HOMOTOPY GROUPS OF SPHERES: A VERY BASIC INTRODUCTION

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Abstract. We define fiber bundles and discuss the long exact sequence of homotopy groups of a fiber bundle, and we give the Hopf bundles as examples. We also prove the Freudenthal suspension theorem for spheres. All results are applied immediately to homotopy groups of spheres.

1. Introduction

The purpose of this note is to give a very basic introduction to the homotopy groups of spheres, assuming only knowledge of the long exact sequence of homotopy groups of a pair. For ease of reading, we avoid all generality which does not apply immediately to homotopy groups of spheres. The primary reference for this note is Hatcher's *Algebraic Topology*, and we use the notation contained therein. These notes may be useful to students having read pages 337 to 345 in Hatcher, but looking for a slightly more gentle introduction to homotopy theory before continuing their reading in Hatcher.

The higher homotopy groups $\pi_k(X, x_0)$ of a space X are generalizations of the fundamental group $\pi_1(X, x_0)$. For $k > 1$, these groups are easily seen to be abelian. If $A \subset X$, the relative homotopy groups $\pi_k(X, A, x_0)$ are also defined, and we have the compression criterion: A map $f : (D^n, S^n, s_0) \to (X, A, x_0)$ represents zero in $\pi_k(X, A, x_0)$ if and only if it is homotopic $\operatorname{rel} S^{n-1}$ to a map with image contained in A. This compression criterion is used to derive the long exact sequence of homotopy groups of the pair (X, A, x_0):

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots \to \pi_0(X, x_0).$$

In this sequence i and j are the inclusions $(A, x_0) \hookrightarrow (X, x_0)$ and $(X, x_0, x_0) \hookrightarrow (X, A, x_0)$, and ∂ is a sort of restriction operator. Recall that $\pi_k(X, A, x_0)$ is guaranteed to be a group only if $k \geq 2$, and is guaranteed to be abelian only if $k \geq 3$. Furthermore, the final terms in

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the exact exact sequence, like $\pi_0(X, x_0)$, the set of path connected components of X, are not usually groups. However, exactness still makes sense: the image of each map is the kernel of the next.

We commonly represent maps in $\pi_i(X, A, x_0)$ as maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$, using Hatcher's definitions $I^{n-1} = I^{n-1} \times \{0\} \subset I^n$ and $J^{n-1} = \mathrm{cl}(I^n - I^{n-1})$.

2. Fiber Bundles

Fiber bundles are a sort of short exact sequence $F \to E \xrightarrow{p} B$ of spaces, in which the fibers $p^{-1}(b)$ are all homeomorphic to F. F is called the fiber of the fiber bundle, E the total space, and B the base space. Formally, a fiber bundle is such a map $E \xrightarrow{p} B$, for which B has an open cover U_α such that $p^{-1}(U_\alpha) \simeq U_\alpha \times F$.

Of course, a projection map $B \times F \to B$ gives rise to a fiber bundle $F \to B \times F \to B$. Perhaps the simplest example of a fiber bundle that is not of this form is the bundle in which the Mobius strip is the total space, the fiber is the closed unit interval, and the base space the circle. Also, every covering space map $X \to Y$ is a fiber bundle with a discrete fiber.

We now proceed to construct a fiber bundle with total space S^{2n+1}, fiber S^1 and base space $\mathbb{C}P^n$. View S^{2n+1} as the set of unit length vectors in $\mathbb{C}^{n+1}$, and consider the quotient map $q : S^{2n+1} \to \mathbb{C}P^n$ given by the equivalence relation $v \sim \lambda v$ if $|\lambda| = 1$. All the fibers $q^{-1}(p)$ are circles. Furthermore, $\mathbb{C}P^n = \cup U_k$, where $U_k = \{[v_1, \cdots, v_{n+1}] | v_k \neq 0\}$. We have

$$\begin{aligned} q^{-1}(U_k) &= \{v = (v_1, \cdots, v_{n+1}) \in S^{2n+1} : v_k \neq 0\} \\ &\cong U_k \times S^1 \end{aligned}$$

so $S^1 \to S^{2n+1} \xrightarrow{q} \mathbb{C}P^n$ is indeed a fiber bundle. Consider in particular $\mathbb{C}P^1$. As a manifold, it has an atlas of the two charts $U_1 \cong \mathbb{C}$ and $U_2 \cong \mathbb{C}$ with transition map $z \mapsto 1/z$. Now consider the sphere $S^2 = \{(a, b, c) : a^2 + b^2 + c^2 = 1\} \subset \mathbb{R}^3$. If we consider the a, b plane to be $\mathbb{C}$, then the two charts of S^2 given by the stereographic projections from the poles, have transition map $z \mapsto 1/\overline{z}$. Hence if we compose one of the charts with conjugation, we get that the new transition map is $z \mapsto 1/z$. Thus, $\mathbb{C}P^1 \cong S^2$. So, in particular, we get the fiber bundle $S^1 \to S^3 \to S^2$, often called the Hopf fibration.

There are also fiber bundles $S^3 \to S^{4n+3} \to \mathbb{H}P^n$ and $S^7 \to S^{8n+7} \to \mathbb{O}P^n$ given by the quaternions and the octonians. In particular, in the case $n = 1$, we have fiber bundles $S^3 \to S^7 \to S^4$ and $S^7 \to S^{15} \to S^8$, which are also often called Hopf fibrations. It is know that there are no other fiber bundles in which the fiber, base space, and total space are all spheres.

The long exact sequence takes a nicer form for pairs (F, E) coming from a fiber bundle. But before we can prove this, we need to discuss some technical details.

A map $p : E \to B$ is said to have the lift extension property for (X, A) if every map $f : X \to B$ lifts to a map $g : X \to E$ extending a given lift $g : A \to E$. The lift extension properties for the pair $(D^n \times I, D^n \times \{0\} \cup \partial D^n \times I)$ will be particularly useful to us. Note that the lift extension property for the pairs $(D^n \times I, D^n \times \{0\} \cup \partial D^n \times I)$ and $(D^n \times I, D^n \times \{0\})$ are equivalent, since these pairs are homeomorphic. This common property is know as the homotopy extension property for disks when it is true for all n.

Proposition. *A fiber bundle $F \to E \xrightarrow{p} B$ has the homotopy extension property for disks.*

Proof. The disk D^n is homeomorphic to the cube I^n. Let $H : I^n \times I \to B$ be the map whose lift $G : I^n \times 0 \cup \partial I^n \times I \to E$ we wish to extend. Choose an open cover $\{U_\alpha\}$ of B with homeomorphisms $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F$. By dividing I^n into small cubes C, and partitioning I into small intervals $I_j = [t_j, t_{j+1}]$, we may assume each product $C \times I_j$ is mapped into a single U_α by H. We can assume by induction on j that G has already been constructed on $C \times [0, t_j]$, and we can further assume, by induction on n, that G has been defined on $\partial C \times [t_j, t_{j+1}]$ for each C. Thus it suffices to construct the lift on each such $C \times [t_j, t_{j+1}]$. We have thus reduced the problem to the case where no subdivision is necessary, and $H(I^n \times I)$ is contained in a single U_α.

To summarize, we have a map $H : I^n \times I \to U_\alpha$, and we want to lift it to a map $G : I^n \times I \to h_\alpha^{-1}(U_\alpha) = U_\alpha \times F$. We are given a map $G : I^n \times \{0\} \cup \partial I^n \times I \to E$ that this map must extend. Now we choose a retract $R : I^n \times I \to I^n \times \{0\} \cup \partial I^n \times I$, and we define $G(x) = (H(x), G(R(x)))$. This is the desired lift of H. □

Theorem. *Suppose $p : E \to B$ has the homotopy lifting property for disks. Choose a base point $b_0 \in B$ and $x_0 \in F = p^{-1}(b_0)$. Then the map $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0, b_0) = \pi_n(B, b_0)$ is an isomorphism. Hence if B is path connected, there is a long exact sequence*

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0).$$

In particular, by the preceding proposition, a fiber bundle with path connected base space has this long exact sequence.

Proof. First we prove that p_* is onto. Represent an element of $\pi_n(B, b_0)$ by a map $f : (I^n, \partial I^n) \to (B, b_0)$. The constant map to x_0 provides a lift of f to E over the subspace $J^{n-1} \subset I^n$, so the homotopy extension property for D^{n-1} (not D^n) extends this to a lift $g : I^n \to E$, and this lift satisfies $g(\partial I^n) \subset F$ since $f(\partial I^n) = b_0$. Then g represents an element of $\pi_n(E, F, x_0)$ with $p_*([g]) = [f]$ since $pg = f$.

Injectivity of p_* is similar. Given $g_0, g_1 : (I^n, \partial I^n, J^{n-1}) \to (E, F, x_0)$ such that $p * ([g_0]) = p * ([g_1])$, let $g : I^n \times I, \partial I^n \times I) \to (B, b_0)$ be a homotopy from pg_0 to pg_1. We have a partial lift G given by g_0 on $I^n \times \{0\}$, g_1 on $I^n \times \{1\}$ and the constant map x_0 on $J^{n-1} \times I$. The homotopy lifting property for D^n extends this to a lift $G : I^n \times I \to E$, giving a homotopy $g_t : (I^n, \partial I^n, J^{n-1}) \to (E, F, x_0)$ from g_0 to g_1. So p_* is infective.

For the last statement of the theorem we plug $\pi_n(B, b_0) \cong \pi_n(E, F, x_0)$ in the long exact sequence for the pair (E, F). □

If we consider the fiber bundle $Z \to \mathbb{R} \to S_1$ given by the covering space map $\mathbb{R} \to S_1$, the long exact sequence gives that $\pi_k(S^1) = 0$ for all $k > 1$. This fact can also be deduced from the lifting criterion of covering space theory. Using this fact, if we consider the Hopf bundle $S^1 \to S^3 \to S^2$, we get $\pi_k(S^3) = \pi_k(S^2)$ for $k \geq 2$. We will soon prove that $\pi_3(S^3) = \mathbb{Z}$, so this will give $\pi_3(S^2) = Z$, a very surprising result! We also get that $\pi_2(S^2) = \mathbb{Z}$ providing that $\pi_1(S^2) = \pi_2(S^3) = 0$. This detail will be proved in the next section.

In general, if we have a bundle $F \to E \to B$ such that the inclusion $F \hookrightarrow E$ is null-homotopic, we get that in the long exact sequence the map $\pi_i(F) \to \pi_i(E)$ is 0, since it is induced by the inclusion $F \to E$. So the long exact sequence breaks into short exact sequences

$$0 \to \pi_i(E) \to \pi_i(B) \to \pi_{i-1}(F) \to 0.$$

Furthermore, above we have an isomorphism $\pi_n(E, F, x_0) \to \pi_i(B, b_0)$. Restricting this to a map $\pi_n(E, x_0, x_0) \to \pi_i(B, b_0)$ shows that our exact sequence is in fact split

$$0 \to \pi_i(E) \rightleftarrows \pi_i(B) \to \pi_{i-1}(F) \to 0.$$

Hence we get isomorphisms $\pi_i(B) = \pi_i(E) \oplus \pi_{i-1}(F)$. Applying this to the Hopf bundles $S^3 \to S^7 \to S^4$ and $S^7 \to S^{15} \to S^8$ give isomorphisms $\pi_i(S^4) = \pi_i(S^7) \oplus \pi_{i-1}(S^3)$ and $\pi_i(S^8) = \pi_i(S^{15}) \oplus \pi_{i-1}(S^7)$.

3. Major Theorems

In this section we will require two difficult technical lemmas. The first is used in the proof of the second.

Lemma. *Let $f : I^n \to Z$ be a map, where Z is obtained from a subspace W by attaching a cell e^k. Then f is homotopic rel $f^{-1}(W)$ to a map f_1 for which there is a simplex $\Delta^k \subset e^k$ with $f_1^{-1}(\Delta^k)$ a union, possibly empty, of finitely many convex polyhedra, on each of which f_1 is the restriction of a linear surjection $\mathbb{R}^n \to \mathbb{R}^k$.*

Corollary. *$\pi_i(S^n) = 0$ for $i < n$.*

Proof of Corollary. In the preceding lemma, set $W = w_0$ to be a point, so $Z = S^n$ is W with a cell e^n attached. Represent an element of $\pi_i(S^n, w_0)$ by a map $f : (I^i, \delta I^i) \to (Z, w_0)$. We then find $\Delta^n \in Z$ and $f_1 \simeq f$ as above. Since there are no linear surjections from $\mathbb{R}^i$ to $\mathbb{R}^n$, $f_1^{-1}(\Delta^n) = \emptyset$. It is now easy to homotope f_1 away from Δ^n to the constant map, showing that $[f] = 0$ in $\pi_i(S^n)$. □

Proof of Lemma. Identifying e^k with $\mathbb{R}^k$, let $B_1, B_2 \subset e_k$ be the closed balls of radius 1 and 2 centered at the origin. By uniform continuity, we can subdivide I^n into small cubes so that the image of each cube has diameter less that 1/2. Let K_1 be the union of all cubes meeting B_1, and let K_2 be the union of all cubes meeting K_1, so that $f^{-1}(B_1) \subset K_1 \subset K_2 \subset f^{-1}(B_2)$.

We can find a subdivision of the cubes in K_2 that is a simplicial complex. Let $g : K_2 \to e_k = R^k$ be the map that equals f on all vertices in that subdivision and is linear on each simplex. Let $\phi : K_2 \to [0, 1]$ be any map with $\phi(K_1) = 1$ and $\phi(\partial K_2) = 0$. We wish to homotope f to be g on K_1, but such a homotopy would need to change K_2 as well, in order to preserve continuity. It would also be ideal if such a homotopy was constant on ∂K_2, so that we can extend it to all of Z. So we define a homotopy $f_t : K_2 \to e_k$ to be $f_t = (1 - t\phi)f + t\phi g$. We check that $f_0 = f$, $f_1|K_1 = g$, and f_t is constant on ∂K_2.

After our homotopy, it is possible that some points of $K_2 - K_1$ map into B_1. This is a problem, because we want the pre-image of a simplex in B_1 to be contained in K_1, where f_1 is linear on simplices. Points in the complement of K_2 are no problem, since $f_1 = f$ there, so they map to points outside of B_1. For points of $K_2 - K_1$, we consider a simplex of the subdivision containing that point. That simplex is mapped by f into a ball of radius 1/2. Since that ball is convex, that simplex is also mapped into that ball by f_1. Thus, $K_2 - K_1$ cannot map into the ball $B_{\frac{1}{2}}$ of radius 1/2. Now, we pick any $\Delta^k \in B_{\frac{1}{2}}$. The pre-image $f_1(\Delta^k) \subset K_1$ is the union of its intersections with simplices σ of K_1, and each intersection is a convex polyhedron since it is the intersection of σ with the convex polyhedron $L_\sigma^{-1}(\Delta^k)$, where $L_\sigma : \mathbb{R}^n \to \mathbb{R}^k$ is the linear map restricting to f_1 on σ. To finish the proof it therefore suffices to choose Δ^k to be disjoint from the images of all the non-surjective L_σ's, which is certainly possible since these images consist of finitely many hyperplanes of dimension less that k. $\square$

This lemma is in fact strikingly powerful. It is used, in Hatcher, to prove the Cellular Approximation Theorem, which stats that a map $f : X \to Y$ between CW complexes is homotopic to a map which caries the k skeleton of X into the k skeleton of Y for all k. Such a map, where $f(X^k) \subset Y^k$ for all k, is called cellular.

Now that we have proven this lemma, we can prove a weak version of excision for homotopy groups, which in turn allows us to prove the Freudenthal Suspension Theorem for spheres. It is called excision because it relates the homotopy groups of a subspace to those of the entire space.

Lemma (Weak Excision Lemma)**.** *Let C be a simplicial complex, let X be C with two $m+1$ cells e_1, e_2 attached, and let $A = C \cup e_1$, $B = C \cup e_2$. The map $\pi_i(A, C) \to \pi_i(X, B)$ induced by inclusion is an isomorphism for $i < 2n$ and a surjection for $i = 2n$.*

Proof of Weak Excision Lemma. To show surjectivity of $\pi_i(A, C) \to \pi_i(X, B)$ we start with a map $f : (I^i, \partial I^i, J^{i-1}) \to (X, B, x_0)$. By the preceding lemma, we can homotope f and find $\Delta_1 \subset e_1$ and $\Delta_2 \subset e_2$ so that $f^{-1}(\Delta_1)$ and $f^{-1}(\Delta_2)$ are finite unions of convex polyhedra, on each of which f is the restriction of a linear surjection from $\mathbb{R}^i$ onto $\mathbb{R}^{n+1}$.

Claim: If $i \leq 2n$, then there exists points $p_1 \in \Delta_1, p_2 \in \Delta_2$, and a map $\phi : I^{n-1} \to [0, 1)$ such that

(a) $f^{-1}(p_2)$ lies below the graph of ϕ in $I^{i-1} \times I = I^i$.
(b) $f^{-1}(p_1)$ lies above the graph of ϕ.
(c) $\phi = 0$ on ∂I^{n-1}.

Given this, let f_t be a homotopy of f excising the region under the graph of ϕ by restricting f to the region above the graph of $t\phi$ for $0 \leq t \leq 1$. By (b), $f_t(I^{i-1})$ is disjoint from p_1 for all t, and by (a), $f_1(I^i)$ is disjoint from p_2. This allows us to push f_1 off of e_2 by homotoping away from p_2. The resulting map, obtained from f through homotopies of the form $f_t : (I^i, \partial I^i, J^{i-1}) \to (X, B, x_0)$, is in $\pi_i(A, C)$. Thus the image of this element of $\pi_i(A, C)$ gives $[f] \in \pi_i(X, B)$ and surjectivity is proved.

Now we prove the claim. For any $p_2 \in \Delta_2$, $f^{-1}(p_2)$ is a finite union of convex polyhedra of dimension less than or equal to $i - n - 1$, since f is the restriction of a linear surjection $\mathbb{R}^i \to \mathbb{R}^{n+1}$ on each of these polyhedra. Thus, if $\pi : I^{i-1} \times I \to I^{i-1}$ is the canonical projection,

$T = \pi^{-1}\pi f^{-1}(\Delta_2)$ is a finite union of convex polyhedra of dimension less than or equal to $i-n$. Since linear maps cannot increase dimension, $f(T) \cap \Delta_1$ is a finite union of convex polyhedra of dimension less than or equal to $i - n$. Thus if $n + 1 > i - n$, there is a point $p_1 \in \Delta_1$ not in $f(T)$. This point p_1 has $f^{-1}(p_1) \cap T = \emptyset$. Hence we can pick a neighbourhood U of $\pi(f^{-1}(p_2))$ in I^{n-1} disjoint from $\pi(f^{-1}(p_1))$. Thus there exists $\phi : I^{i-1} \to [0, 1)$ having support in U, with $f^{-1}(p_2)$ lying under the graph of ϕ. This verifies the claim, and so finishes the proof of surjectivity.

For injectivity the argument is very similar. Suppose we have two maps $f_0, f_1 : (I^i, \partial I^i, J^{i-1}) \to (A, C, x_0)$ representing elements of $\pi_i(A, C, x_0)$ having the same image in $\pi_i(X, B, x_0)$. Thus there is a homotopy from f_0 to f_1 in the form of a map $F : (I^i, \partial I^i, J^{i-1}) \times [0, 1] \to (X, B, x_0)$. After a preliminary deformation, as before, we can find $p_1 \in \Delta_1$, $p_2 \in \Delta_2$ and construct a function $\phi : I^i \times I \to [0, 1)$ separating $F^{-1}(p_2)$ from the set $F^{-1}(p_1)$ as before. Thus allows us to excise $F^{-1}(p_2)$ from the domain of F, from which is follows that f_0 and f_1 represent the same element of $\pi_i(A, C, x_0)$. Since $I^i \times I$ now plays the role of I^i, the dimension i is replaced by $i + 1$ and the dimension restriction becomes $i < 2n$. □

One of the most elegant applications of this Excision Lemma is the following theorem, which talks about suspension. The cone of a space X is $CX = X \times [0, 1]/(X \times \{0\})$. All cones CX are contractible to their tip, $X \times \{0\}$. The suspension of a space X is

$$\Sigma X = (X \times [0, 1])/(X \times \{0\} \cup X \times \{1\}).$$

This is also just two cones C^-X, C^+X glued together. The suspension of a circle is two normal cones glued together, and is thus homeomorphic to the sphere. In general it is easy to see that the suspension of S^n is S^{n+1}. Also, given a map $f : X \to Y$, there is a corresponding map $\Sigma f : \Sigma X \to \Sigma Y$ that maps the two end points of ΣX to those of ΣY and is simply equal to f on each slice $X \times \{x_0\}, 0 < x_0 < 1$.

Corollary (Freudenthal Suspension Theorem)**.** *The suspension map* $\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$ *is an isomorphism for* $i < 2n-1$ *and a surjection for* $i = 2n - 1$*. Consequently,* $\pi_{n+k}(S^n)$ *does not depend on* n *for* $n \geq k + 2$.

Proof. We apply the Weak Excision Lemma with $C = S^n, A = C^-S^n, B = C^+S^n$ and $X = \Sigma S^n = C^-S^n \cup C^+S^n = S^{n+1}$. This gives that the inclusion map

$$\pi_i(C^-S^n, S_n) \to \pi_i(S^{n+1}, C^+S^n)$$

is an isomorphism for $i < 2n$ and a surjection for $i = 2n$. Now, since cones are contractible, the long exact sequence of the pairs (C^-S^n, S^n) and (S^{n+1}, C^+S^n) gives $\pi_i(C^-S^n, S^n) = \pi_{i-1}(S^n)$ and $\pi_i(S^{n+1}, C^+S^n) = \pi_i(S^{n+1})$. Thus we have

$$\pi_{i-1}(S^n) \cong \pi_i(C^-S^n, S_n) \to \pi_i(S^{n+1}, C^+S^n) \cong \pi_i(S^{n+1})$$

and it remains only to see that this map is the suspension map. This is left as an exercise to the reader, as it is easily explained but tricky to write out.

From this point, we see that $\pi_{n+k}(S^n) = \pi_{n+1+k}(S^{n+1})$ if $n + k < 2n-1$, ie if $n \geq k+2$, so $\pi_{n+k}(S^n)$ does not depend on n if $n \geq k+2$. □

The groups $\pi_{n+k}(S^n)$ for $n \geq k + 2$, often denoted π_k^s, are called the stable homotopy groups of spheres, and it is often said that their computation is the largest open problem in algebraic topology.

Corollary. $\pi^n(S^n) = \mathbb{Z}$. *Furthermore, the map* $\pi^n(S^n) \to \mathbb{Z} : [f] \mapsto \deg f$ *given by mapping degree is an isomorphism.*

Proof. Earlier we computed $\pi_2(S^2) = Z$ using the long exact sequence of the Hopf bundle. The previous corollary gives that in the sequence of suspension maps

$$\pi_1(S^1) \to \pi_2(S^2) \to \pi_3(S^3) \to \cdots$$

the first map is surjective, and the rest are isomorphisms. Since the first map as a surjective map $\mathbb{Z} \to \mathbb{Z}$, it is an isomorphism. Now, the map $\deg : S^1 \to \mathbb{Z}$ is an isomorphism. $\pi_1(S^1) = \{[z \mapsto z^k]\}$, so we must have that $\pi^n(S^n)$ is the set of suspensions of these maps $z \to z^k$. The suspensions all have degree k, so we get that $\deg : \pi^n(S^n) \to \mathbb{Z}$ is an isomorphism. □

4. Conclusion

We have computed

- $\pi_i(S^1) = 0$ when $i > 1$
- $\pi_n(Z^n) = \mathbb{Z}$
- $\pi_3(S^2) = \mathbb{Z}$
- $\pi_i(S^2) = \pi_i(S^3)$ for $i \geq 3$
- Given the Hopf bundles, $S^3 \to S^7 \to S^4$ and $S^7 \to S^{15} \to S^8$, $\pi_i(S^4) = \pi_i(S^7) \oplus \pi_{i-1}(S^3)$ and $\pi_i(S^8) = \pi_i(S^{15}) \oplus \pi_{i-1}(S^7)$.

All the groups we have computed have been either 0 or $\mathbb{Z}$. This pattern does not continue, as we can see in this table, taken from Hatcher's *Algebraic Topology.*

$\pi_i(S^n)$

$n \downarrow$ \ $i \rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

It was once hoped that $\pi_i(S^n) = 0$ for $i > n$, and in fact the first counterexample to this was the Hopf fibration, discovered by Heinz Hopf in 1931. Out of this counterexample was born a giant, complex, and mysterious problem: the calculation of the homotopy groups of spheres. Serre prove that $\pi_i(S^n)$ is finite for $i > n$ except for $\pi_{4k-1}(S^{2k})$, which is a direct sum of $\mathbb{Z}$ with finite group. Hence these groups are determined by their p components, where p is prime, that is, the subgroups of elements with order p^k for some k. Hence, much modern research focuses on the p components of the stable homotopy groups of spheres. The primary tool in these computations are spectral sequences, a complex way of organising large amounts of algebraic information.

References

[1] A. Hatcher, *Algebraic Topology.* Cambridge University Press, Cambridge, 2001.