A BRIEF SUMMARY OF OTAL'S PROOF OF MARKED LENGTH SPECTRUM RIGIDITY

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ABSTRACT. We outline Otal's proof of marked length spectrum rigidity for negatively curved surfaces. We omit all technical details, and refer the interested reader to the original **[Ota90]** or the course notes **[Wil]** for details, and to **[Cro90]** for different approach. (Actually the course notes **[Wil]** combine the approaches in **[Ota90, Cro90]**.)

The author thanks Amie Wilkinson for explaining this proof to him. This informal note was written while the author was Amie Wilkinson's teaching assistant for her course on the same topic at the Park City Math Institute, 2012. The author thanks Jenny Wilson for producing the figures.

Consider two negatively curved closed surfaces S and S'. Fix a homeomorphism from S to S', or alternatively consider S and S' to be two Riemannian structures on the same topological surface. Due to negative curvature, every closed curve is homotopic to a unique closed geodesic, called the *geodesic representative* of the homotopy class. Let $\mathcal{C}$ denote the set of homotopy classes of closed curves. The *marked length spectrum* of S is defined as the function $\ell_S : \mathcal{C} \to \mathbb{R}_{>0}$ which assigns to each homotopy class of curve the length of its geodesic representative.

Theorem 1 (Otal, Annals 1990)**.** *Let S and S' be two negatively curved closed marked surfaces. If S and S' have the same marked length spectrum, they are isometric.*

Step 1: Coarse geometry gives a correspondence of geodesics. Let $\tilde{S}$ and $\tilde{S}'$ denote the universal covers of S and S'. The homeomorphism $\text{Id} : S \to S'$ lifts to a homeomorphism

$$\tilde{\text{Id}} : \tilde{S} \to \tilde{S}',$$

which is in fact a quasi-isometry. Again due to negative curvature, both $\tilde{S}$ and $\tilde{S}'$ have boundaries, which are homeomorphic to a circle. The quasi-isometry $\tilde{\mathrm{Id}} : \tilde{S} \to \tilde{S}'$ induced a homeomorphism on boundaries

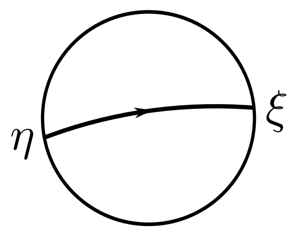
$$\hat{\mathrm{Id}} : \partial\tilde{S} \to \partial\tilde{S}'.$$

The image of any geodesic under a quasi-isometry is a quasi-geodesic, and in negative curvature every quasi-geodesic is a bounded distance from a genuine geodesic. Hence we get a correspondence ϕ between geodesics in $\tilde{S}$ and geodesics in $\tilde{S}'$: given a geodesic γ in $\tilde{S}$, we define $\phi(\gamma)$ to be the unique geodesic which lies within bounded distance from the quasi-geodesic $\tilde{\mathrm{Id}}(\gamma)$.

The space $\mathcal{G}$ of geodesics in $\tilde{S}$ is identified naturally with

$$\partial\tilde{S} \times \partial\tilde{S} \setminus \Delta,$$

where Δ is the diagonal. The identification sends a geodesic to the ordered pair of its forward and backward endpoints and infinity. In



these coordinates, we have that

$$\phi(\xi, \eta) = (\hat{\mathrm{Id}}(\xi), \hat{\mathrm{Id}}(\eta)).$$

In other words, given a geodesic γ in $\tilde{S}$ with endpoints $\xi, \eta \in \partial\tilde{S}$ at infinity, we may map the endpoints to $\hat{\mathrm{Id}}(\xi), \hat{\mathrm{Id}}(\eta) \in \partial\tilde{S}'$ and the geodesic $\phi(\gamma)$ in $\tilde{S}'$ is simply the unique geodesic from $\hat{\mathrm{Id}}(\xi)$ to $\hat{\mathrm{Id}}(\eta)$.

Lemma 2. *The correspondence of geodesics ϕ sends intersecting geodesics to intersecting geodesics.*

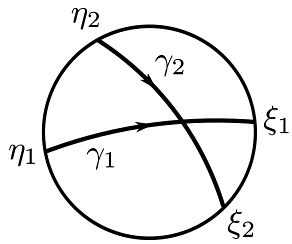
Proof. Suppose we have geodesics γ_1, γ_2 in $\tilde{S}$, which may be described in terms of their endpoints at infinity as

$$\gamma_1 = (\xi_1, \eta_1), \quad \gamma_2 = (\xi_2, \eta_2).$$

Suppose that γ_1 and γ_2 intersect. We will treat the case where $(\xi_1, \xi_2, \eta_1, \eta_2)$ are cyclically ordered on the circle. (There is one other case which is identical: the endpoints at infinity may intertwine in a different order.)

The correspondence

$$\phi : \partial\tilde{S} \times \partial\tilde{S} \setminus \Delta \to \partial\tilde{S}' \times \partial\tilde{S}' \setminus \Delta'$$



is induced by the homeomorphism $\hat{\text{Id}} : \partial\tilde{S} \to \partial\tilde{S}'$. The boundary is homeomorphic to a circle, and any homeomorphism of a circle preserves the cyclic order of quadruples of points. Hence

$$(\hat{\text{Id}}(\xi_1), \hat{\text{Id}}(\xi_2), \hat{\text{Id}}(\eta_1), \hat{\text{Id}}(\eta_2))$$

are cyclically ordered on the circle $\partial\tilde{S}'$. It follows that the geodesics $\phi(\gamma_1)$ and $\phi(\gamma_2)$ intersect. ■

Step 2: Marked length spectrum determines the Liouville current. The *Liouville current* λ of $\tilde{S}$ is a measure on the space $\mathcal{G}$ of geodesics with the following property. If α is a bounded geodesic arc in $\tilde{S}$, then the measure of the set $\mathcal{G}_\alpha$ of geodesics intersecting α is exactly the length of α, that is

$$\lambda(\mathcal{G}_\alpha) = \text{length}(\alpha).$$

Otal proves that marked length spectrum completely determines the Liouville current ("Crofton's Formula"). As a result, we get that ϕ preserves the Liouville current:

Lemma 3. *If $Q \subset \mathcal{G}$ is a set of geodesics, then $\lambda(Q) = \lambda'(\phi(Q))$.*

Step 3: Understanding change in angle. Two intersecting geodesics in $\tilde{S}$ are mapped via the correspondence ϕ to intersecting geodesics in $\tilde{S}'$ (Lemma 2). We will see that negative curvature restricts the change in angle.

To make this precise, let us define a function

$$\theta' : T^1\tilde{S} \times [0, \pi] \to [0, \pi],$$

where $T^1\tilde{S}$ is the unit tangent bundle to $\tilde{S}$. Given $v \in T^1\tilde{S}$, denote by γ_v the geodesic through v, and given $\theta \in [0, 2\pi]$ let θv denote the tangent vector v rotated by θ.

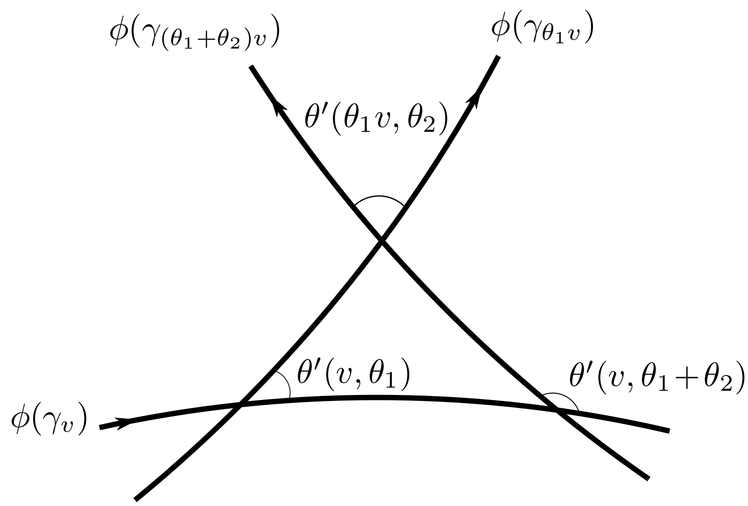
We consider the intersecting geodesics γ_v and $\gamma_{\theta v}$, and the define $\theta'(v, \theta)$ to be the angle between $\phi(\gamma_v)$ and $\phi(\gamma_{\theta v})$. The notation is intuitive because θ is the angle between γ_v and $\gamma_{\theta v}$ in $\tilde{S}$, and θ' is the angle between the corresponding geodesics in $\tilde{S}'$.

Lemma 4. *Let $\theta(v)$ denote the rotation of v by angle θ. Then we have the following superadditivity relation:*

$$\theta'(v, \theta_1 + \theta_2) \geq \theta'(v, \theta_1) + \theta'(\theta_1 v, \theta_2).$$

Equality is only possible if the corresponding geodesics in $\tilde{S}'$ of the three geodesics through $v, \theta_1 v$ and $(\theta_1 + \theta_2)v$ all intersect in a point.

Proof. The proof is by picture. Since the angles in a triangle add up



to strictly less than π in negative curvature, we obtain

$$\theta'(v, \theta_1) + \theta'(\theta_1(v), \theta_2) + (\pi - \theta'(v, \theta_1 + \theta_2)) < \pi,$$

which gives the superadditivity. Equality is only possible in the degenerate case when the triangle above is actually a point. ∎

Step 4: The correspondence ϕ of geodesics sends triples of geodesics intersecting in a single point to triples of geodesics intersecting in a single point. Roughly speaking, we wish to show that we are always in the equality case of the super-additivity relation Lemma 4. To do so, we will have to average θ'.

Indeed, using the Liouville current, we can further constrain the *average* change in angle. We must first note that θ' descends to a well defined function on $T^1S \times [0, \pi]$. The unit tangent bundle T^1S can be equipped with a natural volume measure vol, called the *Liouville measure*. It is with respect to this measure that we average.

We define

$$\Theta'(\theta) = \frac{1}{\mathrm{vol}(T^1S)} \int_{T^1S} \theta'(v,\theta) d \,\mathrm{vol}\,.$$

Lemma 4 implies a corresponding superadditivity relation for Θ'.

Lemma 5.

$$\Theta'(\theta_1 + \theta_2) \geq \Theta'(\theta_1) + \Theta'(\theta_2),$$

with equality if and only if ϕ sends triples of geodesics intersecting in a single point to triples of geodesics intersecting in a single point

What is harder is to show the following.

Proposition 6. *For all continuous convex functions $F : [0,\pi] \to \mathbb{R}$ we have*

$$\int_0^\pi F(\Theta'(\theta)) \sin\theta d\theta \leq \int_0^\pi F(\theta) \sin\theta d\theta.$$

We omit the proof, but make a few remarks:

- The $\sin\theta$ term appears naturally in the expression of the Liouville current on $\mathcal{G}$, in certain natural local coordinates.
- The proof begins with Jensen's inequality.
- The key step uses that the Liouville measure is preserved.
- Using this, Otal computes the average of

  $$\int_0^\pi F(\theta'(v,\theta) \sin\theta d\theta$$

  over ever closed orbit.
- The average of a continuous function on T^1S over all closed geodesics determines its average with respect to the Liouville measure vol.

Otal deduces that Θ' is constant by applying the following elementary result on functions.

Lemma 7. *Let Θ be an increasing homeomorphisms from $[0,\pi]$ to itself satisfying*

(1) *Θ is super-additive and symmetric ($\Theta(\pi-\theta) = \pi - \Theta(\theta)$), and*
(2) *for all continuous convex functions $F : [0,\pi] \to \mathbb{R}$ we have*

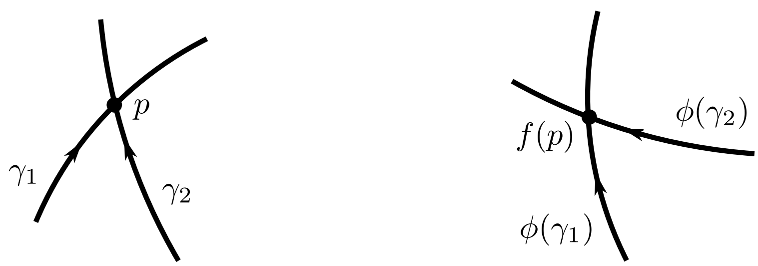
$$\int_0^\pi F(\Theta(\theta)) \sin\theta d\theta \leq \int_0^\pi F(\theta) \sin\theta d\theta.$$

Then Θ is the identity.

Since Θ' is the identity, it follows in particular that equality in Lemma 5 is achieved.

Step 5: Constructing an isometry $\tilde{S} \to \tilde{S}'$. To establish Theorem 1, it suffices to construct an isometry $f : \tilde{S} \to \tilde{S}'$ which is equivariant with respect to deck transformations (the action of the fundamental group).

The map f is defined as follows. Given $p \in \tilde{S}$, we pick any two geodesics γ_1, γ_2 intersecting at p, and set $f(p)$ to be the unique point of intersection of the geodesics $\phi(\gamma_1), \phi(\gamma_2)$.

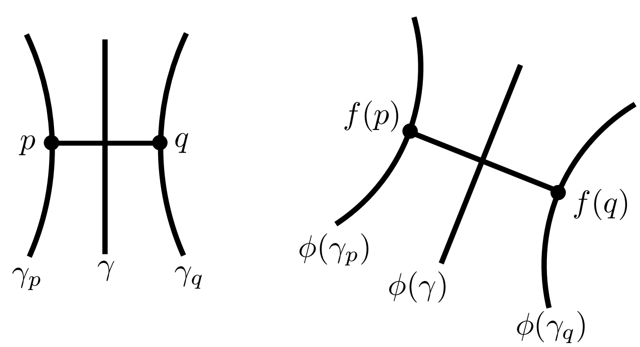


By Step 4, the result does not depend on which two geodesics through p are chosen; f is well defined.

Lemma 8. *Let $p, q \in \tilde{S}$, and let $\alpha = [p, q]$ be the geodesic arc from p to q. Furthermore let $\alpha' = [f(p), f(q)]$ denote the geodesic arc in $\tilde{S}'$ from $f(p)$ to $f(q)$.*

Then if γ is any geodesic in $\tilde{S}$ intersecting α, then the corresponding geodesic $\phi(\gamma)$ in $\tilde{S}'$ intersects α'.

Proof. Let γ_p and γ_q be geodesics through p and q respectively, each not intersecting γ. The geodesic γ is thus "in between" γ_p and γ_q. As



in Lemma 2, since the correspondence ϕ of geodesics is induced by a homeomorphism on boundaries at infinity, $\phi(\gamma)$ lies in between $\phi(\gamma_p)$ and $\phi(\gamma_q)$. However, since $\phi(\gamma_p)$ contains $f(p)$ and $\phi(\gamma_q)$ contains $f(q)$, it follows that $\phi(\gamma)$ must intersect α'. ∎

The proof of the Theorem 1 is completed using the Liouville current again.

Proof that f is an isometry. Let p, q, α, α' be as above. Let $\mathcal{G}_\alpha$ be the set of all geodesics in $\tilde{S}$ intersecting α; similarly $\mathcal{G}_{\alpha'}$ is the set of all geodesics in $\tilde{S}'$ intersecting α'. Lemma 8 gives that

$$\phi(\mathcal{G}_\alpha) = \mathcal{G}_{\alpha'}.$$

We know that the Liouville current measure of $\mathcal{G}_\alpha$ is the length of $\mathcal{G}_\alpha$:

$$\lambda(\mathcal{G}_\alpha) = \text{length}(\alpha),$$

and similarly for $\mathcal{G}_{\alpha'}$.

Now, using that the Liouville current is preserved (Lemma 3), we get

$$\begin{aligned} \text{dist}_{\tilde{S}}(p, q) &= \text{length}(\alpha) \\ &= \lambda(\mathcal{G}_\alpha) \\ &= \lambda'(\phi(\mathcal{G}_\alpha)) \\ &= \lambda'(\mathcal{G}_{\alpha'}) \\ &= \text{length}(\alpha') \\ &= \text{dist}_{\tilde{S}'}(f(p), f(q)). \end{aligned}$$

That is, we have shown that $f : \tilde{S} \to \tilde{S}'$ preserves distances. The isometry f descends to an isometry $S \to S'$, completing the proof. ∎

References

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[Ota90] Jean-Pierre Otal, *Le spectre marqué des longueurs des surfaces à courbure négative*, Ann. of Math. (2) **131** (1990), no. 1, 151–162.

[Wil] Amie Wilkinson, *Pcmi lecture notes on marked length spectrum rigidity.*

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