞ of X_n is a noded Riemann surface, equal to the genus $g-1$ surface Y (left) glued to a torus (right) at a node. The limit of the ω_n is supported on Y, where it is equal to ω.

A point of the augmented Teichmüller space, resp. Deligne-Mumford compactification is a marked noded Riemann surface, resp. noded Riemann surface. Topologically, a noded Riemann surface can be thought of as the result of pinching a set of disjoint curves on a surface of genus g. Roughly speaking, a marking on noded Riemann surface a map from the reference surface S which collapses some disjoint curves. See [**Bai07**] Section 5.2 for the precise definition. The Deligne-Mumford compactification is obtained from augmented Teichmüller space by quotient by the mapping class group (or, equivalently, by forgetting markings). If, as is the case for X_∞, only separating curves are pinched, the noded Riemann surface is said to have *geometric genus g*.

Pinching a set of homologically trivial curves does not effect (co)homology, and so the bundle H^1 over Teichmüller space (whose fiber over a Riemann surface X is $H^1(X, \mathbb{R})$) extends continuously to points in the augmented Teichmüller space where only separating curves have been pinched.

The bundle $H^{1,0}$ also extends continuously to all (marked) noded Riemann surfaces, but for us it will be easier to think only about the Riemann surfaces of geometric genus g. The fiber of $H^{1,0}$ over a noded Riemann surface X of geometric genus g is isomorphic to the space of holomorphic one forms on the disconnected smooth Riemann surface obtained by separating all the nodes of X. In particular, $H^{1,0}(X_\infty)$ is isomorphic to the direct sum of $H^{1,0}(Y)$ and $H^{1,0}$ of the torus which is glued to Y.

The bundle $H^{1,0}$ is a continuous subbundle of H^1 over the locus of possibly noded (marked) Riemann surfaces of geometric genus g.

We will show that there are shrinking neighborhoods U_n of the node on X_∞, and conformal maps $f_n : X_\infty \setminus U_n \to X_n$ such that $(f_n)^*(\omega_n) \to \omega$. (This is the definition of convergence of Abelian differentials, as we will recall in the next section.) Again we say that planes $P_n \subset H^1(X_n, \mathbb{R})$ converge to a plane $P \subset H^1(X_\infty, \mathbb{R})$ if the maps f_n can be

chosen so that $(f_n)^*(P_n) \to P$. Again, if the surfaces are marked, these maps will be required to commute with the markings.

We need the following version of Lemma 3.5.

Lemma 5.7. *Suppose $P_n \subset H^1(X_n, \mathbb{R})$ is a sequence of planes converging to $P \subset H^1(X_\infty, \mathbb{R})$. If $(P_n)_{\mathbb{C}} \cap H^{1,0}(X_n) \neq \{0\}$ for all n, then $P_{\mathbb{C}} \cap H^{1,0}(X_\infty) \neq \{0\}$.*

Proof. This follows as in Lemma 3.5, because $H^{1,0}$ is a continuous sub-bundle of $H^1 \otimes \mathbb{C}$ over the locus of possibly noded (marked) Riemann surfaces of geometric genus g. □

We also need the following version of Lemma 3.6, whose proof is also deferred to the end of the next section.

Lemma 5.8. *Suppose $P_n \subset H^1(X_n, \mathbb{R})$ is a sequence of planes converging to $P \subset H^1(X_\infty, \mathbb{R})$. Fix any $h \in SL(2, \mathbb{R})$. Then $h(P_n)$ converges to P', where P' satisfies $h(\iota^*(P)) = \iota^*(P')$.*

Proof of Proposition 5.6. This follows directly from Lemmas 5.7 and 5.8. □

Proof of Proposition 5.3. Let $\mathcal{H}$ be a connected component of the stratum $\mathcal{H}(2g - 2)$, and let $\mathcal{H}'$ be the connected component of $\mathcal{H}(2g - 4)$ given by Proposition 5.4. We are assuming that every translation surface in $\mathcal{H}$ has k Hodge-Teichmüller planes.

Let $(Y, \omega) \in \mathcal{H}'$. Let (X_1, ω_1) be the result of bubbling a square handle from (Y, ω), and let (X_n, ω_n) be a sequence of translation surfaces obtained from the same bubbling construction but letting the size of the square handle go to zero. By Proposition 5.5, given $h \in SL(2, \mathbb{R})$, one has that X_n^h converges in the Deligne-Mumford compactification to Y^h joined at a node to a torus, and ω_n^h converges to the Abelian differential which is equal to ω^h on Y^h and is zero on the torus.

Let $P_n^{(1)}, \ldots, P_n^{(k)}$ be orthogonal Hodge-Teichmüller planes at (X_n, ω_n). After passing to a subsequence, we may assume that $P_n^{(i)}$ converges to a plane $P^{(i)}$ at X_∞.

The Hodge inner product extends continuously to nodal surfaces of genus g. (The definition given in section 2.7 is equally valid for nodal surfaces of genus g. One could also use the symplectic pairing here.) The planes $P^{(i)}$ are Hodge orthogonal at X_∞, and hence at most one can be contained in $\ker(\iota^*)$. Without loss of generality, assume that $P^{(i)}$ is not contained in $\ker(\iota^*)$ for $i < k$.

Then Proposition 5.6 gives that the set of $\iota^*(P^{(i)})$ for $1 \leq i \leq k - 1$ are orthogonal Hodge-Teichmüller planes at (Y, ω). □

Remark 5.9. In this proof we see the theoretical possibility of loosing a Hodge-Teichmüller plane to $\ker(\iota^*)$ in the inductive step. It is precisely this which has stopped us from resolving Problem 7.1 below.

6. PROOF OF PROPOSITION 5.5

We will proceed in two steps. First we will build the candidate X_∞ using flat geometry, and then we will prove that X_n converges to this X_∞. Once the proof of Proposition 5.5 is complete, we will sketch the proof of Lemma 5.8.

The techniques and results in this section are not new: compare to [**EKZ**, Section 4], which uses work of Rafi [**Raf07**].

6.1. **Construction of X_∞.** The proposition states that X_∞ is Y joined at a node to a torus. We will specify the torus using flat geometry. More precisely, we will construct an infinite volume flat structure which is given by a meromorphic Abelian differential on a torus, and it is this torus that we will join to Y at a node to construct X_∞. The flat infinite volume torus will be obtained by bubbling a handle from an infinite volume flat surface on the sphere.

We begin by constructing the infinite volume flat surface on the sphere, with a zero of some order $m > 0$. Take $m + 1$ copies of the upper half plane, and $m + 1$ copies of the lower half plane. Glue them to each other along the segments $(-\infty, 0]$ and $[0, \infty)$, to construct an infinite volume flat surface with one singularity of cone angle $(m+1)2\pi$. This is in fact a meromorphic Abelian differential on the sphere; there is a pole at infinity.
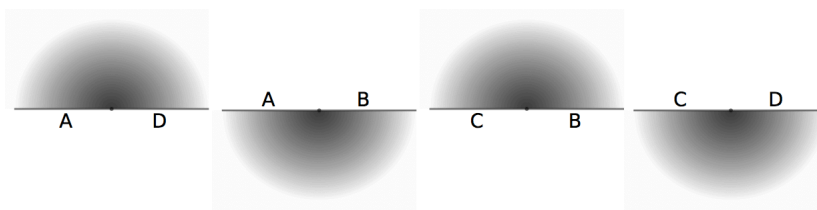

FIGURE 6.1. When $m = 1$, we take two copies each of the upper and lower half planes, and glue them together along the lines $(-\infty, 0]$ and $[0, \infty)$ to create a single cone point with angle 4π. (The shading indicates either the upper or lower half plane, and each labelled line segment is either $(-\infty, 0]$ or $[0, \infty)$.)

From this flat structure we bubble a square handle of size 1 to obtain an infinite volume flat structure on a torus T. The handle should be

bubbled in the same way as on Y: when splitting the zero there is a finite choice of how to split the zero into two zeros of order m' and m'' with $m' + m'' = m$. This choice should be made the same way for the bubbling of the infinite volume sphere as for Y.
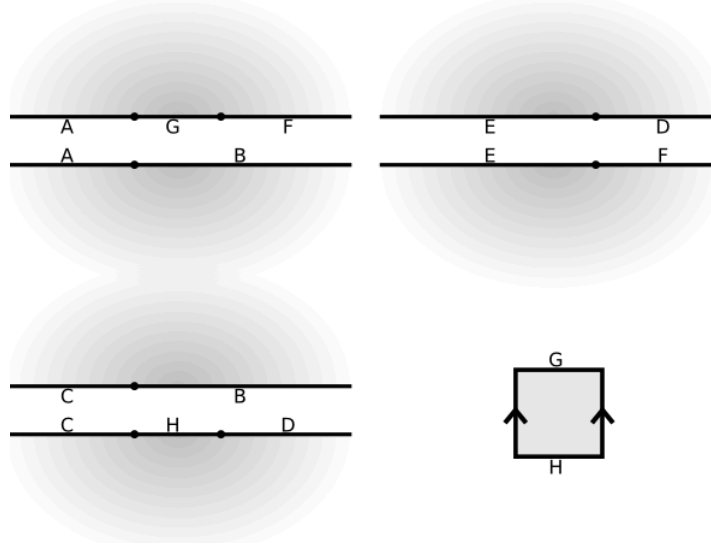

FIGURE 6.2. An infinite volume flat torus with zero of order $m = 2$. (There is a pole at infinity which is not visible but which gives infinite volume.) The segments H and G have length one, and the remaining segments are bi-infinite.

Now define X_∞ to be Y glued to the torus T at a node placed at the zero of Y and the pole of T.

6.2. **Convergence to X_∞.** For this section our primary reference is [**Bai07**], Sections 5.2-5.3, where Bainbridge discusses the topology on the augmented Teichmüller space, the Deligne-Mumford compactification and the bundle of stable Abelian differentials, recalling results of Abikoff [**Abi77**], Bers [**Ber74**], and others as well as results which are well known but not written in the literature.

The topology near X_∞ may be described as follows. Let $U \subset X_\infty$ be any small neighborhood of the node in X_∞. Consider the set V_U of all nodal Riemann surfaces into which there is a holomorphic injection from $X_\infty - \overline{U}$. The V_U form a neighborhood basis at X_∞.

Proof of Proposition 5.5. We will show that X_n converges to X_∞ by showing it is in the neighborhood V_U above, where U gets smaller as n gets larger.

Indeed, suppose that on X_n the size of the square handle is ε_n, where $\varepsilon_n \to 0$ as $n \to \infty$. Let $H_n \subset X_n$ be the set of points whose flat distance to the square handle (with respect to ω_n) is less than $\sqrt{\varepsilon_n}$, and let $C_n = X_n \setminus \overline{H_n}$.

First we claim that there is a conformal map $\phi_n^H : H_n \to T$ for n large enough. Indeed, this map simply rescales the flat metric by ε_n^{-1} so that the handle now has size 1, and then maps as a flat isometry into T with the infinite volume flat metric given above. The image of ϕ_n is all points whose flat distance from the handle on T is at most $\sqrt{\varepsilon_n^{-1}}$.

Next we claim that there is a conformal map $\phi_n^C : C_n \to Y$. Indeed, C_n with the ω_n metric is flat isometric to the complement of a neighborhood of the zero in (Y, ω). This is because bubbling a handle into (Y, ω) is a *local* surgery, and does not change the flat metric outside of a neighborhood of the zero of ω.

Let $U_n = X_\infty \setminus \overline{\phi_n^C(C_n) \cup \phi_n^H(H_n)}$. Then U_n is a neighborhood of the node in X_∞, and the intersection of all the U_n is equal to the node. Define a map $\psi_n : X_\infty \setminus \overline{U_n} \to X_n$ by setting $\psi_n(z) = (\phi_n^C)^{-1}(z)$ for $z \in \phi_n^C(C_n)$, and $\psi_n(z) = (\phi_n^H)^{-1}(z)$ for $z \in \phi_n^H(H_n)$. This map is by definition a conformal injection $X_\infty \setminus \overline{U_n} \to X_n$, showing that $X_n \to X_\infty$.

The claim that ω_n converges to the Abelian differential ω supported on $Y \subset X_\infty$ is immediate from the description of the topology on the bundle of stable Abelian differentials given in [**Bai07**] (see p.60, under "Long cylinders").

This proof applies equally well after applying a fixed $h \in SL(2, \mathbb{R})$ (the only difference is that the handles are no longer squares). $\qquad \square$

Proof of Lemma 5.8. Let $h \in SL(2, \mathbb{R})$. In fact, the arguments in the previous three paragraphs above shows that there is a well defined nodal Riemann surface X_∞^h. Here X_∞^h is the result of acting linearly by h on the infinite volume flat structure on X_∞ and taking the underlying Riemann surface structure. In particular, X_n^h converges to X_∞^h in the Deligne-Mumford compactification. Here we are using the notation $h(X_n, \omega_n) = (X_n^h, \omega_n^h)$ and $h(Y, \omega) = (Y^h, \omega^h)$.

The bundle $H_\mathbb{R}^1$ is trivial over augmented Teichmüller space. Set P' to be the parallel transport of P to X_∞. (We are implicitly fixing markings here.) It is now straightforward to see that $h(P_n)$ converges to P'.

Indeed, for n sufficiently large depending on h, there is a natural identification of $H_1(X_n^h, \mathbb{R})$ with $H^1(Y^h, \mathbb{R}) \oplus \mathbb{R}^2$

Note that $H^1(X_\infty^h, \mathbb{R}) = H_1(X_\infty^h, \mathbb{R})^*$, and the map ι^* is restriction to $H_1(Y^h, \mathbb{R})$. Using the above identification, we also have restriction maps $\iota^* : H^1(X_n^h, \mathbb{R}) \to H^1(Y^h, \mathbb{R})$, for n sufficiently large.

Since the action of h on the homology group $H_1(X_n, \mathbb{R})$ is trivial we see that $\iota^* \circ h = h \circ \iota^*$.

Thus $\iota^*(hP_n) = h\iota^*(P_n)$ for all n sufficiently large. Taking a limit we get that $\iota^*(P') = h\iota^*(P)$.

$\square$

7. Open problems

Here we list three problems arising directly from our work.

Problem 7.1. Show that for all connected components of strata in genus at least 3 there is at least one surface whose only Hodge-Teichmüller plane is the tautological plane.

A solution to Problem 7.1 would help in the study of primitive but not algebraically primitive Teichmüller curves, as would a solution to the following.

Problem 7.2. Let $\mathcal{Q}$ be a connected component of a stratum of quadratic differentials, and let $\mathcal{M}$ be the corresponding affine invariant submanifold of translation surfaces that are holonomy double covers of half-translation surfaces in $\mathcal{Q}$. Show that except in certain small genus cases (such as Prym loci) there is a translation surface in $\mathcal{M}$ whose only Hodge-Teichmüller plane is the tautological plane.

Problems 7.1 and 7.2 are closely related to the ongoing study of the Kontsevich-Zorich cocycle.

Problem 7.3. In each connected component of each stratum in genus g at least 3, give an explicit open set of translation surfaces without g orthogonal Hodge-Teichmüller planes.

The simplest case is probably the hyperelliptic connected component of $\mathcal{H}(4)$. Even in that case, we do not know how to do Problem 7.3, however we do not feel that it is completely hopeless. For example, a study of the second derivative of the period matrix might be helpful.

References

[Abi77]   William Abikoff, *Degenerating families of Riemann surfaces*, Ann. of
          Math. (2) **105** (1977), no. 1, 29–44.
[ANW]   David Aulicino, Duc-Manh Nguyen, and Alex Wright, *Classification of
          higher rank orbit closures in $\mathcal{H}^{\mathrm{odd}}(4)$*, preprint, arXiv 1308.5879 (2013).

[Bai07]   Matt Bainbridge, *Euler characteristics of Teichmüller curves in genus two*, Geom. Topol. **11** (2007), 1887–2073.

[Ber74]   Lipman Bers, *On spaces of Riemann surfaces with nodes*, Bull. Amer. Math. Soc. **80** (1974), 1219–1222.

[BM10]   Irene I. Bouw and Martin Möller, *Teichmüller curves, triangle groups, and Lyapunov exponents*, Ann. of Math. (2) **172** (2010), no. 1, 139–185.

[BM12]   Matt Bainbridge and Martin Möller, *The Deligne-Mumford compactification of the real multiplication locus and Teichmüller curves in genus 3*, Acta Math. **208** (2012), no. 1, 1–92.

[Cal04]   Kariane Calta, *Veech surfaces and complete periodicity in genus two*, J. Amer. Math. Soc. **17** (2004), no. 4, 871–908.

[EKZ]   Alex Eskin, Maxim Kontsevich, and Anton Zorich, *Sum of lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow*, preprint, arXiv 1112.5872 (2012).

[EM]   Alex Eskin and Maryam Mirzakhani, *Invariant and stationary measures for the $SL(2, \mathbb{R})$ action on moduli space*, preprint, arXiv 1302.3320 (2013).

[EMM]   Alex Eskin, Maryam Mirzakhani, and Amir Mohammadi, *Isolation theorems for $SL(2, \mathbb{R})$-invariant submanifolds in moduli space*, preprint, arXiv:1305.3015 (2013).

[EMZ03]   Alex Eskin, Howard Masur, and Anton Zorich, *Moduli spaces of abelian differentials: the principal boundary, counting problems, and the Siegel-Veech constants*, Publ. Math. Inst. Hautes Études Sci. (2003), no. 97, 61–179.

[Fila]   Simion Filip, *Semisimplicity and rigidity of the kontsevich-zorich cocycle*, preprint, arXiv:1307.7314 (2013).

[Filb]   ―――, *Splitting mixed Hodge structures over affine invariant manifolds*, preprint, arXiv:1311.2350 (2013).

[FMZ]   Giovanni Forni, Carlos Matheus, and Anton Zorich, *Lyapunov spectrum of invariant subbundles of the Hodge bundle*, preprint, arXiv: 1112.0370 (2011), to appear in Ergodic Theory Dynam. Systems.

[GJ00]   Eugene Gutkin and Chris Judge, *Affine mappings of translation surfaces: geometry and arithmetic*, Duke Math. J. **103** (2000), no. 2, 191–213.

[Hoo13]   W. Patrick Hooper, *Grid graphs and lattice surfaces*, Int. Math. Res. Not. IMRN (2013), no. 12, 2657–2698.

[HS01]   P. Hubert and T. A. Schmidt, *Invariants of translation surfaces*, Ann. Inst. Fourier (Grenoble) **51** (2001), no. 2, 461–495.

[HS06]   Pascal Hubert and Thomas A. Schmidt, *An introduction to Veech surfaces*, Handbook of dynamical systems. Vol. 1B, Elsevier B. V., Amsterdam, 2006, pp. 501–526.

[KS00]   Richard Kenyon and John Smillie, *Billiards on rational-angled triangles*, Comment. Math. Helv. **75** (2000), no. 1, 65–108.

[KZ03]   Maxim Kontsevich and Anton Zorich, *Connected components of the moduli spaces of Abelian differentials with prescribed singularities*, Invent. Math. **153** (2003), no. 3, 631–678.

[LN]   Erwan Lanneau and Duc-Mahn Nguyen, *Teichmüller curves generated by weierstrass prym eigenforms in genus three and genus four*, preprint, arXiv:1111.2299 (2011).

[Mas82]   Howard Masur, *Interval exchange transformations and measured foliations*, Ann. of Math. (2) **115** (1982), no. 1, 169–200.

[McM03a] Curtis T. McMullen, *Billiards and Teichmüller curves on Hilbert modular surfaces*, J. Amer. Math. Soc. **16** (2003), no. 4, 857–885 (electronic).

[McM03b] _____, *Teichmüller geodesics of infinite complexity*, Acta Math. **191** (2003), no. 2, 191–223.

[McM05]   _____, *Teichmüller curves in genus two: discriminant and spin*, Math. Ann. **333** (2005), no. 1, 87–130.

[McM06a] _____, *Prym varieties and Teichmüller curves*, Duke Math. J. **133** (2006), no. 3, 569–590.

[McM06b] _____, *Teichmüller curves in genus two: torsion divisors and ratios of sines*, Invent. Math. **165** (2006), no. 3, 651–672.

[McM07]   _____, *Dynamics of* $SL_2(\mathbb{R})$ *over moduli space in genus two*, Ann. of Math. (2) **165** (2007), no. 2, 397–456.

[McM09]   _____, *Rigidity of Teichmüller curves*, Math. Res. Lett. **16** (2009), no. 4, 647–649.

[Möl]   Martin Möller, *Prym covers, theta functions and Kobayashi curves in Hilbert modular surfaces*, arXiv:1111.2624 (2011), to appear in Amer. Journal of Math.

[Möl06a] _____, *Periodic points on Veech surfaces and the Mordell-Weil group over a Teichmüller curve*, Invent. Math. **165** (2006), no. 3, 633–649.

[Möl06b] _____, *Variations of Hodge structures of a Teichmüller curve*, J. Amer. Math. Soc. **19** (2006), no. 2, 327–344 (electronic).

[Möl08]   _____, *Finiteness results for Teichmüller curves*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 1, 63–83.

[Muka]   Ronen Mukamel, *Fundamental domains and generators for lattice Veech groups*, http://www.stanford.edu/~ronen/papers/FundamentalDomains.pdf, preprint.

[Mukb]   _____, *Orbifold points on Teichmuller curves and Jacobians with complex multiplication*, http://www.stanford.edu/~ronen/papers/OrbifoldPts.pdf, to appear in Geom. Topol.

[MY10]   Carlos Matheus and Jean-Christophe Yoccoz, *The action of the affine diffeomorphisms on the relative homology group of certain exceptionally symmetric origamis*, Journal of Modern Dynamics **4** (2010), no. 3, 453–486.

[NW]   Duc-Manh Nguyen and Alex Wright, *Non-veech surfaces in* $\mathcal{H}^{\mathrm{hyp}}(4)$ *are generic*, preprint, arXiv:1306.4922 (2013).

[Raf07]   Kasra Rafi, *Thick-thin decomposition for quadratic differentials*, Math. Res. Lett. **14** (2007), no. 2, 333–341.

[SW04]   John Smillie and Barak Weiss, *Minimal sets for flows on moduli space*, Israel J. Math. **142** (2004), 249–260.

[SW10]   _____, *Characterizations of lattice surfaces*, Invent. Math. **180** (2010), no. 3, 535–557.

[Vee82]   William A. Veech, *Gauss measures for transformations on the space of interval exchange maps*, Ann. of Math. (2) **115** (1982), no. 1, 201–242.

[Vee89]   W. A. Veech, *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards*, Invent. Math. **97** (1989), no. 3, 553–583.

[Vee90]   William A. Veech, *Moduli spaces of quadratic differentials*, J. Analyse Math. **55** (1990), 117–171.

[Vee95]   _____, *Geometric realizations of hyperelliptic curves*, Algorithms, fractals, and dynamics (Okayama/Kyoto, 1992), Plenum, New York, 1995, pp. 217–226.

[Wria]   Alex Wright, *Cylinder deformations in orbit closures of translation surfaces*, preprint, arXiv:1302.4108(2013).

[Wrib]   _____, *The field of definition of affine invariant submanifolds of the moduli space of abelian differentials*, arXiv:1210.4806 (2012), to appear in Geom. Topol.

[Wri12]   _____, *Schwarz triangle mappings and Teichmüller curves: abelian square-tiled surfaces*, J. Mod. Dyn. **6** (2012), no. 3, 405–426.

[Wri13]   _____, *Schwarz triangle mappings and Teichmüller curves: the Veech-Ward-Bouw-Möller curves*, Geom. Funct. Anal. **23** (2013), no. 2, 776–809.

UNIVERSITÉ PARIS 13, SORBONNE PARIS CITÉ, LAGA, CNRS (UMR 7539), F-93430, VILLETANEUSE, FRANCE
  E-mail address: `matheus.cmss@gmail.com`

MATH DEPARTMENT, UNIVERSITY OF CHICAGO, 5734 SOUTH UNIVERSITY AVENUE, CHICAGO, IL 60637
  E-mail address: `alexmwright@gmail.com`