Random surfaces of large genus

Alex Wright

Joint work with Michael Lipnowski

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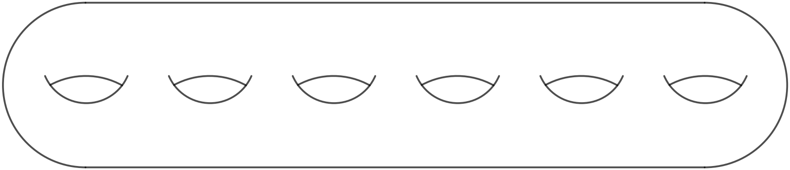
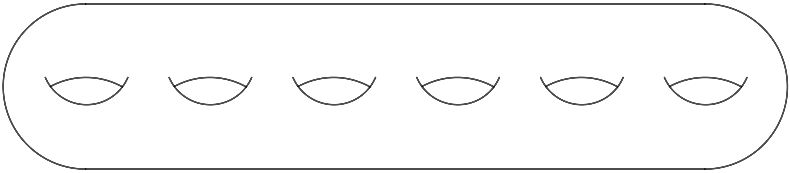


Like this? (Linear diameter, linear number of curves of length at most 100, easy to cut in two, . . .)

What does a “typical” surface of large genus look like?



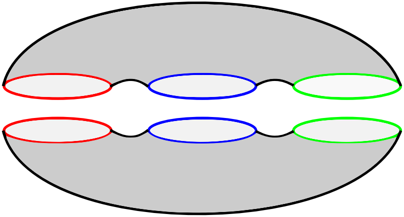
Like this? (Linear diameter, linear number of curves of length at most 100, easy to cut in two, . . .)

No!

We are interested in hyperbolic surfaces.

Given any three positive numbers, can build a unique pants with those cuff lengths.

If, for example, we pick all cuff lengths to be a fixed constant, we can glue together.



Each pants has three cuffs, so, gluing pattern is given by a three regular graph.

Vertices correspond to pants, edges to pairs of cuffs glued together.

Number of edges, vertices, and genus related by:

$$e = 3g - 3, \quad v = 2g - 2, \quad e = \frac{3}{2}v.$$

More elementary question: What does a typical large 3-regular graph look like?

(Or d-regular with $d \geqslant 3$. We take $d = 3$ only for concreteness.)

Theorem (Bollobas & de la Vega '82)

Diameter typically about $\log_2 v$.

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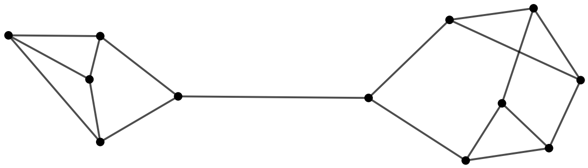
Let N_i denote the number of i-cycles.

Theorem (Bollobas, Wormald '80)

The $N_i, i \geqslant 3$ *are asymptotically independent Poisson random variables with means* $\lambda_i = 2^i/(2i)$.

Deeper questions: How hard is it to cut graph in two? How fast does random walk mix?



Both questions related to eigenvalues.

Define the adjacency matrix A as the v by v matrix with a 1 in position (i, j) if there is an edge from vertex i to vertex j.

Define the Laplacian of a 3-regular graph to be

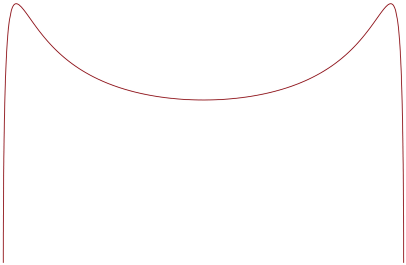
$$I - \frac{1}{3}A.$$

It is closely related to the matrix $\frac{1}{3}A$ governing the random walk on the graph.

Denote the spectrum of the Laplacian by $0 = \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{v-1}$.

Theorem (McKay '81)

The proportion of eigenvalues λ_i in an interval is typically close to the weight assigned to this interval by an explicit distribution supported on $[1-\frac{2\sqrt{2}}{3}, 1+\frac{2\sqrt{2}}{3}]$.



The Cheeger constant can be bounded above and below by λ_1, and the bigger λ_1 is the faster the random walk mixes.

Big λ_1 is desirable, and sequences of graphs with λ_1 bounded from below are called expanders.

Theorem (Friedman ‘02)

Typically $\lambda_1 \geqslant (1 - \frac{2\sqrt{2}}{3}) - \epsilon$.

Proved an ‘86 conjecture of Alon.

All 3-regular graphs with enough vertices have $\lambda_1 \leqslant 1 - \frac{2\sqrt{2}}{3} + \epsilon$ (Alon & Boppana ‘86).

Graphs with $\lambda_1 \geqslant 1 - \frac{2\sqrt{2}}{3}$ are called Ramanujan.

Want analogues of all these results for random surfaces. Instead of "large number of vertices/edges" we consider "large genus".

But the set $\mathcal{M}_g$ of hyperbolic surfaces of genus g is infinite! It is a $6g - 6$ dimensional manifold.

How to define a random surface?

Random graph model: fix $\ell > 0$, and use a random 3-regular graph as a guide to glue together pants all of whose boundaries have length ℓ.

Brooks-Makover model: use a random 3-regular graph as a guide to gluing together ideal triangles, then fill in cusps.

Random cover model: fix a hyperbolic surface, and then take a random cover.

All these models use the uniform measure on a finite set.

So these models see only finitely many points in the $6g-6$ dimensional manifold $\mathcal{M}_g$ of genus g surfaces.

Is there a natural, tractable measure whose support is all of $\mathcal{M}_g$?

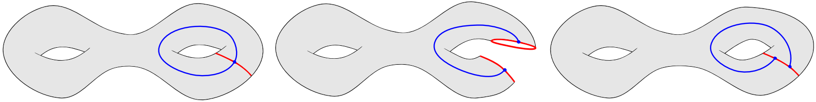
Yes! The Weil-Petersson measure. Its study in the context of random surfaces was pioneered by Mirzakhani.

Based on Fenchel-Nielsen local coordinates for $\mathcal{M}_g$:

Fix $X \in \mathcal{M}_g$. Consider a pants decomposition for X, which you can think of as a collection of $3g - 3$ disjoint simple geodesics; they cut the surface up into $2g - 2$ pants.

The lengths of the $3g - 3$ cuffs give $3g - 3$ local coordinates.

Then there are $3g - 3$ “twist” coordinates that keep track of how the two pants are glued together at each cuff.



In total, $6g - 6$ local coordinates for $\mathcal{M}_g$.

Each point of $\mathcal{M}_g$ is contained in infinitely many local coordinate charts, since each surface has infinitely many pants decompositions.

"Magic" fact (Wolpert): The standard volume form is well defined (does not depend on choice of local F-N coordinate chart.)

Back to main question: What does random surface of large genus look like?

The answers to most questions do not depend too much on the model.

Theorem

In all models, diameter is typically logarithmic in g.

BM: Brooks & Makover '04, Budzinski, Curien & Petri '19.

WP: Mirzakhani '13.

Let $N_{[a,b]}$ denote the number of primitive geodesics of length in $[a, b]$ for a WP random surface of genus g.

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Theorem (Mirzakhani & Petri '17)

If the intervals $[a_1, b_1], \ldots, [a_k, b_k]$ are disjoint, then the $N_{[a_i,b_i]}$ are asymptotically independent Poisson random variables with means

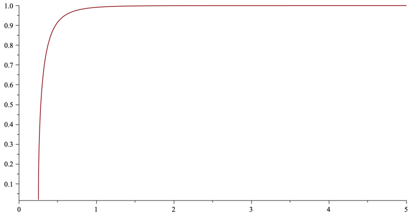
$$\lambda_i = \int_{a_i}^{b_i} \frac{e^t + e^{-t} - 2}{2t} dt.$$

For eigenvalues we use the Laplace-Beltrami operator, which has eigenvalues $0 = \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$.

λ_1 is related to the Cheeger constant, and governs the speed of mixing for Brownian motion and geodesic flow.

Theorem (Monk '20)

The number of eigenvalues λ_i *in an interval, divided by* g*, is WP-typically close to the weight assigned to this interval by an explicit distribution supported on* $[\frac{1}{4}, \infty)$.



Related results going back to Wallach '76.

What about λ_1?

Conjecture

λ_1 *is typically greater than* $\frac{1}{4} - \varepsilon$.

It is known that in high enough genus, no surface can have $\lambda_1 > \frac{1}{4} + \varepsilon$.

Conjecture is open in all models.

Just this month, Hide & Magee shows there is a sequence of surfaces S_n with $\lambda_1(S_n) \to \frac{1}{4}$ and genus going to infinity.

This had been open since ‘84!

Brooks & Makover ‘04 shows BM-typically $\lambda_1 > C$, constant C not explicit.

Mirzakhani ’13 showed WP-typically $\lambda_1 > 0.002$.

Theorem (Lipnowski-W, Wu-Xue)

WP-typically $\lambda_1 > \frac{3}{16} - \varepsilon$.

A $\frac{3}{16}$ result was also proved for random covers by Magee, Naud & Puder '20.

Anantharaman and Monk working on related topics.

How to prove bounds on λ_1?

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For surfaces, use the Selberg trace formula. Relates eigenvalues to primitive closed geodesics.

To prove theorem, end up needing to understand the average number of geodesics of length at most $C \log(g)$ on a surface of genus g.

Problem: We cannot compute the WP-average of this number over $\mathcal{M}_g$.

Mirzakhani's thesis: Can compute the average number of simple, non-separating geodesics of length at most L over $\mathcal{M}_g$.

Simple means no self-intersections.

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As long as $L \ll \sqrt{g}$, the answer is about e^L/L (Mirzakhani-Petri).

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As long as $L \ll \sqrt{g}$, the answer is about e^L/L (Mirzakhani-Petri).

Compare: On a fixed surface, the number of geodesics of length less than L is asymptotic to e^L/L as $L \to \infty$.

Conjecture (Lipnowski-W)

If L $\ll \sqrt{g}$*, then on most surfaces in* $\mathcal{M}_g$*, most geodesics of length at most L are simple and non-separating.*

If L $\gg \sqrt{g}$*, then on most surfaces in* $\mathcal{M}_g$*, most geodesics of length at least L are not simple.*

Compare: On a fixed surface, most geodesics of large enough length are very far from simple!

At moderate length scales, somehow get the expected *number* of geodesics, but they don't have the expected *shape*.

Conjecture (Lipnowski-W)

If $L \ll \sqrt{g}$, *then on most surfaces in* $\mathcal{M}_g$, *most geodesics of length at most* L *are simple and non-separating.*

If $L \gg \sqrt{g}$, *then on most surfaces in* $\mathcal{M}_g$, *most geodesics of length at least* L *are not simple.*

We only need an "averaged" statement.

We only need to work at the $\log(g)$ length scale. Mirzakhani-Petri worked at bounded length scales.

But, the error terms are central in our analysis. Basically want to show the average number of geodesics is e^L/L with very small error; if you get a smaller error, you can replace $\frac{3}{16}$ with something closer to $\frac{1}{4}$.

Recap: To bound λ_1 using the trace formula, need to know the average number of geodesics of length at most $C \log(g)$.

Mirzakhani tells us the average number of simple geodesics of this length.

We suspect most geodesics of this length are simple. How to prove this?

Central idea: Every closed geodesic either fills the whole surface, or fills a subsurface.

A geodesic fills a subsurface if its complement in the subsurface consists of simply connected regions and annular regions about the boundary of the subsurface.

A geodesic of length at most $C \log(g)$ is way too short to fill the whole surface (isoperimetric inequality).

Ideas of Mirzakhani give bounds for the average number of subsurfaces that such a geodesic can fill.

So it suffices to show that most of the relevant subsurfaces don't have too many filling geodesics.

Theorem (Lipnowski-W)

A "tangle-free" subsurface has "very few" closed geodesics.

Analogue of "tangle-free" condition used for graphs. Also studied by Monk & Thomas.

Tangle-free means no pants or one-holed tori with boundary less than some constant.

For typical surfaces in $\mathcal{M}_g$, the constant can be taken to be $\log(g)$ in size.

Final summary:

Theorem (Lipnowski-W, Wu-Xue)

WP-typically $\lambda_1 > \frac{3}{16} - \varepsilon$.

1. The Selberg trace formula relates closed geodesics to spectrum.
2. Need to understand geodesics of length at a scale slowly growing with genus.
3. There are an understandable number of subsurfaces that geodesics of this length can fill.
4. And each such subsurface has few geodesics.
5. So suffices to use Mirzakhani's count of simple geodesics.

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