SMILLIE'S THEOREM ON CLOSED $SL_2(\mathbb{R})$ ORBITS OF QUADRATIC DIFFERENTIALS

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The purpose of this note is to give a proof of Smillie's theorem. Originally proven but not published by John Smillie, it was first announced by Veech in [Vee95], where a short proof outline was given. The proof used Ratner's theorem, and the key fact that the orbit of certain circle measures under the godesic flow is compact in the weak-* topology. Because this says that certain weak-* limits of probability measures have mass 1, instead of having mass less than 1, this property is called *no loss of mass*. It is a dynamical property of the $SL_n(\mathbb{R})$ action on the space of quadratic differentials. Veech claims that this follows from the techniques of Kerckhoff, Masur and Smillie's paper [KMS86], but it can be more directly seen from [EM01].

We will not give that proof, but rather one suggested by Smillie and Weiss in [SW04] that avoids the use of Ratner's theorem and uses the *quantitative recurrence of horocycle flow* of Minsky and Weiss [MW02].

Theorem 1. *Let Ω be the space of quadratic differentials on a closed surface of fixed positive genus. Say $(X,\omega) \in \Omega$ and $SL_2(\mathbb{R}) \cdot (X,\Omega)$ is closed. Then the stabilizer $SL(X,\Omega)$ is a lattice in $SL_2(\mathbb{R})$.*

Here is an outline of our proof:

1. Show that the orbit $SL_2(\mathbb{R}) \cdot (X,\Omega)$ is an embedded copy of $SL_2(\mathbb{R})/SL(X,\omega)$; So we think of the orbit as (the unit tangent bundle to) a hyperbolic orbifold.
2. By [MW02], the horocycle flow on Ω is quantitatively recurrent.
3. A Mautner type computation will give that whenever the horocycle flow on a hyperbolic orbifold is quantitatively recurrent, the orbifold has finite hyperbolic volume.
4. Hence $SL_2(\mathbb{R})/SL(X,\omega)$ has finite volume, so $SL(X,\omega)$ is a lattice.

Any dynamical property of the $SL_2(\mathbb{R})$ action on Ω that holds at every point of Ω but does not hold for an infinite volume hyperbolic orbifold could similarly be used to prove Smillie's theorem.

If $SL(X,\Omega)$ were finitely generated (which is known *not* to be the case) our job would be much easier. If $SL(X,\Omega)$ is not a lattice but is finitely generated, then $SL_2(\mathbb{R})/SL(X,\omega)$ has a flare, and hence has very few nice dynamical properties. For example, on such a surface, it is not true that geodesic flow in almost every direction is recurrent. However, it is shown in [KMS86] that geodesic flow in almost every direction starting at any point of Ω is recurrent.

We now state quantitative recurrence, and then proceed to the details of our proof.

A flow h_t on a space Y is said to be *quantitatively recurrent* if there is an exhaustion of Y by compact sets K_n, so that for each K_n there is another compact set K_n' and a $\delta_n > 0$ so that

$$\liminf_{T\to\infty} \frac{1}{T} m(\{t \in [0,T] : h_t(y) \subset K_n'\}) > \delta_n F,$$

for all $y \in K_n$. (Note m is just Lebesgue measure on $\mathbb{R}$.) By [MW02], positive horocycle flow h_t on Ω is quantitatively recurrent. In fact [MW02] prove a stronger version of quantitative recurrence than this, but this is all we need.

Lemma 2. *Suppose G is a locally compact, second countable Hausdorff group, and Y is a locally compact Hausdorff space. If G acts transitively on Y, and $y \in Y$, then the natural map $f : G/G_y \to Y$ is a homeomorphism.*

The following proof is adapted from [AM07].

Proof. Let U be a compact neighborhood of $e \in G$, and W be a smaller compact neighborhood with $W \cdot W^{-1} \subset U$. Since G is second countable, we can pick a countable subcover of $\{g \cdot W : g \in G\}$, say $\{g_1 \cdot W, g_2 \cdot W, \ldots\}$. Each of $f(g_i \cdot W)$ is compact, and $\cup_{i=1}^{\infty} f(g_i \cdot W) = Y$, so there is some i so that $f(g_i \cdot W) = g_i \cdot f(W)$ contains an open set. Hence $f(W)$ contains an open set, since g_i acts as a homeomorphism.

Pick $h \in W$ so that $f(h)$ is in the interior of $f(W)$. Then $h^{-1}f(W) \subset f(U)$ contains an neighborhood of y. This gives that whenever U is a neighborhood of e, $f(U)$ contains a neighborhood of y. Now if U is a neighborhood of $g \in G$, then $f(g^{-1}U)$ contains a neighborhood of y, so $f(U)$ contains a neighborhood of gy. Hence the map f is open. ☐

Lemma 3. *Let $\mathcal{H}$ be a strongly continuous unitary representation of a topological group G. Fix $v \in \mathcal{H}$. If $h_n, h_n' \in G$ fix v for all $n > 0$, and there are $g_n \in G$ so that $h_n g_n h_n' \to 1$, $g_n \to g$, then g also fixes v.*

The following proof of this variant of the Mautner Lemma is taken from [Mar91].

Proof. Note g fixes v if and only if $\langle gv, v\rangle = \langle v, v\rangle$, since g acts unitarily. Now

$$\langle g_n v, v\rangle \to \langle gv, v\rangle$$

and

$$\langle g_n v, v\rangle = \langle h_n g_n h'_n v, v\rangle \to \langle v, v\rangle,$$

so the result follows. □

Lemma 4. *Given a unitary representation of $SL_2(\mathbb{R})$, any vector fixed v by positive horocycle flow is fixed by all of $SL_2(\mathbb{R})$.*

Proof. Compute

$$\begin{pmatrix} 1 & c^{-1}(1-a) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & c^{-1}(1-a^{-1}) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}.$$

Letting $c \to 0$ we get that $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ fixes v.

Now, apply the previous claim again, using the fact that geodesic flow contracts horocycle flow, to get that negative horocycle flow fixes v. Explicitly, we compute

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a^{-2}c & 1 \end{pmatrix}$$

and we let $a \to \infty$.

It remains only to verify that $SL_2(\mathbb{R})$ is generated by positive and negative horocycle flows along with geodesic flow. Note

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} = \begin{pmatrix} 1+cd & c \\ d & 1 \end{pmatrix}.$$

Every matrix in $SL_2(\mathbb{R})$ with a 1 in the lower right entry is of this form. Every matrix in $SL_2(\mathbb{R})$ with non-zero lower right entry can be brought into this form by multiplying by geodesic flow. And of course any non-zero matrix in $SL_2(\mathbb{R})$ can be made to have non-zero lower right entry by multiplying by positive or negative horocycle flow. □

Lemma 5. *If a space Y admits a $SL_2(\mathbb{R})$ action with a locally finite invariant $SL_2(\mathbb{R})$ measure μ, and satisfies quantitative horocycle recurrence, then μ is in fact a finite measure.*

Proof. Pick a compact set $K \subset \Omega$ of positive measure, so that there is a compact set K' and a $\delta > 0$ so that

$$\liminf_{T\to\infty} \frac{1}{T} m(\{t \in [0,T] : h_t(y) \subset K'\}) > \delta$$

for all $y \in K$.

Let $\chi_{K'}$ be the characteristic function of K'. Define

$$S = \left\{ y \in Y : \liminf_{T\to\infty} \frac{1}{T} \int_0^T \chi_{K'}(h_t(y))dt \geq \delta \right\}.$$

Note $K \subset S$ so S has positive measure. But by the Birkhoff ergodic theorem, the function

$$y \mapsto \lim_{T\to\infty} \frac{1}{T} \int_0^T \chi_{K'}(h_t(y))dt$$

exists almost everywhere and has integral $\mu(K)$. So we see that $\mu(S)\cdot\delta \leq \mu(K') < \infty$, so S has finite measure. Of course, S is horocycle flow invariant.

Now, χ_S is an invariant vector of the unitary representation of $SL_2(\mathbb{R})$ on $L^2(\mu)$. So, by the previous lemma, χ_S is in fact $SL_2(\mathbb{R})$–invariant. Hence S is all of Y, and μ is in fact finite. □

Now, we can easily piece together the proof of Smillie's theorem.

Proof of Smillie's Theorem. Since $SL_2(\mathbb{R})\cdot(X,\omega)$ is closed, the action of $SL_2(\mathbb{R})$ on $SL_2(\mathbb{R})\cdot(X,\omega)$ satisfies the conditions of Lemma 2, we get that it is an embedded copy of $SL_2(\mathbb{R})/SL(X,\omega)$. (If the orbit were not closed, the transitive action on the orbit would be an action on a space which is not locally compact.) In particular, it is the unit tangent bundle to a hyperbolic orbifold, and admits a locally finite $SL_2(\mathbb{R})$–invariant measure. This hyperbolic orbifold has quantitative recurrence of horocycle flow, and so must be finite volume. □

References

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