# LUCK'S THEOREM

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Warning: These are the authors personal notes for a talk in a learning seminar (October 2015). There may be incorrect or misleading statements. Corrections welcome.

## 1. CONVERGENCE OF BETTI NUMBERS

The purpose of this note is to outline a proof of the following.

**Theorem 1.1** (Luck, 1994). *Let $X$ be a finite CW complex with universal cover $\tilde{X}$, and let $\Gamma_m, m = 1, 2, \ldots$ be normal finite index subgroups of $\Gamma = \pi_1(X)$ such that $\Gamma_{m+1} \trianglelefteq \Gamma_m$ and $\cap \Gamma_m$ is the trivial group. Then for any $p$, the normalized $p$-th Betti numbers of $X_m = \tilde{X}/\Gamma_m$ converge: the limit*

$$\lim_{m \to \infty} \frac{b_p(X_m)}{[\Gamma : \Gamma_m]}$$

*exists, where $b_p(X_m) = \dim H^p(X_m, \mathbb{C})$.*

In fact the limit is the much studied $p$-th $L^2$-Betti number of $X$. Luck's Theorem answered a question of Gromov. That the limsup is at most the $L^2$-Betti number was previously established by Kazhdan.

In this note we will try to see in an elementary way why this limit exists. The reader should look elsewhere for more general and powerful statements, as well as for context and applications. The author consulted the following sources while preparing this note.

- Luck's original article "Approximating $L^2$-invariants by their finite-dimensional analogues."
- Pansu's "Introduction to $L^2$ -Betti numbers."
- Chapter 8 on $L^p$-cohomology in Gromov's "Asymptotic Invariants of Infinite Groups."
- Luck's book "$L^2$-Invariants: Theory and Applications to Geometry and $K$-Theory."

## 2. From Betti numbers to traces

Consider the cellular cochain complex of $\tilde{X}$. For each $p$, there are infinitely many $p$-cells in $\tilde{X}$, but only finitely many $\Gamma$ orbits of $p$-cells. Indeed, if $X$ has $n_p$ many $p$-cells, then as a $\mathbb{C}[\Gamma]$ module the space of $p$-cells in $\tilde{X}$ is $\mathbb{C}[\Gamma]^{n_p}$. The differential $\partial_p$ from $p$-cells to $(p+1)$-cells is given by a $n_{p+1}$ by $n_p$ matrix $B_p$ with entries in $\mathbb{Z}[\Gamma]$. (The fact that the entires are in $\mathbb{Z}[\Gamma]$ and not just $\mathbb{C}[\Gamma]$ will be important.)

The cochain complex of $X_m$ has $\mathbb{C}$-modules $\mathbb{C}[\Gamma/\Gamma_m]^{n_p}$, and the differentials $\partial_{p,m}$ can be described by the same matrices $B_p$ with entries in $\mathbb{Z}[\Gamma]$ as for $\tilde{X}$. Using the usual Rank-Nullity Theorem for $\mathbb{C}$-vector spaces we see that

$$b_p(X_m) = \dim \ker \partial_{p,m} - n_{p-1}[\Gamma : \Gamma_m] + \dim \ker \partial_{p-1,m}.$$

Thus it suffices to fix $p$ and show that

$$\frac{\dim \ker(\partial_{p,m} : \mathbb{C}[\Gamma/\Gamma_m]^{n_p} \to \mathbb{C}[\Gamma/\Gamma_m]^{n_{p+1}})}{[\Gamma : \Gamma_m]}$$

converges as $m \to \infty$, and this exactly what we will do.

The kernel of $\partial_{p,m}$ is equal to that of $g_m = \partial_{p,m}^* \partial_{p,m}$. Note $g_m$ is self-adjoint, and hence there is a well-defined orthogonal projection onto the kernel of $g_m$. The dimension $\dim \ker(\partial_{p,m})$ is given by the trace of this projection.

This projection is, by basic linear algebra, a polynomial in $g_m$. However, this polynomial depends on $m$. The first key idea of the proof is to approximate $\dim \ker(\partial_{p,m})$ by the trace of a polynomial $p$ not depending on $m$ in $g_m$. The next section will explain why the limiting behavior as $m \to \infty$ of traces of fixed polynomials in $g_m$ is accessible.

## 3. $\mathbb{C}[\Gamma]$-trace

Recall that all the $\partial_{p,m}$ can be represented by the same matrix $B_p$ with coefficients in $\mathbb{Z}[\Gamma]$. Thus all the $g_m$ can be represented by $B_p^* B_p$, where $(B_p^*)_{i,j} = \overline{(B_p)_{j,i}}$ and $\overline{\sum c_\gamma \gamma} = \sum c_\gamma \gamma^{-1}$. (It isn't important what the formula for this matrix is, just that there is an expression that is independent of $m$.)

Define the $\mathbb{C}[\Gamma]$-trace of an element of $\mathbb{C}[\Gamma]$ to be the coefficient of the identity group element, and the $\mathbb{C}[\Gamma]$-trace of a matrix with entries in $\mathbb{C}[\Gamma]$ to be the sum of the traces of its diagonal elements. So

$$\mathrm{tr}_{\mathbb{C}[\Gamma]} \left( \sum_{\gamma \in \Gamma} c_\gamma \gamma \right) = c_e \qquad \text{and} \qquad \mathrm{tr}_{\mathbb{C}[\Gamma]}(B_p^* B_p) = \sum_i \mathrm{tr}_{\mathbb{C}[\Gamma]}(B_p^* B_p)_{i,i}.$$

The following lemma is the only place that we use that the $\Gamma_n$ are *normal* subgroups of $\Gamma$.

**Lemma 3.1.** *For any fixed polynomial $P$, and for $m$ large enough depending on $P$,*

$$\mathrm{tr}(P(g_m)) = \mathrm{tr}_{\mathbb{C}[\Gamma]}\left(P(B_p^* B_p)\right)[\Gamma : \Gamma_m].$$

The proof works for any matrix with coefficients in $\mathbb{C}[\Gamma]$.

*Proof.* Let $S \subset \Gamma$ be a finite set so that all entries of $P(B_p^* B_p)$ are linear combinations of elements of $S$. Pick $m$ large enough so that $S \cap \Gamma_m = \{e\}$. For any $\gamma \in S, \gamma \neq e$, the normality of $\Gamma_m$ in $\Gamma$ implies that the left multiplication action of $\gamma$ on $\mathbb{C}[\Gamma/\Gamma_m]$ has zero trace. This is simply the fact that if

$$\gamma \gamma_0 \Gamma_m = \gamma_0 \Gamma_m \qquad \text{then} \qquad \Gamma_m \gamma^{-1} = \gamma_0 \Gamma_m \gamma_0^{-1} = \Gamma_m.$$

Contributions to $\mathrm{tr}(P(g_m))$ thus come from constant terms in the diagonal. Each such constant term $c$ acts on all of $\mathbb{C}[\Gamma/\Gamma_m]$ by multiplication by $c$, thus contributing $c[\Gamma : \Gamma_m]$ to the trace. $\square$

## 4. Picking polynomial approximations

Recall that if $T$ is any linear transformation with eigenvalues $\lambda_i$ (counted with multiplicity), and $P$ is any polynomial, then

$$\mathrm{tr}(P(T)) = \sum P(\lambda_i).$$

Here we wish to approximate the dimension of the kernel of $g_m$. Since $g_m = \partial_{p,m}^* \partial_{p,m}$, all its eigenvalues are non-negative, but if we want to get a good approximation, we need to make sure the contributions of the non-zero eigenvalues are not too significant. (The non-negativity of the eigenvalues, or even the fact that they are real, is not an important point.) To do this it is crucial to bound independently of $m$ the size of the eigenvalues, which we do via operator norm.

**Lemma 4.1.** *There exists $K > 0$ so that $\|g_m\| < K$ for all $m$.*

*Proof.* This follows from the fact that, independently of $m$, all the $g_m$ can be described by multiplication by the matrix $B_p^* B_p$. You can give an explicit $K$ in terms of the coefficients of this matrix. $\square$

Let $P_{\lambda,\varepsilon}$ to be any polynomial (illustrated on the next page) that is between 1 and $1 + \varepsilon$ on $[0, \lambda]$, is between 0 and $1 + \varepsilon$ on $[\lambda, \lambda + \varepsilon]$, and is between 0 and $\varepsilon$ on $[\lambda + \varepsilon, K]$. (Such polynomials exist by the Stone-Weierstrass Theorem.) This polynomial is intended to be a good approximation for the characteristic function of $[0, \lambda]$, and $\mathrm{tr}(P_{\lambda,\varepsilon}(g_m))$ is intended to approximate the sum of the dimensions of
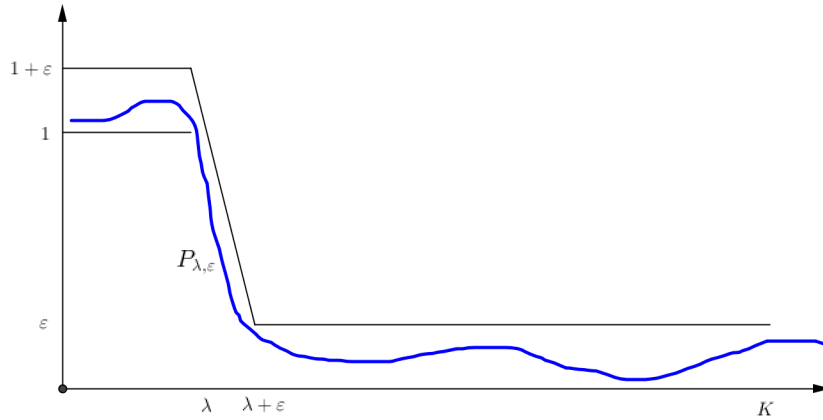
eigenspaces with eigenvalue at most $\lambda$. The point is to pick $\lambda$ small, because ultimately we want only the zero eigenspace, and $\varepsilon$ small, so that the error is small.

The result of applying Lemma 3.1 to $P_{\lambda,\varepsilon}$ is that, for all $m$ large enough, the sum of dimensions of the eigenspaces with eigenvalue less than $\lambda$, divided by $[\Gamma : \Gamma_m]$, is approximately independent of $m$. Making $\varepsilon$ small reduces the error, but we must still account for the difference between eigenvalues less than $\lambda$ and eigenvalues equal to 0. (The number of eigenvalues in $(\lambda, \lambda + \varepsilon)$ is also a source of error above, but this is also handled by the following lemma, which can be applied for $\lambda + \varepsilon$ as well as $\lambda$.)

**Lemma 4.2.** *Let $T$ be any square $d$ by $d$ integer matrix of operator norm at most $K$. Then the sum of the dimensions of the eigenvalues in $(0, \lambda)$, divided by the dimension $d$, goes to zero as $\lambda \to 0$ uniformly in $d$.*

*Proof.* The product of all non-zero eigenvalues is equal to one of the terms of the characteristic polynomial. Since the matrix is integral, this product is at least 1 in absolute value. If there are $r$ eigenvalues in $(0, \lambda)$, we get $\lambda^r K^{d-r} \geq 1$, so $\lambda^r K^d \geq 1$ (we can assume $K > 1$), and

$$r \log \lambda + d \log K \geq 0.$$

Thus

$$r/d \leq \log K / (-\log \lambda).$$

Since $-\log \lambda \to \infty$ as $\lambda \to 0$ this gives the result. $\qquad\square$

Now, if $\lambda$ is small enough, and $\varepsilon$ is small enough, we see that the value of $\operatorname{tr}(P_{\lambda,\varepsilon})/[\Gamma : \Gamma_m]$ is both a good approximation for $\dim \ker(g_m)$ as well as eventually constant in $m$, giving the result.

More precisely,

$$\frac{\mathrm{tr}(P_{\lambda,\varepsilon}(g_m)) - \dim \ker(g_m)}{[\Gamma : \Gamma_m]}$$

is bounded above by a constant times $\varepsilon$, plus $(1 + \varepsilon)$ times the function of $\lambda$ of the previous lemma that goes to zero as $\lambda \to 0$. Since $\mathrm{tr}(P_{\lambda,\varepsilon}(g_m))$ is eventually constant in $m$, this shows that the sequence $(\dim \ker(g_m))/[\Gamma : \Gamma_m]$ is Cauchy, and Luck's Theorem is proved. (Formally: In the definition of Cauchy, you are given some arbitrarily small constant. You pick $\lambda, \varepsilon$ so that the error in the display math above is less than that constant over 2. Then you use $m$ large enough so that $\mathrm{tr}(P_{\lambda,\varepsilon}(g_m))$ does not depend on $m$.)

## 5. The limit

We have that

$$\lim_{m\to\infty} \frac{\mathrm{tr}(P_{\lambda,\varepsilon}(g_m))}{[\Gamma : \Gamma_m]} = \mathrm{tr}_{\mathbb{C}[\Gamma]}(P_{\lambda,\varepsilon}(B_p^* B_p)),$$

where $B_p$ is the matrix for the differential $\partial_p$ for the cochain complex of $\tilde{X}$. Thus

$$\lim_{m\to\infty} \frac{\dim \ker(g_m)}{[\Gamma : \Gamma_m]} = \lim_{\varepsilon\to 0, \lambda\to 0} \mathrm{tr}_{\mathbb{C}[\Gamma]}(P_{\lambda,\varepsilon}(B_p^* B_p)).$$

Instead of considering $B_p^* B_p$ as an operator from $\mathbb{C}[\Gamma]^{n_p}$ to itself, we can consider it as an operator on $\ell^2[\Gamma]^{n_p}$, where as usual $\ell^2[\Gamma]$ denotes the set of $\sum c_\gamma \gamma$ with $\sum |c_\gamma|^2 < \infty$. This is a so-called Hilbert module, which is simply a nice (in a technical sense) Hilbert space with an isometric action $\Gamma$. For bounded $\Gamma$ equivariant operators on this Hilbert module there is a notion of trace, which coincides with $\mathrm{tr}_{\mathbb{C}[\Gamma]}$ for $P_{\lambda,\varepsilon}(B_p^* B_p)$.

This theory is sufficiently well behaved that the traces of $P_{\lambda,\varepsilon}(B_p^* B_p)$ converge to the trace of the projection onto the kernel of $B_p^* B_p$, which is the kernel of $\partial_p$. The trace of this projection is by definition the von Neumann dimension of the kernel. Thus the limit of $b_p(X_m)/[\Gamma : \Gamma_m]$ is the $p$-th $L^2$-Betti number of $X$, which can be defined as the von Neumann dimension of the $p$-th (reduced) $L^2$-cohomology of $X$.

We will not enter into the theory of dimension here, except to say that dimension is always the trace of a projection, and to give the following example. If $\Gamma = \mathbb{Z}$, then the set of bounded $\Gamma$-equivariant operators on $\ell^2(\Gamma)$ is, via Fourier transform, equal to $L^\infty(S^1)$, and the trace is given by integration against Haar measure.