PLANE CUBICS

ALEX WRIGHT

These are my personal notes on some of the classical background material for my paper with Mukamel and McMullen on cubic curves and totally geodesic subvarieties of moduli space.

Linear series and projective embeddings. Let A be a Riemann surface of genus 1. By Riemann-Roch, for any divisor D of degree $d \geq 1$ we have $h^0(D) = \dim H^0(\mathcal{O}_D) = d$, where $H^0(\mathcal{O}_D)$ is the dimension of the space of global sections of the sheaf $\mathcal{O}_D$ of meromorphic functions h on A with $(h) + D \geq 0$. If the degree d is at least 3, the linear system gives an embedding into $\mathbb{P}^{d-1}$. The image of this embedding has degree d. We will be especially interested in the case $d = 3$, that is, in realizations of A as cubic plane curves.

Suppose $A = Z(f) \subset \mathbb{P}^2$ is a cubic plane curve. Here f is a homogeneous polynomial of degree 3 in three variables. By Riemann-Roch, we immediately see that this linear series is complete. That is, if D is the intersection of a line in $\mathbb{P}^2$ with A, then $h^0(D) = 3$. So every embedding of A into $\mathbb{P}^2$ comes from a degree 3 divisor. Since two divisors that are linearly equivalent give projectively equivalent embeddings into projective space, and since A has a unique divisor of degree 3 up to automorphisms of A and linear equivalence, we see that the embedding of A into $\mathbb{P}^2$ is unique up to pre-composition with automorphisms of A and post-composition by projective maps of $\mathbb{P}^2$. (Given two divisors D, D' on A of the same degree, one can replace one with a translate, i.e. its image under an automorphism, so that $D - D' = 0 \in \mathrm{Jac}(X)$.)

It is obvious for the usual reason that if D' is the intersection of a different line with A, then D and D' are linearly equivalent. (If the first line is defined by a linear function ϕ, and the second line is defined by a linear function ϕ', then ϕ/ϕ' is a meromorphic function on A with divisor $D - D'$.) Completeness means exactly that if D' is an effective divisor and D' is linearly equivalent to the intersection D of A with a line, then D' is also the intersection of A with a line.

The linear series of quadrics on a cubic $A \subset \mathbb{P}^2$ is complete. Recall that $\mathbb{P}^2$ has a line bundle $\mathcal{O}(k)$, for which the space of global sections has dimension $\binom{k+2}{2}$ and consists of the homogeneous polynomials of degree k. For later use, we record the following.

Lemma 1. *Let D denote the intersection of A with a quadric, so D has degree 6.*

The restriction map from global sections of $\mathcal{O}(2)$ to the global sections of $\mathcal{O}(D)$ on A is an isomorphism.

Proof. The space of global sections of $\mathcal{O}(2)$ has dimension 6. Since A doesn't lie on a conic, the restriction map is injective. By Riemann-Roch, $h^0(D) = 6$, so the restriction map must be surjective. □

Points on $\mathbb{P}^2$. As a point of reference, we mention that 2 points determine a line, 5 points determine a conic, and 9 points determine a cubic. In general, $(d+2)(d+1)/2 - 1$ points determine a curve of degree d, which is expected since this is the dimension of the space of homogeneous polynomials of degree d up to scaling.

Degree three maps $A \to \mathbb{P}^1$. Suppose $\pi : A \to \mathbb{P}^1$ is degree 3. We may choose an embedding of A in $\mathbb{P}^2$ so that any given fiber of π is the intersection of A with a line. (Precomposing a given map $A \to \mathbb{P}^1$ with an automorphism of A changes which triples of points lie on a line.) Since all fibers of π are linearly equivalent, this means that all fibers of π will be intersections of A with a line. Take two such lines arising from fibers, and let S be their intersection. Let $\pi_S : A \to \mathbb{P}^1$ be the projection from S.

By definition, π and π_S are equal on two different fibers. Any two maps with this property must be equal up to projective transformation, since two meromorphic functions with the same zeros and poles must be multiples of each other. Hence every map $A \to \mathbb{P}^1$ arrises from a map π_S for some embedding $A \subset \mathbb{P}^2$.

Note, π_S is degree 3 with $S \notin A$ but degree 2 when $S \in A$. The codomain $\mathbb{P}^1$ can be considered as the space of lines through S, and if $S \in A$ then $\pi_S(S)$ is equal to the tangent line to A at S.

Polars. Let f be a degree d homogeneous polynomial in $n+1$ variables, and let $Z(f) \subset \mathbb{P}^n$ denote its zero set. Fix $s = (s_0, \ldots, s_n) \in \mathbb{C}^{n+1}$ and consider the corresponding point $S = [s] \in \mathbb{P}^n$.

Let consider the projection from $\mathbb{C}^{n+1}$ to the perp space of s. This projection is linear and hence degree 1 homogenous, and hence induces a projection π_S from $\mathbb{P}^n \setminus \{S\}$ to the projective space on the perp space of s. The range of π_S can be viewed intrinsically as the set of all lines through S.

The kernel of the projection from $\mathbb{C}^{n+1}$ to the perp space of s is spanned by s. The tangent space to $Z(f)$ at $[x]$ is given by the perp space to $\nabla f(x)$. Hence, $Z(\langle \nabla f(x), s\rangle)$ gives the set of critical points of π_S restricted to $Z(f)$. We define $Z(\langle \nabla f(x), s\rangle)$ to be the polar of A, and denote it $\mathrm{Pol}(A, S)$.

By definition, $\mathrm{Pol}(A, S)$ intersects A in the critical points of π_S. The polar of f has degree 1 less than that of f, the number of critical points of π_S is $d(d-1)$. For plane cubics, the six critical points of π_S lie on the conic $\mathrm{Pol}(A, S)$.

Dual curve. For a plane algebraic curve, the dual curve is the closure of the set of tangent lines (at smooth points) to the curve. The dual curve is naturally a plane algebraic curve in the dual projective space. If the curve has degree d, the dual curve has degree $d(d-1)$. Indeed, the degree of the dual curve is the number of tangent lines passing through a generic point S, and that is exactly the number of intersections of $\mathrm{Pol}(A, S)$ with A. Typically, dual curves of smooth curves have singularities.

The if C is a plane algebraic curve, the double dual of C is equal to C. This is intuitive. Indeed, the tangent line L to C at a point p is the line that goes through p and comes as close as possible to all the nearby points of C. The tangent to the dual curve at L is given by a point which is contained in L and comes as close as possible being contained in all the nearby tangent lines to C. This obviously should be p.

Satellite. Let $A = Z(f) \subset \mathbb{P}^2$ be a plane cubic. The Polar of A with respect to $S \subset \mathbb{P}^2$ is a conic which contains the 6 critical points of π_S. We now claim that there is a conic that contains the 6 co-critical points of π_S.

Say the critical points are C_i, and the co-critical points are D_i. We have $2C_i + D_i$ is given by the zeros of a section of $\mathcal{O}(1)$ intersected with A, since these points lie on a line (through S). We have that $\sum C_i$ is similarly given by the section of $\mathcal{O}(2)$ given by the quadric defining $\mathrm{Pol}(A, S)$.

Now, $\sum D_i = \sum(2C_i + D_i) - 2\sum C_i$ is given by a section of $\mathcal{O}(1)^{\otimes 6} \otimes \mathcal{O}(2)^{\otimes(-2)} = \mathcal{O}(2)$ restricted to A. Since $\mathcal{O}(2)$ on $\mathbb{P}^2$ restricted to A is a complete linear series, this means that there is a quadratic polynomial whose intersection with A is $\sum D_i$. The zero set of this quadratic polynomial will be denoted $\mathrm{Sat}(A, S)$.

The symmetry property of polars. Let D_a denote directional derivative in the direction a. Let f be a homogeneous polynomial of degree d. Then we have

$$k!((D_a)^k f)(b) = (d-k)!((D_b)^{d-k} f)(a).$$

We will give a proof when a and b are not collinear. Let $X_1, \ldots,$ be a basis of degree 1 homogeneous polynomials such that

$$X_i(a) = 1 \text{ if } i = 1 \text{ and } 0 \text{ otherwise}$$

$$X_i(b) = 1 \text{ if } i = 2 \text{ and } 0 \text{ otherwise}$$

Write $f = \sum X_1^i X_2^j g_{i,j}$, where $g_{i,j}$ is a polynomial in the X_i with $i > 2$. Then

$$(D_a^k f)(b) = k! g_{k,d-k}$$

and

$$(D_b^{d-k} f)(a) = (d-k)! g_{k,d-k}.$$

The Hessian. Every conic is defined by a quadratic form. For cubics, the symmetry property of polars gives

$$b \in \mathrm{Pol}(Z(f), a) = Z(D_a f) \iff a \in Z((D_b)^2 f),$$

which by definition means that the quadratic form given by the Hessian matrix $\mathrm{Hess}(f)$ evaluated at a is zero at b. Hence, the quadratic form defining $\mathrm{Pol}(Z(f), a)$ is $\mathrm{Hess}(f)$ evaluated at a.

A conic is singular if and only if the quadratic form is singular, in which case the conic is a cone. In particular $\mathrm{Pol}(Z(f), a)$ is singular if and only if $a \in Z(\det(\mathrm{Hess}(f)))$, in which case it is a union of two lines. When $A = Z(f)$, we will will refer to $Z(\det(\mathrm{Hess}(f)))$ as HA.

If $\mathrm{Pol}(A, p)$ is singular, then certainly π_p has three collinear critical points. (In fact it has two pairs of three collinear critical points.) Conversely, suppose that π_p has three collinear critical points. Then $\mathrm{Pol}(A, p)$ intersects a line in three points. Since a quadratic intersects a line in only two points, this means that $\mathrm{Pol}(A, p)$ is singular. Hence we see that $Z(\mathrm{Hess}(f))$ is exactly the locus where there are three collinear critical points.

The Hessian HA is of course cubic.

Flexes. We now claim that $Z(\mathrm{Hess}(f)) \cap A$ gives the nine flexes of A. Indeed, saying that $S \in A$ is a flex is the same as saying that π_S has a point of total ramification at S. So if L is the tangent line, then $A \cap L$ should be S with multiplicity three. The polar must also intersect A at S with multiplicity two (since total ramification counts for two simple ramifications). So therefor L and $\mathrm{Pol}(A, S)$ must be tangent to multiplicity two. But a quadratic and a line can't be tangent to order two. So $L \subset \mathrm{Pol}(A,S)$ and hence $\mathrm{Pol}(A,S)$ is singular. Conversely if $\mathrm{Pol}(A,S) = L \cup L'$ is singular and S in $L \cap A$, we have that $L \cap A = 3S$. This is because the degree 2 map π_S can't have two points of branching over one point. Hence S is a flex.

Cayleyan. The Cayleyan CA is the locus of lines contained in singular polars. Either line L or L' contained in $\mathrm{Pol}(A,S)$ determines S, since S is the intersection of the tangent lines to A at the three points $A \cap L$. So there is a two-to-one covering map from CA to HA.

Consider the variety in $\mathbb{P}^2$ times the dual $\mathbb{P}^2$ given by pairs $S \in HA$ and $L \subset \mathrm{Pol}(A,S)$. We have just proven that the projection onto the first factor is two-to-one.

We claim CA is a degree 3 curve in the dual projective space. Indeed, it suffices to show that any point q is contained in exactly three singular polars (with multiplicity). We will chose to do this computation for some $q \in A$. We then wish to find $S \in HA$ so $q \in \mathrm{Pol}(A,S)$. These S are nothing other than the intersections of the tangent line to A at q with HA. Hence the claim that CA is degree three follows from the fact that HA is degree three.

Lattès maps. For a point x on A, let x' denote the other intersection of the tangent line to A at x with A. Let L be a line, and let a, b, c be the intersections of A with L. We claim that a', b', c' also lie on a line. Indeed, $a + b + c$ is given by a section of $\mathcal{O}(1)$ of $\mathbb{P}^2$, and $2a + a'$ by a section of $\mathcal{O}(1)$. So

$$a' + b' + c' = (2a + a') + (2b + b') + (2c + c') - 2(a + b + c)$$

is given by a section of $\mathcal{O}(1)^{\otimes 3} \otimes \mathcal{O}(1)^{\otimes(-2)} = \mathcal{O}(1)$.

The map from the line L to the line L' spanned by a', b', c' is called a Lattès map $\delta = \delta_A$. It is a map of the dual $\mathbb{P}^2$.

Fix an origin on A that is a flex point, so A becomes an elliptic curve. Consider the endomorphism of $A_0 = \{(a_1, a_2, a_3) \in A^3 : \sum a_i = 0\}$ given by

$$(a_1, a_2, a_3) \mapsto (-2a_1, -2a_2, -2a_3).$$

Note that A_0/S_3 is the space of lines in $\mathbb{P}^2$, i.e. it is the dual $\mathbb{P}^2$. The given endomorphism covers δ, and so we conclude that δ has topological degree 16. (Note the times 2 map on A has topological degree 4, and that $A_0 \simeq A^2$.)

Satellite Cayleyan. If $S \in HA$ then the critical points lie on two lines. Hence by the above the co-critical points also lie on two lines. That is, $\mathrm{Pol}(A,S)$ singular implies $\mathrm{Sat}(A,S)$ singular.

The converse is true if S is not in A. Indeed, if three co-critical points a', b', c' are on a line, then $a' + b' + c'$ in $\mathcal{O}(1)$ and $2a + a'$ in $\mathcal{O}(1)$

so $2(a+b+c)$ is in $\mathcal{O}(2)$. So there is a quadratic tangent to A at a, b, c. However, all these tangent lines go through a single point, namely S. For a smooth conic, the dual curve is degree 2, so at most two tangent lines go through a given point. Hence the conic is a double line, and we get that a, b, c lie on a line.

Let SA denote the set of lines contained in singular $\mathrm{Sat}(A, S)$ for some $S \in HA$. Note $SA = \delta(CA)$.

Note that if $S \in HA$ is not on a flex of A, and if $L' \subset \mathrm{Sat}(A, S)$, then L' intersects A transversely. Indeed, if L' is tangent to A, then $\mathrm{Sat}(A, S) \cap A$ has at most 5 distinct points, so $\mathrm{Pol}(A, S) \cap A$ has at most 5 distinct points, because the map from critical points to co-critical points is injective. So π_S has a point of total ramification, so S is on a flex. (A similar argument shows that if $S \in A$ then $\mathrm{Sat}(A, S)$ is singular.)

In particular a generic $L' \in SA$ intersects A transversely.

Does $L' \in SA$ determine S? Consider the extent to which a line $L \in SA$ determines S with $L' \subset \mathrm{Sat}(A, S)$. First note that if additionally $L' \subset \mathrm{Sat}(A, T)$, and $L_S \subset \mathrm{Pol}(A, S)$ and $L_T \subset \mathrm{Pol}(A, T)$ are the corresponding lines in the polars, then we can write

$$L_S \cap A = \{a_S, b_S, c_S\} \quad \text{and} \quad L_T \cap A = \{a_T, b_T, c_T\}$$

so that $(a_S)' = (a_T)', (b_S)' = (b_T)', (c_S)' = (c_T)'$ are the three points of $L' \cap A$. Hence $a_S - a_T, b_S - b_T, c_S - c_T$ are the three two-torsion points of A. In particular, since L_T determines T, this shows that there are always at most 7 choices of S for any $L \in SA$. (These corresponding to picking an ordering of the three non-trivial two torsion points, and adding them to a_S, b_S, c_S. This produces at most 6 new a_T, b_T, c_T.)

It is apparently also true that a line $L \in SA$ generically determines S. I wish I new a soft proof of this. One way to do it, apparently, is to compute some example of HA and see that it is degree 12. The relation between the degree of HA and whether $L \in SA$ determines S follows.

SA is degree 12. This follows from the facts that $SA = \delta(CA)$ has degree 12, CA has degree 3, δ has topological degree 16, and the general fact that if X subset $\mathbb{P}^n$, then

$$\deg(f(X)) = \deg(f)^{\dim(X)/n} \deg(X)/\deg(f|X).$$

The fact that $L \in SA$ generically determines S gives that $\delta|CA$ has degree 1.

Note, that since CA has genus 1 (it is a smooth cubic), and since $\delta|CA$ has degree 1, we get that $SA = \delta(CA)$ has genus 1 also.

Formula for satellite quadric. We now want to show that $\mathrm{Sat}(A, S)$ is defined by

$$\langle x, \nabla f(s)\rangle^2 - 4f(s)\langle s, \nabla f(x)\rangle.$$

Consider the one variable monic polynomial $g(t) = (t-a)^2(t-b)$, and compute

$$\begin{aligned}
g'(t) &= 2(t-a)(t-b) + (t-a)^2 \\
g''(t) &= 2(t-a) + 2(t-b) + 2(t-a) = 4(t-a) + 2(t-b) \\
g''(t)^2 &= 16(t-a)^2 + 16(t-a)(t-b) + 4(t-b)^2 \\
g''(t)^2 - 16g'(t) &= -16(t-a)(t-b) + 4(t-b)^2.
\end{aligned}$$

Conclude that b is a root of $g''(t)^2 - 16g'(t)$.

Note that the leading order term of $t \mapsto f(x_0 + ts)$ is

$$\lim_{t\to\infty} \frac{f(x_0+ts)}{t^3} = \lim_{t\to\infty} f((x_0+ts)/t) = f(s),$$

using that f is homogeneous of degree 3.

Now, consider the monic cubic polynomial $g(t) = f(x_0+ts)/f(s)$. If $x_0 \in Z(f)$ is a critical point of π_S, that means the line $x_0 + ts$ is tangent to $Z(f)$ at x_0, and hence $g(t)$ has a double root at $t = 0$. The third root therefor occurs at a t such that $g''(t)^2 - 16g'(t) = 0$. We have

$$g'(t) = \langle \nabla f(x_0+ts)), s\rangle / f(s), \quad g''(t) = \langle (\Delta f(x_0+ts))s, s\rangle / f(s)$$

so the co-critical points of π_S occur in

$$\langle (\Delta f(x))s, s\rangle^2 / f(s)^2 - 16\langle \nabla f(x), s\rangle / f(s).$$

Now we use the fact that

$$\langle (\Delta f(x))s, s\rangle = D_s^2(f)(x) = 2D_x(f)(s) = 2\langle \nabla f(s), x\rangle.$$