

**COURSE NOTES FOR MATH 636:
INTRODUCTION TO SYMPLECTIC TOPOLOGY**

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Introduction: These are notes on a graduate topics course at the University of Michigan in Winter 2026. Corrections are welcome and may be sent via email to Alex Wright.

This course covered roughly chapters 1, 2, 3, 4, 6, 7, 9, 10 from da Silva's book "Lectures on Symplectic Geometry" [CdS01] and roughly chapters 1, 2, 3, 4, 6, 7, 8 from Evans' book "Lectures on Lagrangian Torus Fibrations", with interludes to at least mention some key results covered in [MS17] and to discuss Legendrian knots.

Authorship: For each lecture, one course participant was designated as the author, and another as the editor. The notes for each lecture are labeled with the initials of the author followed by the initials of the editor. In addition to the listed contributed, Ritwick Bhargava was the editor for one lecture.

Warning: These notes may contain errors and are not a substitute for more reliable sources.

1. SYMPLECTIC FORMS (01/09, YL, MN)

This lecture covers material from [CdS01, Chapter 1].

1.1. Symplectic Linear Algebra. Unless mentioned otherwise, we work over the field $\mathbb{R}$, and all vector spaces are assumed to be finite-dimensional.

Definition 1.1. A bilinear map $\Omega: V \times V \rightarrow \mathbb{R}$ is called **skew-symmetric** if

$$\Omega(u, v) = -\Omega(v, u) \quad \text{for all } u, v \in V.$$

Theorem 1.2. *For any skew-symmetric bilinear map Ω , there exists a basis of V in which*

$$\Omega(u, v) = \begin{pmatrix} - & u & - \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I_m \\ 0 & -I_m & 0 \end{pmatrix} \begin{pmatrix} | \\ v \\ | \end{pmatrix},$$

where I_m is the $m \times m$ identity matrix for some $m \in \mathbb{N}$.

Proof. If $\Omega = 0$, the result is trivial with $m = 0$. Otherwise, choose $e, f \in V$ such that $\Omega(e, f) = 1$. This is always possible: since $\Omega \neq 0$, there exist $e', f' \in V$ with $\Omega(e', f') \neq 0$, and we may take

$$e = e', \quad f = \frac{f'}{\Omega(e', f')}.$$

Define the subspace

$$W := \{w \in V \mid \Omega(e, w) = 0 = \Omega(f, w)\} = \ker(\Omega(e, \cdot)) \cap \ker(\Omega(f, \cdot)).$$

We claim that

$$V = \langle e, f \rangle \oplus W \quad \text{and} \quad \Omega = \begin{pmatrix} 0 & 1 & \mathbf{0} \\ -1 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Omega|_W \end{pmatrix}.$$

Thus $V = \langle e, f \rangle + W$. If $ae + bf \in W$, then

$$0 = \Omega(e, ae + bf) = b, \quad 0 = \Omega(f, ae + bf) = -a,$$

so $a = b = 0$. Hence $\langle e, f \rangle \cap W = \{0\}$, and the decomposition is direct.

The result now follows by induction on $\dim W$. □

Definition 1.3. For any bilinear map $\Omega: V \times V \rightarrow \mathbb{R}$, define

$$\tilde{\Omega}: V \rightarrow V^*, \quad \tilde{\Omega}(v) := \Omega(v, \cdot).$$

Note that

$$\ker(\tilde{\Omega}) = \{v \in V \mid \Omega(v, u) = 0 \text{ for all } u \in V\}$$

may be nontrivial.

Definition 1.4. A skew-symmetric bilinear map Ω is called **symplectic** (or **non-degenerate**) if $\tilde{\Omega}$ is a bijection. In this case, Ω is called a **linear symplectic structure** on V , and (V, Ω) is called a **symplectic vector space**.

If Ω is symplectic, then:

- $\dim V = 2n$ for some $n \in \mathbb{N}$;
- there exists a basis $e_1, \dots, e_n, f_1, \dots, f_n$ such that

$$\Omega(e_i, f_j) = \delta_{ij}, \quad \Omega(e_i, e_j) = 0 = \Omega(f_i, f_j),$$

for all $i, j \in \{1, \dots, n\}$.

In such a basis,

$$\Omega(u, v) = \left(\begin{array}{c|c} - & u & - \end{array} \right) \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} | \\ v \\ | \end{pmatrix}.$$

Such a basis is called a **symplectic basis**.

Definition 1.5. Let (V, Ω) be a symplectic vector space. A subspace $W \subset V$ is

- **symplectic** if $\Omega|_W$ is non-degenerate;
- **isotropic** if $\Omega|_W = 0$.

Example 1.6. If $e_1, \dots, e_n, f_1, \dots, f_n$ is a symplectic basis of (V, Ω) , then $\langle e_1, \dots, e_n \rangle$ is isotropic, while $\langle e_1, f_1 \rangle$ is symplectic.

Remark 1.7. A generic even subspace $W \subset V$ is symplectic, because being symplectic is a non-vanishing condition on $\det \Omega|_W$. On the other hand it is very special for a subspace to be isotropic.

Definition 1.8. The **standard symplectic vector space** is

$$(\mathbb{R}^{2n}, \Omega_{\text{std}}), \quad \Omega_{\text{std}} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Definition 1.9. A **symplectomorphism** between symplectic vector spaces (V, Ω) and (V', Ω') is a linear isomorphism $l: V \rightarrow V'$ such that

$$l^* \Omega' = \Omega, \quad \text{i.e., } \Omega'(l(u), l(v)) = \Omega(u, v) \text{ for all } u, v \in V.$$

Every symplectic vector space (V, Ω) is symplectomorphic to $(\mathbb{R}^{2n}, \Omega_{\text{std}})$.

1.2. Symplectic Manifolds. Let ω be a differential 2-form on a manifold M . For each $p \in M$, the map

$$\omega_p: T_p M \times T_p M \rightarrow \mathbb{R}$$

is a skew-symmetric bilinear form.

Definition 1.10. A 2-form ω is **symplectic** if

- (1) $d\omega = 0$;
- (2) ω_p is non-degenerate for all $p \in M$.

Definition 1.11. A **symplectic manifold** is a pair (M, ω) where ω is a symplectic 2-form.

Example 1.12 (The standard symplectic structure on $\mathbb{R}^{2n}$). Let $M = \mathbb{R}^{2n}$ with coordinates

$$(x_1, \dots, x_n, y_1, \dots, y_n).$$

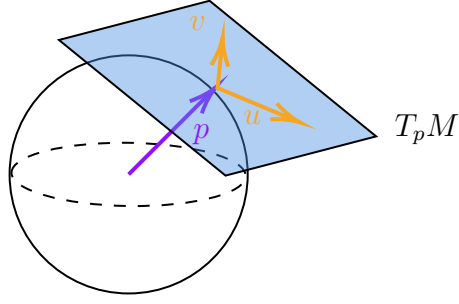


FIGURE 1

Define

$$\omega_{\text{std}} := \sum_{i=1}^n dx_i \wedge dy_i.$$

Then $(\mathbb{R}^{2n}, \omega_{\text{std}})$ is a symplectic manifold. Moreover, for each $p \in \mathbb{R}^{2n}$, the ordered basis

$$\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right\}$$

is a symplectic basis of $T_p \mathbb{R}^{2n}$.

Example 1.13 (The standard symplectic structure on $\mathbb{C}^n$). Let $M = \mathbb{C}^n$ with complex coordinates $(z_1, \dots, z_n)$, where

$$z_k = x_k + iy_k, \quad k = 1, \dots, n.$$

Define

$$\omega := \frac{i}{2} \sum_{k=1}^n dz_k \wedge d\bar{z}_k.$$

Then $(\mathbb{C}^n, \omega)$ is a symplectic manifold. Indeed, a direct computation shows

$$\begin{aligned} \frac{i}{2} \sum_{k=1}^n dz_k \wedge d\bar{z}_k &= \frac{i}{2} \sum_{k=1}^n (dx_k + i dy_k) \wedge (dx_k - i dy_k) \\ &= \sum_{k=1}^n dx_k \wedge dy_k, \end{aligned}$$

so this symplectic structure agrees with the standard one on $\mathbb{R}^{2n}$ under the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

Example 1.14 (The symplectic structure on $\mathbb{S}^2$). Let

$$M = \mathbb{S}^2 := \{p \in \mathbb{R}^3 \mid \|p\| = 1\}.$$

For $p \in \mathbb{S}^2$, the tangent space is

$$T_p \mathbb{S}^2 = \{v \in \mathbb{R}^3 \mid \langle p, v \rangle = 0\}.$$

See Figure 1. Define a 2-form ω on $\mathbb{S}^2$ by

$$\omega_p(u, v) := \langle p, u \times v \rangle = \det \begin{pmatrix} - & p & - \\ - & u & - \\ - & v & - \end{pmatrix}.$$

This form is closed since it is a top-degree form on $\mathbb{S}^2$. It is also non-degenerate: $\omega_p(u, v) = 0$ if and only if u and v are linearly dependent, i.e. the parallelogram they span has zero area. Hence $(\mathbb{S}^2, \omega)$ is a symplectic manifold.

2. COTANGENT BUNDLES (01/13, MN, YW)

This lecture covers material from [CdS01, Chapters 1,2,3].

2.1. Symplectomorphisms.

Definition 2.1. Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. A diffeomorphism

$$\varphi: M_1 \rightarrow M_2$$

is called a **symplectomorphism** if

$$\varphi^* \omega_2 = \omega_1.$$

Equivalently, for every $p \in M_1$, the differential

$$(d\varphi)_p: (T_p M_1, \omega_{1,p}) \xrightarrow{\sim} (T_{\varphi(p)} M_2, \omega_{2,\varphi(p)})$$

is a linear symplectic isomorphism.

Example 2.2. Suppose M is a surface, i.e. $\dim M = 2$, and ω is a 2-form on M . Then

$$\begin{aligned} (M, \omega) \text{ is symplectic} &\iff \omega \text{ is nowhere vanishing} \\ &\iff \omega \text{ is an area form.} \end{aligned}$$

In this case, symplectomorphisms are precisely area-preserving diffeomorphisms. In higher dimensions, symplectomorphisms are volume-preserving diffeomorphisms, but the converse does not hold.

For a Riemannian manifold (M, g) , we can define local invariants (curvature) for a point $p \in (M, g)$ as the obstruction to finding a coordinate chart $(U, x_1, \dots, x_n)$ about p where the metric can be expressed in the standard form

$$g_q = \sum dx_i \otimes dx_i \quad \forall q \in U.$$

In contrast, for a symplectic manifold (M, ω) , there are no such local invariants because the following theorem says that for every point $p \in (M, \omega)$, there exists a coordinate chart $(U, x_1, \dots, x_n, y_1, \dots, y_n)$ around p where the symplectic form can be expressed in the standard form

$$\omega_q = \sum dx_i \wedge dy_i \quad \forall q \in U.$$

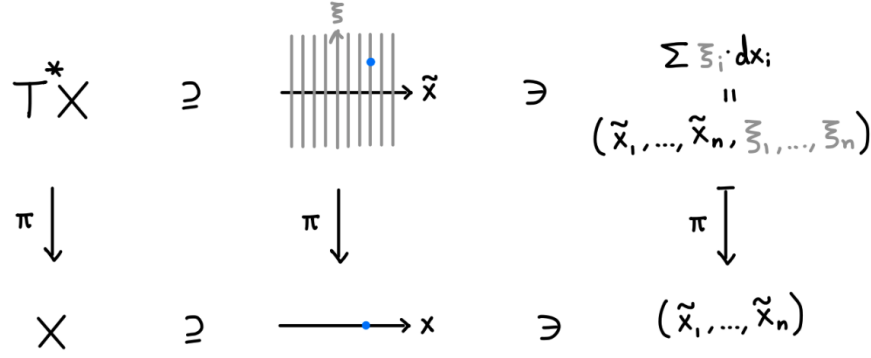


FIGURE 2. Natural coordinates $(\tilde{x}_1, \dots, \tilde{x}_n, \xi_1, \dots, \xi_n)$ on T^*X induced from coordinates $(x_1, \dots, x_n)$ on X .

Theorem 2.3 (Darboux). *Let (M^{2n}, ω) be a symplectic manifold and let $p \in M$. Then there exist local coordinates*

$$(x_1, \dots, x_n, y_1, \dots, y_n)$$

defined near p such that

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i$$

in these coordinates.

Remark 2.4. Darboux's theorem is the first time we use the condition that ω is a closed form. In fact, Darboux's theorem tells us that a non-degenerate 2-form ω is closed if and only if there exists local coordinates where ω can be written in the standard form.

2.2. Symplectic Form on the Cotangent Bundle. Let X be a smooth manifold and let T^*X denote its cotangent bundle. Let

$$\begin{aligned} \pi: T^*X &\rightarrow X \\ (x, \xi) &\mapsto x \end{aligned}$$

be the natural projection, where a point of T^*X is written as (x, ξ) with $x \in X$ and $\xi \in T_x^*X$.

Definition 2.5. The **tautological (Liouville) 1-form** α on the cotangent bundle T^*X (not the base manifold X) is defined by

$$\alpha_{(x, \xi)}(v) := \xi((d\pi)_{(x, \xi)}(v)),$$

where $(x, \xi) \in T^*X$ and $v \in T_{(x, \xi)}(T^*X)$.

The following diagram unpacks the above definition:

$$\begin{array}{ccccc} T_{(x, \xi)}(T^*X) & \xrightarrow{(d\pi)_{(x, \xi)}} & T_x X & \xrightarrow{\xi} & \mathbb{R} \\ v & \longmapsto & (d\pi)_{(x, \xi)}(v) & \longmapsto & \xi((d\pi)_{(x, \xi)}(v)) \end{array}$$

The above definition is coordinate-free. We now look at the tautological 1-form in coordinates.

Lemma 2.6. *The tautological 1-form on T^*X takes the form*

$$\alpha = \sum_{i=1}^n \xi_i dx_i$$

*in natural coordinates of T^*X .*

Proof. The expression $\alpha = \sum_{i=1}^n \xi_i dx_i$ is confusing because dx_i is a 1-form on X , while α is a 1-form on T^*X . To alleviate this confusion we proceed slowly and spell everything out. Start with coordinates $(x_1, \dots, x_n)$ on an open set $U \subset X$. This induces coordinates $(\tilde{x}_1, \dots, \tilde{x}_n, \xi_1, \dots, \xi_n)$ on $\pi^{-1}(U) \subset T^*X$ as follows.

To start, we define the coordinate function $\tilde{x}_i$ as the composition of the projection $T^*X \rightarrow X$ (restricted to $\pi^{-1}(U)$) and the coordinate function x_i . Most people let the projection be implicit, and use the notation x_i instead of $\tilde{x}_i$, and eventually we'll adopt this mild but standard abuse of notation. But for now we keep the tildes. So, given a point $(p, \xi) \in T^*X$, we have $\tilde{x}_i(p, \xi) = x_i(p)$.

We extend these coordinates to a coordinate system $(\tilde{x}_1, \dots, \tilde{x}_n, \xi_1, \dots, \xi_n)$ by defining $\xi_i(p, \xi)$ to be the coefficient of dx_i when ξ is written in $(x_1, \dots, x_n)$ coordinates.

Having been at least momentarily careful to distinguish the coordinate functions (for example x_i) with their values (for example $x_i(p)$), we now adopt the usual practice of using the same notation for both, even though this risks some confusion. Thus, the natural projection π in these coordinates is given by

$$\pi: (\tilde{x}_1, \dots, \tilde{x}_n, \xi_1, \dots, \xi_n) \mapsto (\tilde{x}_1, \dots, \tilde{x}_n).$$

Next, suppose $p \in T^*X$ has coordinates $(\tilde{x}_1, \dots, \tilde{x}_n, \xi_1, \dots, \xi_n)$ and $v \in T_p(T^*X)$ is a tangent vector given by

$$v = \sum_{i=1}^n a_i \frac{\partial}{\partial \tilde{x}_i} + \sum_{i=1}^n b_i \frac{\partial}{\partial \xi_i} \in T_p(T^*X).$$

Then, we can compute $\alpha(v)$ as the composition

$$\begin{array}{ccccc} T_p(T^*X) & \xrightarrow{(d\pi)_p} & T_{\pi(p)}X & \xrightarrow{\xi} & \mathbb{R} \\ \sum_{i=1}^n a_i \frac{\partial}{\partial \tilde{x}_i} + \sum_{i=1}^n b_i \frac{\partial}{\partial \xi_i} & \longmapsto & \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} & \longmapsto & \sum_{i=1}^n \xi_i a_i \end{array}$$

This establishes that

$$\alpha = \sum_{i=1}^n \xi_i d\tilde{x}_i.$$

Finally, we remark again that everybody abuses notation and writes dx_i to mean $d\tilde{x}_i$ even though $x_i \circ \pi = \tilde{x}_i$. $\square$

The next Lemma explains why α is called the tautological 1-form on T^*X . In fact, it is the unique 1-form that satisfies the following.

Lemma 2.7. *Let μ be a 1-form on X , and let*

$$\begin{aligned} s_\mu: X &\rightarrow T^*X \\ x &\mapsto (x, \mu_x) \end{aligned}$$

be the associated section. Then

$$s_\mu^* \alpha = \mu.$$

Proof. We check the equality at a point $x \in X$:

$$\begin{aligned} (s_\mu^* \alpha)_x &= (ds_\mu)_x^* \alpha_{(x, \mu_x)} && \text{(definition of pullback)} \\ &= (ds_\mu)_x^* (d\pi)_{(x, \mu_x)}^* \mu_x && \text{(equality } \alpha_{(x, \xi)} = (d\pi)_{(x, \xi)}^* \xi \text{)} \\ &= (d(\pi \circ s_\mu))_x^* \mu_x && \text{(functoriality)} \\ &= (d(\text{id}_X))^* \mu_x && \text{(property of a section } \pi \circ s_\mu = \text{id}_X \text{)} \\ &= \mu_x \end{aligned}$$

The formulation $\alpha_{(x, \xi)} = (d\pi)_{(x, \xi)}^* \xi$ is equivalent to Definition 2.5. $\square$

Definition 2.8. The **canonical symplectic form** on T^*X is

$$\omega := -d\alpha.$$

In local coordinates $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ the canonical symplectic form takes the standard form:

$$\begin{aligned} \omega &= -d\alpha \\ &= -d\left(\sum_{i=1}^n \xi_i dx_i\right) \\ &= \sum_{i=1}^n dx_i \wedge d\xi_i. \end{aligned}$$

This shows, ω is a symplectic form. Further, notice that $\omega = d(-\alpha)$ is an exact differential form.

2.3. Lagrangian Submanifolds.

Definition 2.9. In this course, a **submanifold** Y of a manifold M is a manifold together with a proper injective immersion

$$i: Y \hookrightarrow M.$$

We typically identify Y with its image and view i as an inclusion map.

Definition 2.10. A submanifold Y of a symplectic manifold (M, ω) is called **Lagrangian** if:

- (1) $i^* \omega = \omega|_Y = 0$,
- (2) $\dim Y = \frac{1}{2} \dim M$.

Equivalently, for every $p \in Y$, the subspace $T_p Y \subset T_p M$ is a Lagrangian subspace.

Remark 2.11. If (M, ω) is two-dimensional, then every one-dimensional submanifold $Y \subset M$ is Lagrangian. This is because every one-dimensional subspace of $(\mathbb{R}^2, \Omega_{\text{std}})$ is a Lagrangian subspace. This is a miracle of low dimension. In higher dimensions, Lagrangian submanifolds are rare (non-generic) and special, analogous to how Lagrangian subspaces of $(\mathbb{R}^{2n}, \Omega_{\text{std}})$ are special for $n \geq 2$.

Example 2.12. The zero section

$$X_0 := \{(x, 0) \mid x \in X\} \subset T^*X$$

is a Lagrangian submanifold of (T^*X, ω) . In local coordinates, it is given by $\xi_j = 0$ for all j and hence

$$i^* \left(\sum_{i=1}^n dx_i \wedge d\xi_i \right) = \left(\sum_{i=1}^n dx_i \wedge d\xi_i \right) \Big|_{\xi_j=0} = 0.$$

Later we shall see that $X_0 \subset T^*X$ is a local model for all compact Lagrangian submanifolds.

Next, we ask what are the other Lagrangian submanifolds near $X_0 \subset T^*X$. Start by thinking about what are all the ordinary submanifolds near $X_0 \subset T^*X$ and notice,

$$\left\{ \begin{array}{c} \text{perturbations of} \\ \text{the submanifold} \\ X_0 \subset T^*X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{sections of the normal bundle} \\ s: X_0 \longrightarrow N_{T^*X} X_0 \simeq T^*X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{1-forms on } X \\ \mu \in \Gamma(T^*X) \end{array} \right\}$$

Among all perturbations, the ones that give Lagrangian submanifolds are precisely the closed 1-forms.

More concretely, given any section

$$s_\mu: X \rightarrow T^*X,$$

we can view it as:

- (1) a 1-form μ on X ;
- (2) a submanifold $X_\mu := \{(x, \mu_x) \mid x \in X\} \subset T^*X$.

Lemma 2.13. *The submanifold X_μ is Lagrangian if and only if μ is closed.*

Proof. First, recall $\omega = -d\alpha$ is the symplectic form on T^*X , where α is the tautological 1-form. Next,

$$\begin{aligned} X_\mu \subset T^*X \text{ is Lagrangian} &\iff \omega|_{X_\mu} = 0 \\ &\iff (d\alpha)|_{X_\mu} = 0 \\ &\iff d(\alpha|_{X_\mu}) = 0 \\ &\iff \alpha|_{X_\mu} \text{ is a closed 1-form on } X_\mu \\ &\iff s_\mu^* \alpha \text{ is a closed 1-form on } X \\ &\iff \mu \text{ is a closed 1-form on } X. \end{aligned}$$

The penultimate equivalence follows because s_μ induces a diffeomorphism from X to $X_\mu \subseteq T^*X$ and the last equivalence is a consequence of Lemma 2.7. Note that here when we talk about the restriction of a form to X_μ we mean the pullback via the inclusion of X_μ into T^*X . $\square$

Remark 2.14. Lemma 2.13 provides an abundance of Lagrangian submanifolds of the cotangent bundle, but not all Lagrangian submanifolds necessarily arise in this way. This is closely related to the nearby Lagrangian conjecture by Arnold.

Remark 2.15. One of the main guiding principles of symplectic topology, to which we will return on several occasions, is the following:

Lagrangian creed.

“Everything is Lagrangian (when viewed from the right angle).”

The above lemma can be viewed as a first instance of this creed.

3. GENERATING FUNCTIONS (01/15, YW, JZ)

This lecture covers materials from [CdS01, Chapter 4].

Last time we proved that, for any 1-form μ on X , the following two conditions are equivalent:

$$\mu \text{ is closed} \iff \text{the graph } X_\mu = \{(x, \mu_x)\} \subseteq T^*X \text{ is Lagrangian.}$$

Definition 3.1. Let μ be an exact 1-form on X . Then any function f satisfying $\mu = df$ is called a **generating function for X_μ** .

Question 3.2. Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds. When is a diffeomorphism $\psi : M_1 \rightarrow M_2$ a symplectomorphism?

The answer to this question is our second example of the *Lagrangian creed*. Indeed, consider the diagram

$$\begin{array}{ccc} & & M_1 \times M_2 \\ & \nearrow \gamma & \\ M_1 & & \\ & \nwarrow \text{pr}_1 & \\ & & M_2 \end{array}$$

where pr_1, pr_2 are the projection maps, and $\gamma : M_1 \hookrightarrow M_1 \times M_2$ is the graph embedding given by $p \mapsto (p, \psi(p))$. We adopt the following notations:

- (1) $\text{Im}(\gamma)$ is denoted as Γ_ψ . So we will also use γ to denote the inclusion $\Gamma_\psi \subseteq M_1 \times M_2$;
- (2) $\omega = (\text{pr}_1)^*\omega_1 + (\text{pr}_2)^*\omega_2$ is a symplectic form on $M_1 \times M_2$;
- (3) $\tilde{\omega} = (\text{pr}_1)^*\omega_1 - (\text{pr}_2)^*\omega_2$ is the twisted symplectic form on $M_1 \times M_2$. The sign change will turn out to be convenient.

Proposition 3.3. *The diffeomorphism ψ is a symplectomorphism if and only if Γ_ψ is a Lagrangian submanifold of $(M_1 \times M_2, \tilde{\omega})$.*

Proof. By definition, Γ_ψ is Lagrangian iff $\gamma^*\tilde{\omega} = 0$. The proposition then follows from the computation

$$\begin{aligned} \gamma^*\tilde{\omega} &= \gamma^*\text{pr}_1^*\omega_1 - \gamma^*\text{pr}_2^*\omega_2 \\ &= (\text{pr}_1 \circ \gamma)^*\omega_1 - (\text{pr}_2 \circ \gamma)^*\omega_2 \\ &= \omega_1 - \psi^*\omega_2. \end{aligned}$$

□

3.1. Producing symplectomorphisms. We specialize to the situation when the Lagrangians are $M_1 = T^*X_1, M_2 = T^*X_2$. Recall that each is equipped with the tautological 1-form α_i , and therefore has a canonical symplectic form $\omega_i = -d\alpha_i$ for $i = 1, 2$.

The product $M_1 \times M_2$ is also a cotangent bundle $T^*(X_1 \times X_2)$, which has a tautological 1-form

$$\alpha = \text{pr}_1^* \alpha_1 + \text{pr}_2^* \alpha_2$$

and therefore a canonical form

$$\omega = \text{pr}_1^* \omega_1 + \text{pr}_2^* \omega_2.$$

Definition 3.4. Let $\sigma_2: M_2 \rightarrow M_2$ be the involution given by $(x_2, \xi_2) \mapsto (x_2, -\xi_2)$. Then $\sigma_2^* \alpha_2 = -\alpha_2$, and $\sigma_2^* \omega_2 = -\omega_2$.

Definition 3.5. Let $\sigma: M_1 \times M_2 \rightarrow M_1 \times M_2$ be given by $\sigma = \text{Id}_{M_1} \times \sigma_2$. This

- (1) is an involution, i.e., $\sigma^2 = \text{Id}$,
- (2) exchanges the two natural symplectic structures on $M_1 \times M_2$, i.e.,

$$\sigma^* \omega = \tilde{\omega} \quad \text{and} \quad \sigma^* \tilde{\omega} = \omega,$$

- (3) is therefore a symplectomorphism $(M_1 \times M_2, \omega) \leftrightarrow (M_1 \times M_2, \tilde{\omega})$.

For any submanifold $Y \subseteq M_1 \times M_2$, we let Y^σ denote $\sigma(Y)$. Then $Y \subseteq (M_1 \times M_2, \omega)$ is Lagrangian if and only if $Y^\sigma \subseteq (M_1 \times M_2, \tilde{\omega})$ is. Combined with Proposition 3.3, we have a

recipe for producing symplectomorphisms $M_1 \rightarrow M_2$:

- (1) start with a Lagrangian submanifold $Y \subseteq (M_1 \times M_2, \omega)$,
- (2) twist to get a Lagrangian submanifold $Y^\sigma \subseteq (M_1 \times M_2, \tilde{\omega})$,
- (3) check if Y^σ is the graph of a map $M_1 \rightarrow M_2$; if so, this map is a symplectomorphism.

Let $i: Y^\sigma \hookrightarrow M_1 \times M_2$ denote the inclusion. Concretely, step (3) amounts to checking that $\text{pr}_1 \circ i$ and $\text{pr}_2 \circ i$ are both diffeomorphisms. If this is the case, then

$$\psi := (\text{pr}_2 \circ i) \circ (\text{pr}_1 \circ i)^{-1}$$

is the desired symplectomorphism $M_1 \rightarrow M_2$.

3.2. Generating functions on $M_1 \times M_2 = T^*(X_1 \times X_2)$. We want to further explore when step (3) holds true. To this end, we focus on a particular class of Lagrangians in $(M_1 \times M_2, \omega)$ arising as graphs of exact 1-forms.

Recall that, for any function $f \in C^\infty(X_1 \times X_2)$, we have a closed 1-form df on $X_1 \times X_2$. By Lemma 2.13, the graph

$$Y_f := (X_1 \times X_2)_{df} = \{(x, y, (df)_{(x,y)})\} \subseteq M_1 \times M_2 \text{ is Lagrangian.}$$

We can also split the cotangent coordinates by writing $(df)_{(x,y)} = d_x f + d_y f$, and therefore $(df)_{(x,y)} = (d_x f, d_y f)$. Note this means in local coordinates, $d_x f = \sum \frac{\partial f}{\partial x_i} dx_i$ and similarly $d_y f = \sum \frac{\partial f}{\partial y_i} dy_i$.

Definition 3.6. If $Y_f^\sigma = \sigma(Y_f)$ is the graph of a diffeomorphism $\psi: M_1 \rightarrow M_2$, we say

- (1) ψ is the symplectomorphism generated by f , and

(2) f is the generating function of ψ .

Question 3.7. Given $f \in C^\infty(X_1 \times X_2)$, what is a necessary condition for it to generate a symplectomorphism ψ ?

In addition to finding a necessary condition, we also explore what might be sufficient for ψ to exist locally.

So we work locally. Let $(U_1, x_1, \dots, x_n)$ and $(U_2, y_1, \dots, y_n)$ be coordinate charts of X_1 and X_2 , which induce charts

$$(T^*U_1, x_1, \dots, x_n, \xi_1, \dots, \xi_n) \text{ of } M_1, \text{ and}$$

$$(T^*U_2, y_1, \dots, y_n, \eta_1, \dots, \eta_n) \text{ of } M_2.$$

In order for f to generate some ψ , Y_f^σ must be the graph of said ψ . In other words, the two sets

$$\Gamma_\psi = \{(x, \xi, y, \eta) : \psi(x, \xi) = (y, \eta)\}, \quad Y_f^\sigma = \{(x, y, d_x f, -d_y f)\}$$

are equal. This means for any $(x, \xi) \in M_1$ and $(y, \eta) \in M_2$, we have

$$\psi(x, \xi) = (y, \eta) \iff \xi = d_x f \text{ and } \eta = -d_y f.$$

Let us fix any $(x, \xi) \in T^*U_1$, and try to compute $\psi(x, \xi)$. We have $\xi = d_x f$ and $\eta = -d_y f$ at any $(x, y) \in U_1 \times U_2$; in coordinates, this means

$$\xi_i = \frac{\partial f}{\partial x_i}(x, y) \quad \text{and} \quad \eta_i = -\frac{\partial f}{\partial y_i}(x, y).$$

We need to find an implicit solution to the left hand side that gives y as a function in terms of x, ξ . Then we can feed y into the right hand side to express η in terms of x, ξ as well.

By the implicit function theorem, to locally solve for y in the left hand side, we necessarily need a non-degeneracy condition at point $(x, y) \in U_1 \times U_2$,

$$\det \left(\frac{\partial}{\partial y_j} \left(\frac{\partial f}{\partial x_i} \right) \right)_{i,j=1}^n \neq 0.$$

This guarantees f to **locally** generate ψ , albeit not necessarily globally. This gives an answer to Question 3.7 (and actually shows something a bit stronger, namely a sufficient condition for local generation).

Here is another point of view: We assume ψ exists (generated by f), and ask what this implies. Since it already exists by assumption, we are really using the easier direction of the implicit function theorem, and we should be able to see directly that the Jacobian determinant of some map is nonzero. Write $\psi = (\psi_1, \psi_2)$, i.e., $\psi_1(x, \xi) = y$. By above, this implies $\xi = d_x f$. So locally,

$$\psi_1(x_1, \dots, x_n, \frac{\partial f}{\partial x_1}(x_1, \dots, x_n, y_1, \dots, y_n), \dots, \frac{\partial f}{\partial x_n}(x_1, \dots, x_n, y_1, \dots, y_n)) = (y_1, \dots, y_n).$$

Suppressing the x variables, this just says $\psi_1 \circ (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})(y) = y$. In other words, for fixed x , the map $y \mapsto (\frac{\partial f}{\partial x_1}(x, y), \dots, \frac{\partial f}{\partial x_n}(x, y))$ is a diffeo, so its Jacobian determinant is nonzero. Therefore, this determinant being nonzero is certainly **necessary** for f to generate a symplectomorphism ψ .

4. THE POINCARÉ BIRKHOFF THEOREM (01/20, JZ, ZH)

From last time:

Theorem 4.1. *Let $\varphi : M_1 \rightarrow M_2$ be a diffeomorphism. Then φ is a symplectomorphism if and only if $\Gamma_\varphi \subset (M_1 \times M_2, \tilde{\omega})$ is Lagrangian.*

This is the second instance of the *Lagrangian creed*!

Today, we will discuss the Poincaré-Birkhoff theorem, which ensures the existence of fixed points. Recall: for any map $\varphi : M \rightarrow M$, the number of fixed points is

$$\#\text{Fix}(\varphi) = \#(\Delta \cap \Gamma_\varphi),$$

where $\Delta \subset M \times M$ is the diagonal and Γ_φ is the graph of φ . This works in basically any situation, even just at the level of sets. Note also that if φ is a symplectomorphism, then $\Delta \subset (M \times M, \tilde{\omega})$ is Lagrangian (just check this directly or use that $\Delta = \Gamma_{\text{id}}$). Thus,

$$\#\text{Fix}(\varphi) = \#(\Delta \cap \Gamma_\varphi) = \#(\text{intersection of two Lagrangians}).$$

Major nontrivial theme: Often the symplectic assumption gives much more than topology alone could get you.

Remark 4.2. One of the main pillars in symplectic geometry, the Arnold conjecture, is about counting fixed points. The statement is as follows: Suppose (M, ω) is compact symplectic. Given a function $H : M \rightarrow \mathbb{R}$, we get an associated vector field X_H defined by $\omega(X_H, \cdot) = -dH$ (compare to what is done in the proof of Moser's theorem in next lecture!). Then, given a one-parameter family $H_t : M \rightarrow \mathbb{R}$, we get the associated X_{H_t} which in turn has an associated flow map $\phi_t : M \rightarrow M$ (also discussed in more detail in the next lecture). The Arnold conjecture states that the number of fixed points of ϕ_1 (assuming they are all nondegenerate) is at least as large as the minimal number of critical points of any Morse function on M . More simply, by Morse theory, this implies $\#\text{Fix}(\phi_1) \geq \sum b_k$ where the b_k are the Betti numbers of M . Floer theory was developed to prove this conjecture.

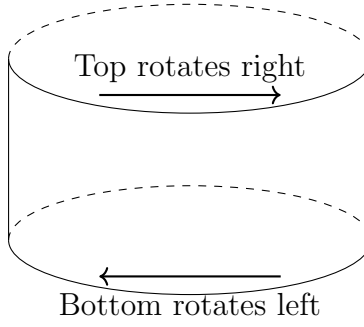
Setup: Diffeomorphisms of the Annulus. Next, we will consider the easiest, most elementary situation, where “symplecto” gets you more fixed points than “diffeo”.

Consider a diffeomorphism of an annulus/cylinder:

$$A = S^1 \times [a, b], \quad S^1 = \mathbb{R}/\mathbb{Z}.$$

Such a diffeomorphism need not have fixed points, for example, just rotate everything.

In order to try to ensure fixed points, we must impose more conditions. The first condition we assume is the “twist condition” that the top of A rotates right while the bottom rotates left:



We encode this in the following notation for a map $\varphi: A \rightarrow A$. Any map

$$\varphi: S^1 \times [a, b] \rightarrow S^1 \times [a, b]$$

can be written in the form $\varphi([x], y) = ([f(x, y)], g(x, y))$ where

$$f: \mathbb{R} \times [a, b] \rightarrow \mathbb{R}, \quad g: \mathbb{R} \times [a, b] \rightarrow [a, b],$$

with

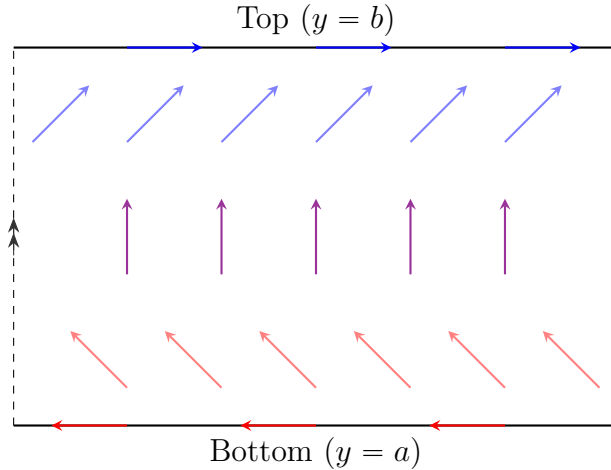
$$f(x+1, y) \equiv f(x, y) \pmod{\mathbb{Z}}, \quad g(x+1, y) = g(x, y).$$

Note that here $[x]$ is the class of x in $S^1 = \mathbb{R}/\mathbb{Z}$. The assumption “top moves right, bottom moves left” says:

$$(\star) \quad f(x, a) < x, \quad f(x, b) > x.$$

Implicit in this is also the fact that $g(x, a) = a$ and $g(x, b) = b$, i.e., the bottom of the strip is mapped to the bottom and the top of the strip is mapped to the top.

Still, this does not guarantee fixed points! For example, we could have that $g(x, y)$ only depends on y , with $g(y) > y$ if $y \neq a, b$, with all interior points moving upward and boundary points moving as prescribed above. Such a map on the cylinder looks like:



However, such examples will not be area-preserving. Once we add this second assumption, we obtain:

Theorem 4.3 (Poincaré–Birkhoff). *An area-preserving diffeomorphism $\varphi: S^1 \times [a, b] \rightarrow S^1 \times [a, b]$ satisfying the twist condition $(\star)$ has at least two fixed points.*

Remark 4.4. Area-preserving is the same as symplectic for dimension 2! But this is just a nice coincidence. In general, symplectic maps are more special than area-preserving in higher dimensions.

Remark 4.5. We will only show there exists one fixed point.

Proof. We work entirely with the map

$$\varphi = (f, g): \mathbb{R} \times [a, b] \rightarrow \mathbb{R} \times [a, b].$$

In other words, just view φ as a map $\tilde{A} := \mathbb{R} \times [a, b] \rightarrow \mathbb{R} \times [a, b]$ that satisfies all the facts we've mentioned above. If this map has a fixed point, then the induced map on A also has a fixed point.

Assume that φ has no fixed points. We can define a “direction moved map”:

$$\begin{aligned} \rho: \mathbb{R} \times [a, b] &\rightarrow S^1 \subset \mathbb{R}^2. \\ (x, y) &\mapsto \frac{\varphi(x, y) - (x, y)}{\|\varphi(x, y) - (x, y)\|} \end{aligned}$$

By the twist condition assumption, $\rho(x, a) = -1$ and $\rho(x, b) = 1$. We use ρ to define a function on paths from the bottom of the cylinder to the top, based on winding number. Consider a path $\gamma: [0, 1] \rightarrow \mathbb{R} \times [a, b]$ from a point in $\mathbb{R} \times \{a\}$ to a point in $\mathbb{R} \times \{b\}$. The composite map $\rho \circ \gamma: [0, 1] \rightarrow S^1$ starts at -1 and goes to 1 . The composite map can be homotoped to a unique constant speed path which either goes clockwise (CW) or counter-clockwise (CCW). Here, we implicitly use that $\pi_1(S^1) = \mathbb{Z}$.

Since the map $\gamma \mapsto \{\text{CW}, \text{CCW}\}$ is continuous and any path $[0, 1] \rightarrow \mathbb{R} \times [a, b]$ from a point in $\mathbb{R} \times \{a\}$ to a point in $\mathbb{R} \times \{b\}$ can be deformed to any other such path, the map $\gamma \mapsto \{\text{CW}, \text{CCW}\}$ does not depend on our choice of γ .

We say that “ φ turns CW” or “ φ turns CCW” depending on whether the image of this map is CW or CCW.

In particular, φ turns all curves CW or it turns all curves CCW. We will show ρ turns both clockwise and counterclockwise, giving a contradiction.

Warm up: If there exists γ such that $\varphi(\gamma(t))$ always has y -component greater than or equal to the y -component of $\gamma(t)$ then γ is forced to be CW. This is because $\rho \circ \gamma$ will necessarily be pointing up, so the winding pattern will be homotopic to a single half turn clockwise.

Now, we “cheat” by replacing φ with the perturbed map $\varphi_\epsilon: \tilde{A} \rightarrow \mathbb{R}^2$ defined by $\varphi_\epsilon = \varphi + (0, \epsilon)$. Note that adding a shift preserves area, so φ_ϵ is still an area-preserving homeomorphism onto its image (the image being the annulus itself, just shifted up slightly). Let

$$\delta = \min_{(x, y) \in A} \|\varphi(x, y) - (x, y)\|.$$

Since the cylinder A is compact, δ exists and it is non-zero because φ has no fixed points. Then for any ϵ with $\epsilon < \delta$, the map φ_ϵ does not have a fixed point. From now on, assume ϵ is always smaller than δ so that φ_ϵ is an area preserving map with no fixed points.

We will use φ_ϵ along with the area-preserving assumption to guarantee existence of curves that are CW. Then, running the same argument but with $\varphi_{-\epsilon}$ will guarantee the existence of curves that are CCW.

Consider the region $D_0 = \{a \leq y < a + \epsilon\} \subset \tilde{A}$. Then φ_ϵ moves the line $\mathbb{R} \times \{a\}$ to $\mathbb{R} \times \{a + \epsilon\}$. It follows that D_0 is disjoint from the iterates $\varphi_\epsilon^k(D_0) =: D_k$ and hence all of the D_k 's are disjoint from each other. Since φ_ϵ is area-preserving, there must be an integer $k \geq 2$ such that $D_{k-1} \not\subset \tilde{A}$. Therefore, there is a point $x_0 \in \mathbb{R}$ such that the y -coordinate of $\varphi_\epsilon^k(x_0, a)$ is greater than b . Here we crucially use that φ was originally a function on the finite area space A to know that the D_k do indeed eventually overflow out of the strip. Let l be a straight line between (x_0, a) and $\varphi_\epsilon(x_0, a)$. Then, by applying iterates of φ_ϵ^j to this line (with $j = 0, \dots, k-1$), define the path γ_0 as the path from (x_0, a) to (x_k, y_k) as seen below in Figure 3.¹

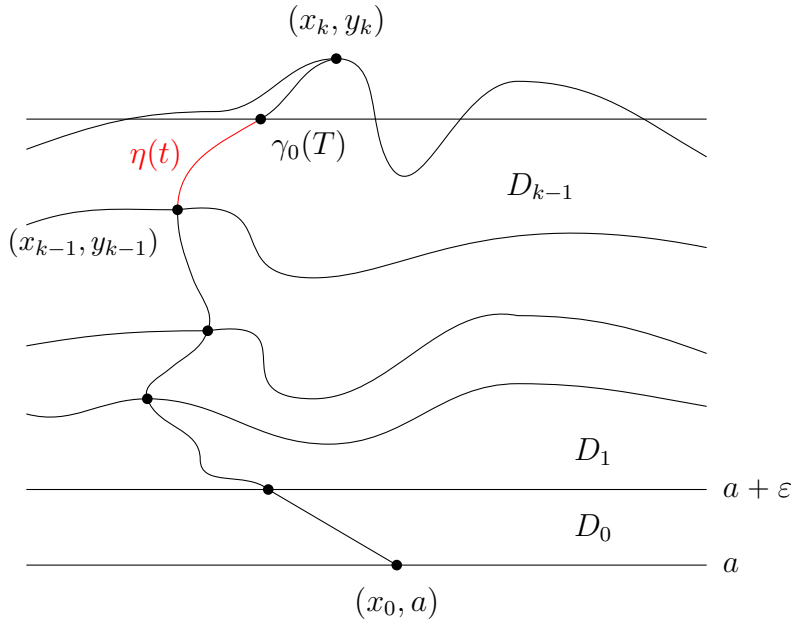


FIGURE 3. Constructing the paths $\gamma_0, \tilde{\gamma}$, and γ

Let T be the first time that $\gamma_0(T) \in \mathbb{R} \times \{b\}$ and let η be the portion of γ_0 that travels from $(x_{k-1}, y_{k-1}) := \varphi_\epsilon^{k-1}(x_0, a)$ to $\gamma_0(T)$. Then, since η is contained in $\tilde{A}$, φ_ϵ is defined on points of η and we can extend γ_0 by concatenating with $\varphi_\epsilon(\eta)$. Call this extended path $\tilde{\gamma}$. Since $\varphi_\epsilon(\eta)$ starts at $\varphi_\epsilon(x_{k-1}, y_{k-1}) = (x_k, y_k)$ which is where γ_0 ends, $\tilde{\gamma}$ is continuous. Finally, let $\gamma = \tilde{\gamma}|_{[0, T]}$ be the path that starts at $y = a$ and ends at $y = b$ at the top of the strip.

For convenience, reparameterize so that $T = 1$, i.e., γ is a function $[0, 1] \rightarrow \tilde{A}$ and $\tilde{\gamma}$ is a function $[0, 1 + \tau] \rightarrow \tilde{A}$ where τ is the amount of time taken to travel from $\gamma_0(1)$ to the end of $\varphi_\epsilon(\eta)$. In total, this setup has ensured that φ_ϵ sends points on γ to later points on γ (i.e., $\varphi_\epsilon(\gamma(t)) = \gamma(s)$ for a unique $s > t$) or to a point above D_{k-1} . Even more precise, φ_ϵ sends points on γ to later points on $\tilde{\gamma}$.

¹Made by Google Gemini with inspiration taken from page 349 of [MS17]

Lemma 4.6. *The path γ is CW.*

Proof. Consider the following maps $\rho_\epsilon, \mu: [0, 1] \rightarrow S^1$:

$$\rho_\epsilon(t) = \frac{\varphi_\epsilon(\gamma(t)) - \gamma(t)}{\|\varphi_\epsilon(\gamma(t)) - \gamma(t)\|}$$

$$\mu(t) = \begin{cases} \frac{\varphi_\epsilon(\gamma(2t)) - \gamma(0)}{\|\varphi_\epsilon(\gamma(2t)) - \gamma(0)\|} & 0 \leq t \leq 1/2 \\ \frac{\varphi_\epsilon(\gamma(1)) - \gamma(2t-1)}{\|\varphi_\epsilon(\gamma(1)) - \gamma(2t-1)\|} & 1/2 \leq t \leq 1. \end{cases}$$

In words, ρ_ϵ is the same “direction moved map” from above. The map μ is the path which is direction from $\gamma(0)$ to $\varphi_\epsilon(\gamma(2t))$ for the first half and then direction from $\gamma(t)$ to $\varphi_\epsilon(\gamma(1))$ in the second half. By the warm-up, $\mu(t)$ can only turn clockwise because $\mu(t)$ always points upwards. In particular, $\gamma(0)$ is at the bottom of the strip so automatically μ points up in the first half; and $\varphi_\epsilon(\gamma(1))$ is ϵ above the strip while $\gamma(2t-1)$ is always in the strip, so μ also must point up in the second half.

For ϵ very small (namely, small compared to $\|\varphi(\gamma(0)) - \gamma(0)\|$), $\mu(0)$ “points” mostly left. Crucially, we are using that φ moves the bottom of the strip to itself so there is no upwards movement. Similarly, if ϵ is small compared to $\|\varphi(\gamma(1)) - \gamma(1)\|$ then $\mu(1)$ points mostly right. So for ϵ small compared to δ , see that $\mu(0) \approx \pi$ and $\mu(1) \approx 0$. In total, letting $\epsilon \rightarrow 0$, see that $\mu(t)$ approaches a CW path from π to 0. By all these observations, μ is CW, or at least, is close to approximating a CW path (so homotopic to one half CW turn, but which might not start exactly at π and end exactly at 0).

Now, we claim $\rho_\epsilon(t)$ is homotopic to μ . This is where it is useful to have the extended curve $\tilde{\gamma}$. By the setup above, both $\gamma(t)$ and $\varphi_\epsilon(\gamma(t))$ are points on $\tilde{\gamma}$. Therefore, both ρ_ϵ and μ are paths of the form “direction from point on $\tilde{\gamma}$ to a later point on $\tilde{\gamma}$ ”. Crucially, any such paths are all homotopic to each other since I is contractible. This is made more precise in the lemma below. Thus, ρ_ϵ and μ are homotopic. Noting that ρ_ϵ and μ agree at end points, it follows that ρ_ϵ also turns CW, starting approximately at π and ending approximately at 0. Letting $\epsilon \rightarrow 0$, see that $\rho_0 = \rho \circ \gamma = \frac{\varphi(\gamma(t)) - \gamma(t)}{\|\varphi(\gamma(t)) - \gamma(t)\|}$ itself must turn clockwise, so $\Phi(\gamma) = \text{CW}$. As mentioned, it follows that Φ sends every curve to CW, i.e., φ is CW. $\square$

Lemma 4.7. *Let $\gamma: [0, b] \rightarrow \mathbb{R}^2$ be an injective map and pick $T \in (0, b)$. For $i = 0, 1$, let $c_i, h_i: [0, T] \rightarrow [0, b]$ be maps such that $h_i(t) > 0$ and $c_i(t) + h_i(t) \leq b$ for all t . For $i = 0, 1$, let $\mu_i: [0, T] \rightarrow S^1$ be the map*

$$\mu_i(t) = \frac{\gamma(c_i(t) + h_i(t)) - \gamma(c_i(t))}{\|\gamma(c_i(t) + h_i(t)) - \gamma(c_i(t))\|}.$$

Then μ_0 is homotopic to μ_1 .

Proof. Crucially, $h_i(t) > 0$ so a straight-line homotopy is well defined, showing the two maps are homotopic. $\square$

Remark 4.8. We now explain how to apply Lemma 4.7 to show that the maps ρ_ϵ and μ are homotopic: We must show both paths are of the form

$$\frac{\tilde{\gamma}(c_i(t) + h_i(t)) - \tilde{\gamma}(c_i(t))}{\|\tilde{\gamma}(c_i(t) + h_i(t)) - \tilde{\gamma}(c_i(t))\|}$$

for some c_i, h_i . For ρ_ϵ , $c_{\rho_\epsilon}(t)$ is just t , although with domain $[0, 1]$ instead of $[0, 1 + \tau]$. Then $h_{\rho_\epsilon}(t)$ is always positive, depending on how much φ_ϵ shifts γ . Specifically, $h_{\rho_\epsilon}(t)$ satisfies $\varphi_\epsilon(\gamma(t)) = \tilde{\gamma}(t + h_{\rho_\epsilon}(t))$ for all $t \in [0, 1]$. Since φ_ϵ shifts D_i to D_{i+1} , the value $h_{\rho_\epsilon}(t)$ is certainly positive. Note that $h_{\rho_\epsilon}(1) = \tau$. For μ , we are forced to set

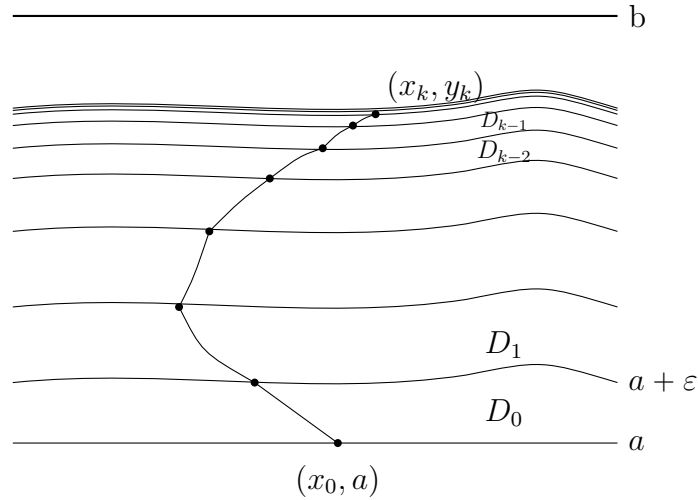
$$c_\mu(t) = \begin{cases} 0 & 0 \leq t \leq 1/2 \\ 2t - 1 & 1/2 \leq t \leq 1 \end{cases}$$

$$h_\mu(t) = \begin{cases} 2t + h_{\rho_\epsilon}(2t) & 0 \leq t \leq 1/2 \\ 1 + \tau - (2t - 1) & 1/2 \leq t \leq 1 \end{cases}.$$

Plugging in and using $\varphi_\epsilon(\gamma(t)) = \tilde{\gamma}(t + h_{\rho_\epsilon}(t))$, see that these are the c and h which work for μ . Since $\tau > 0$, h_μ is clearly positive. Thus, we can apply Lemma 4.7 to show ρ_ϵ and μ are homotopic.

We can now conclude our proof of Poincaré–Birkhoff. With the proof of Lemma 4.6, we can conclude that Φ maps γ and therefore every path to CW. But, by making the same argument with $\varphi_{-\epsilon}$ instead, we also deduce Φ maps everything to CCW, which is a contradiction. Thus, φ must have a fixed point, and we are done. $\square$

Remark 4.9. Without area preservation, one might get stuck at some point without having any iterate D_k land above the strip. The picture is as follows:



Consider the map φ similar to the one mentioned above except that shifts everything down and which does not have a fixed point. Our argument fails because we might get stuck somewhere in the strip and therefore we won't be able to construct γ . In this case, the direction move map always points down, so Φ maps all curves to CCW without a contradiction.

5. MOSER'S TRICK (01/22, YL, MN)

This lecture covers material from [CdS01, Chapters 6,7].

5.1. Review of differential topology. The goal of this section is to review the material needed to justify the key calculation underlying Moser's trick:

$$\begin{aligned}
\left. \frac{d}{ds} (\rho_s^* \omega_s) \right|_{s=t_0} &= \left. \frac{d}{ds} (\rho_s^* \omega_{t_0}) \right|_{s=t_0} + \rho_{t_0}^* \left(\left. \frac{d\omega_s}{ds} \right|_{s=t_0} \right) \\
&= \rho_{t_0}^* (\mathcal{L}_{v_s} \omega_{t_0} |_{s=t_0}) + \rho_{t_0}^* \left(\left. \frac{d\omega_s}{ds} \right|_{s=t_0} \right) \\
&= \rho_{t_0}^* \left(\mathcal{L}_{v_s} \omega_{t_0} |_{s=t_0} + \left. \frac{d\omega_s}{ds} \right|_{s=t_0} \right) \\
&= \rho_{t_0}^* \left(\mathcal{L}_{v_{t_0}} \omega_{t_0} + \left. \frac{d\omega_s}{ds} \right|_{s=t_0} \right) \\
&= \rho_{t_0}^* \left(d\iota_{v_{t_0}} \omega_{t_0} + \iota_{v_{t_0}} d\omega_{t_0} + \left. \frac{d\omega_s}{ds} \right|_{s=t_0} \right).
\end{aligned}$$

We begin by defining the objects involved. Let M be a smooth manifold, and let

$$\rho: M \times \mathbb{R} \rightarrow M$$

be a smooth map. For each $s \in \mathbb{R}$, define $\rho_s(p) := \rho(p, s)$.

Definition 5.1. The map ρ is called an **isotopy** if $\rho_s: M \rightarrow M$ is a diffeomorphism for all $s \in \mathbb{R}$ and $\rho_0 = \text{id}_M$.

An isotopy ρ_s determines a time-dependent vector field v_s via

$$v_s := \frac{d\rho_s}{ds} \circ \rho_s^{-1}.$$

See Figure 4 to understand the role of ρ_s^{-1} . Equivalently,

$$(1) \quad v_{t_0}(p) = \left. \frac{d}{ds} \rho_s(q) \right|_{s=t_0}, \quad q = \rho_{t_0}^{-1}(p).$$

Conversely, given a time-dependent vector field v_s , the Picard–Lindelöf theorem allows us to solve (1) locally to obtain a locally defined isotopy. If M is compact, the solution exists globally and for all time. Thus, we obtain a one-to-one correspondence:

$$\{ \text{Isotopies } \rho_s: M \rightarrow M \} \longleftrightarrow \{ \text{Time-dependent vector fields } v_s \}.$$

Remark 5.2. If $v_s \equiv v$ is independent of s , the associated isotopy is called the **exponential map** of v , denoted $\exp(sv)$, and defined by

$$\begin{aligned}
\exp(sv) \big|_{s=0} &= \text{id}_M, \\
\left. \frac{d}{ds} (\exp(sv)(p)) \right|_{s=t_0} &= v(\exp(t_0 v)(p)).
\end{aligned}$$

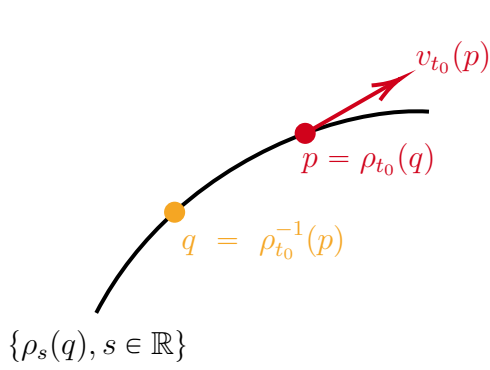


FIGURE 4. Time-dependent vector field at a point p .

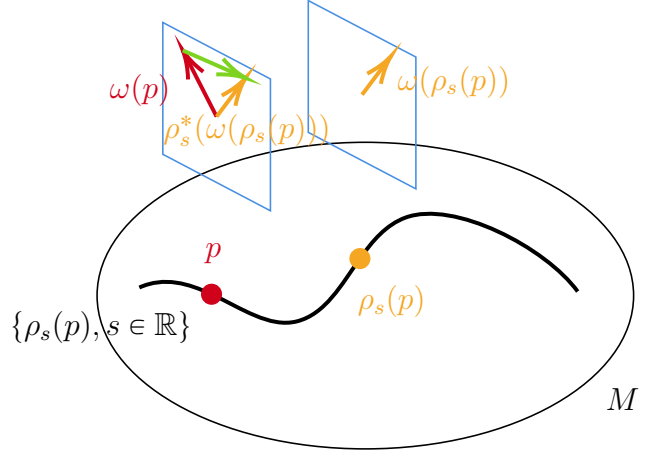


FIGURE 5. The Lie derivative of ω at $s = 0$.

It is also common to call $\exp(sv)$ the **flow** (for time s in the vector v). In fact, many authors even use the word flow for the isotopy corresponding to a time dependent vector field, but [CdS01] uses the word isotopy instead and reserves the word flow for use with vector fields that do not depend on time.

Definition 5.3. Let v_s be a time-dependent vector field generating an isotopy ρ_s . The **Lie derivative** with respect to v_s is defined by

$$\mathcal{L}_{v_s}|_{s=t_0} : \Omega^k(M) \rightarrow \Omega^k(M),$$

where

$$\begin{aligned} \mathcal{L}_{v_s}\omega|_{s=0} &:= \left. \frac{d}{ds}(\rho_s^*\omega) \right|_{s=0}, \\ \mathcal{L}_{v_s}\omega|_{s=t_0} &:= (\rho_{t_0}^{-1})^* \left(\left. \frac{d}{ds}(\rho_s^*\omega) \right|_{s=t_0} \right). \end{aligned}$$

Remark 5.4. Usually, the Lie derivative is defined only for time-independent vector fields. The two notions are related by observing that

$$\mathcal{L}_{v_s}\omega|_{s=t_0} = \mathcal{L}_{v_{t_0}}\omega,$$

where the Lie derivative on the left-hand side is taken with respect to the time-dependent vector field v_s , while the Lie derivative on the right-hand side is taken with respect to the time-independent vector field v_{t_0} . Here, v_{t_0} represents the vector field v_s “frozen” at the time $s = t_0$.

Proposition 5.5 (Cartan’s Magic Formula). *For any differential form $\omega \in \Omega^k(M)$ and a time-independent vector field v , we have*

$$(2) \quad \mathcal{L}_v\omega = \iota_v d\omega + d\iota_v\omega.$$

Remark 5.6. This identity shows that the Lie derivative is chain homotopic to zero; in particular, it induces the zero map on de Rham cohomology.

Remark 5.7. The Lie derivative encapsulates two effects: (i) the variation of the coefficients of the form ω along the flow ρ_s , and (ii) the variation of the coefficients of the vector field v . This explains the two terms on the right-hand side of Cartan's Magic Formula (2). We illustrate this with two examples:

(1) Let $M = \mathbb{R}$, $\omega = f(x)$ be a 0-form, and $v = \frac{\partial}{\partial x}$. Then

$$\mathcal{L}_v \omega = \frac{\partial f}{\partial x} + 0.$$

(2) Let $M = \mathbb{R}$, $\omega = dx$ be a 1-form, and $v = a(x) \frac{\partial}{\partial x}$. Then

$$\mathcal{L}_v \omega = 0 + \frac{\partial a}{\partial x} dx.$$

Now that we have established the necessary tools, we can justify the main calculation appearing in Moser's trick:

$$\begin{aligned} \left. \frac{d}{ds} (\rho_s^* \omega_s) \right|_{s=t_0} &\stackrel{(1)}{=} \left. \frac{d}{ds} (\rho_s^* \omega_{t_0}) \right|_{s=t_0} + \rho_{t_0}^* \left(\left. \frac{d\omega_s}{ds} \right|_{s=t_0} \right) \\ &\stackrel{(2)}{=} \rho_{t_0}^* (\mathcal{L}_{v_s} \omega_{t_0}|_{s=t_0}) + \rho_{t_0}^* \left(\left. \frac{d\omega_s}{ds} \right|_{s=t_0} \right) \\ &\stackrel{(3)}{=} \rho_{t_0}^* \left(\mathcal{L}_{v_s} \omega_{t_0}|_{s=t_0} + \left. \frac{d\omega_s}{ds} \right|_{s=t_0} \right) \\ &\stackrel{(4)}{=} \rho_{t_0}^* \left(\mathcal{L}_{v_{t_0}} \omega_{t_0} + \left. \frac{d\omega_s}{ds} \right|_{s=t_0} \right) \\ &\stackrel{(5)}{=} \rho_{t_0}^* \left(d\iota_{v_{t_0}} \omega_{t_0} + \iota_{v_{t_0}} d\omega_{t_0} + \left. \frac{d\omega_s}{ds} \right|_{s=t_0} \right). \end{aligned}$$

Equality (1) is an instance of the following matrix-calculus rule:

$$\left. \frac{d}{ds} [A(s)v(s)] \right|_{s=t_0} = A'(t_0)v(t_0) + A(t_0)v'(t_0).$$

Equality (2) is the definition of the Lie derivative, equality (3) factors out $\rho_{t_0}^*$, equality (4) follows from Remark 5.4, and equality (5) follows from Cartan's Magic Formula.

5.2. Moser's Theorem.

Theorem 5.8 (Moser). *Let M be a closed smooth manifold, and let $\{\omega_s\}_{s \in [0,1]}$ be a smooth family of symplectic forms such that*

$$[\omega_s] = [\omega_0] \in H^2(M) \quad \text{for all } s \in [0, 1].$$

Then there exists an isotopy $\rho: M \times [0, 1] \rightarrow M$ such that

$$(\rho_s)^* \omega_s = \omega_0 \quad \text{for all } s \in [0, 1].$$

In particular, $\rho_1: (M, \omega_0) \rightarrow (M, \omega_1)$ is a symplectomorphism.

Remark 5.9. The assumption $[\omega_s] = [\omega_0]$ is necessary since ρ_s is isotopic to id_M , and hence $(\rho_s)^*$ acts trivially on cohomology.

Proof. (Moser's Trick) Assume the desired isotopy exists. Then

$$\rho_s^* \omega_s = \omega_0.$$

Differentiating with respect to s and evaluating at $s = t_0$ gives

$$\left. \frac{d}{ds} (\rho_s^* \omega_s) \right|_{s=t_0} = 0.$$

Using the computation above, this becomes

$$0 = \left. \frac{d}{ds} (\rho_s^* \omega_s) \right|_{s=t_0} = \rho_{t_0}^* \left(d\iota_{v_{t_0}} \omega_{t_0} + \iota_{v_{t_0}} d\omega_{t_0} + \left. \frac{d\omega_s}{ds} \right|_{s=t_0} \right).$$

We now use the assumptions:

- (1) Each ω_s is symplectic, hence closed: $d\omega_{t_0} = 0$.
- (2) The cohomology class $[\omega_s]$ is constant with s , that is,

$$\frac{d[w_s]}{ds} = \left[\frac{dw_s}{ds} \right] = 0$$

in $H^2(M)$ and, hence,

$$\left. \frac{d\omega_s}{ds} \right|_{s=t_0} = d\mu_{t_0}.$$

Notice that μ_{t_0} is not unique and, in fact, leads to the non-uniqueness of solution of the isotopy ρ_s . Nonetheless, we can make a choose a smooth family of 1-forms μ_s .

Using these assumptions, we obtain:

$$\begin{aligned} 0 &= \rho_{t_0}^* \left(d\iota_{v_{t_0}} \omega_{t_0} + \iota_{v_{t_0}} d\omega_{t_0} + \left. \frac{d\omega_s}{ds} \right|_{s=t_0} \right) \\ &= \rho_{t_0}^* (d\iota_{v_{t_0}} \omega_{t_0} + 0 + d\mu_{t_0}) \\ &= \rho_{t_0}^* d(\iota_{v_{t_0}} \omega_{t_0} + \mu_{t_0}) \end{aligned}$$

This is what we get by assuming there is a isotopy ρ_{t_0} . Now we turn this on its head by starting with

$$0 = \iota_{v_{t_0}} \omega_{t_0} + \mu_{t_0}$$

and reversing our steps to obtain the required isotopy ρ_{t_0} . Thus, it remains to produce a vector field v_{t_0} such that for each $p \in M$

$$\iota_{v_{t_0}(p)} \omega_{t_0} = -\mu_{t_0}(p).$$

We know that $-\mu_{t_0}(p)$ is a linear functional on $T_p M$, and that $\omega_{t_0}(p)$ is a non-degenerate bilinear form on $T_p M$. Therefore, the map $v \in T_p M \mapsto \omega_{t_0}(p)(v, \cdot) \in (T_p M)^*$ is a bijection, and we can find $v_{t_0}(p)$ as the unique inverse image v of $\omega_{t_0}(p)(v, \cdot) = -\mu_{t_0}(p)$. $\square$

Remark 5.10. A special case is

$$\omega_s = (1 - s)\omega_0 + s\omega_1,$$

for which

$$\frac{d\omega_s}{ds} = \omega_1 - \omega_0$$

is independent of s . In this case, the 1-form μ_s may be chosen independent of s , although the vector field v_s will still depend on s through ω_s .

6. RELATIVE MOSER THEOREM (01/27, AK, JS)

This lecture covers material from [CdS01, Chapters 6.2–6.3, 7.3].

Last time, we studied the **Moser trick**. Let us recall the main ideas: Suppose ω_s is a smooth path of symplectic forms such that the cohomology class $[\omega_s] \in H^2(M)$ is constant with s . This implies that

$$\left. \frac{d\omega_s}{ds} \right|_{s=t_0} = d\mu_{t_0}$$

for some smooth family of 1-forms μ_s . Assuming that the isotopy ρ_s with $\rho_s^*\omega_s = \omega_0$ exists, we derive the equation

$$\iota_{v_{t_0}}\omega_{t_0} = -\mu_{t_0}.$$

Solving this equation gives us a time-dependent vector field v_{t_0} with $\omega_{t_0}(v_{t_0}, \cdot) = -\mu_{t_0}$, and its flow ρ_{t_0} is the desired isotopy.

Now we move on to the **relative Moser theorem**, which is strictly stronger than the previous version and will be used frequently going forward.

Theorem 6.1 (Relative Moser theorem). *Let M be a manifold with compact submanifold $i : X \rightarrow M$. Let ω_0, ω_1 be symplectic forms on M with*

$$\omega_0|_p = \omega_1|_p, \quad \forall p \in X.$$

Then there exists neighborhoods U_0, U_1 of X in M and a diffeomorphism $\varphi : U_0 \rightarrow U_1$ with $\varphi|_X = \text{id}|_X$ and $\varphi^\omega_1 = \omega_0$.*

If $i : X \rightarrow M$ is a submanifold, we can take its normal bundle $\mathcal{N}X$ after fixing a Riemannian metric on X . The *tubular neighborhood theorem* states that there is a neighborhood U_0 of X in $\mathcal{N}X$ which is convex in each fiber, a neighborhood U of X in M , and a diffeomorphism $\psi : U_0 \rightarrow U$ such that the following diagram commutes:

$$\begin{array}{ccc} U_0 & \xrightarrow{\psi} & U \\ \uparrow & \nearrow & \\ X & & \end{array}$$

$U_0 \subset \mathcal{N}X$ and $U \subset M$ are interchangeably called *tubular neighborhoods* of X in M .

Lemma 6.2. *Let $i : X \rightarrow M$ be a connected submanifold and U_0 be a tubular neighborhood of X in M . If $\omega \in \Omega^d(U_0)$ is a closed form such $\omega_x = 0$ for all $x \in X$, then ω is exact, i.e. $\omega = d\mu$ for some $\mu \in \Omega^{d-1}(U_0)$. Moreover, we can choose μ such that $\mu_x = 0$ for all $x \in X$.*

Proof. For every $0 \leq t \leq 1$, define the map

$$\rho_t : U_0 \rightarrow U_0, \quad (x, v) \mapsto (x, tv).$$

So $\rho_1 = \text{id}$, $\rho_0 = i_0 \circ \pi_0$ where $i_0 : X \rightarrow U_0$ is the inclusion and $\pi_0 : U_0 \rightarrow X$ is the projection. Since π_0 is a deformation retraction of U_0 onto X , the maps ρ_0 and ρ_1 induce same maps on cohomology. We build a chain homotopy Q between ρ_0 and ρ_1 , i.e.

$$Q : \Omega^d(U_0) \rightarrow \Omega^{d-1}(U_0)$$

which satisfies

$$(\dagger) \quad \text{id} - (i_0 \circ \pi_0)^* = dQ + Qd.$$

Since $d\omega = 0$ and $i^*\omega = 0$, we get

$$(\text{id} - (i_0 \circ \pi_0)^*)\omega = \omega - (\pi_0^* \circ i_0^*)\omega = \omega;$$

$$(dQ + Qd)\omega = dQ\omega.$$

Hence we have $\omega = dQ\omega = d\mu$ for the $(d-1)$ -form $\mu = Q\omega$.

We will give an explicit chain homotopy map Q which will result in μ vanishing pointwise on X . Define

$$Q\omega = \int_0^1 \rho_t^*(\iota_{v_t}\omega)dt,$$

where the vector v_t at $\rho_t(p)$ is tangent to the path $\rho_s(p)$ at $s = t$. Since $\rho_t|_X = \text{id}_X$ for all t and $\omega_x = 0$ for $x \in X$, v_t vanishes on X and thus $\mu_x = 0$ for all $x \in X$.

So we are done once we verify the chain homotopy property $(\dagger)$ for the map Q defined as above. This is a straightforward computation involving Cartan's magic formula and the definition of the Lie derivative:

$$\begin{aligned} Qd\omega + dQ\omega &= \int_0^1 \rho_t^*(\iota_{v_t}d\omega)dt + d\left(\int_0^1 \rho_t^*(\iota_{v_t}\omega)dt\right) \\ &= \int_0^1 \rho_t^*(\iota_{v_t}d\omega + d\iota_{v_t}\omega)dt \\ &= \int_0^1 \rho_t^*\mathcal{L}_{v_t}\omega dt = \int_0^1 \frac{d}{dt}(\rho_t^*\omega)dt = \rho_1^*\omega - \rho_0^*\omega. \end{aligned}$$

□

Proof of 6.1. Pick a tubular neighborhood U_0 of X . The 2-form $\omega_1 - \omega_0$ is closed on U_0 and $(\omega_1 - \omega_0)_x = 0$ for all $x \in X$. By Lemma 6.2, there exists a 1-form μ on U_0 such that $\mu_x = 0$ for all $x \in X$ and satisfies $\omega_1 - \omega_0 = d\mu$. Taking a path of 2-forms $\omega_t = (1-t)\omega_0 + t\omega_1$, we have

$$\frac{d}{dt}\omega_t = \omega_1 - \omega_0 = d\mu.$$

Now we solve the Moser equation $\iota_{v_t}\omega_t = -\mu$ to obtain a vector v_t such that $\omega_t(v_t, \cdot) = -\mu$. Note that v_t vanishes on X for all t . Following the final steps of the Moser trick, we can integrate v_t (after shrinking U_0 if needed to ensure ω_t is symplectic on U_0 for all

t) to get an isotopy (ρ_t) with $\rho_t^* \omega_t = \omega_0$ and $\rho_t|_X = \text{id}_X$ for all t . Taking $\varphi = \rho_1$ and $U_1 = \rho_1(U_0)$, we are done. $\square$

Now we are in good position to prove Darboux's theorem, which states that all symplectic manifolds are locally symplectomorphic to each other.

Theorem 6.3 (Darboux's theorem). *Let (M, ω) be a symplectic manifold and $p \in M$. Then we can find a coordinates system $(U; x_1, \dots, x_n, y_1, \dots, y_n)$ at p such that ω is standard on U , i.e.*

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i.$$

Proof. Use a symplectic basis for $T_p M$ to construct coordinates $(x'_1, \dots, x'_n, y'_1, \dots, y'_n)$ near p with $\omega_p = \sum_{i=1}^n dx'_i \wedge dy'_i$. Set $\omega_0 = \omega|_{U'}$. Then by the relative Moser theorem with $X = \{p\}$, there are neighborhoods U_0, U_1 of p and a diffeomorphism $\varphi : U_0 \rightarrow U_1$ such that $\varphi(p) = p$ and $\varphi^* \omega' = \omega$, where we take $\omega' = \sum_{i=1}^n dx'_i \wedge dy'_i$ on U_1 . This gives the result with coordinates $x_i = x'_i \circ \varphi$, $y_i = y'_i \circ \varphi$, since

$$\omega = \varphi^* \left(\sum_{i=1}^n dx'_i \wedge dy'_i \right) = \sum_{i=1}^n dx_i \wedge dy_i.$$

$\square$

The relative Moser theorem has the assumption that

$$\omega_0|_p = \omega_1|_p, \quad \forall p \in X$$

which means

$$\omega_0|_p(v, w) = \omega_1|_p(v, w)$$

for all tangent vectors $v, w \in T_p M$. We might like to weaken this to

$$i^* \omega_0 = i^* \omega_1$$

for the inclusion $i : X \rightarrow M$, which is exactly the same statement but for $v, w \in T_p X$. In general, this is too much to ask for. But we will show that it is true if X is Lagrangian for ω_0 and ω_1 , i.e. $i^* \omega_0 = i^* \omega_1 = 0$ and $\dim X = \frac{1}{2} \dim M$.

7. WEINSTEIN'S NEIGHBORHOOD THEOREM (01/29, JS, SC)

7.1. Weinstein's Neighborhood Theorem. In the previous lecture, we saw that the relative Moser theorem requires the strong hypothesis that two symplectic forms agree *pointwise* along a submanifold $X \subset M$, i.e. $\omega_0|_p = \omega_1|_p$ for all $p \in X$. Equivalently, the two forms must agree on all of $T_p M$ for every $p \in X$. In general, this condition cannot be weakened to the equality of pullbacks $i^* \omega_0 = i^* \omega_1$.

However, an important simplification occurs when X is *Lagrangian* for both symplectic forms. Recall that if X is Lagrangian, then $i^* \omega = 0$ and $\dim X = \frac{1}{2} \dim M$. In this case, the symplectic orthogonal satisfies $TX^{\perp \omega} = TX$, so the transverse obstruction disappears. This observation motivates the following theorem, which can be viewed as a strengthening of the relative Moser theorem under a Lagrangian hypothesis.

Theorem 7.1 (Weinstein’s Lagrangian Neighborhood Theorem). *Suppose M is a manifold, $\iota : X \hookrightarrow M$ a submanifold and ω_0, ω_1 symplectic forms on M such that X is Lagrangian on (M, ω_0) and (M, ω_1) . Then there exists neighborhoods U_0, U_1 of X in M and a symplectomorphism from $(U_0, \omega_0|_{U_0})$ to $(U_1, \omega_1|_{U_1})$.*

The proof proceeds by reducing the problem to the relative Moser theorem, after establishing appropriate linear-algebraic preliminaries and the Whitney extension theorem.

We first record two results from symplectic linear algebra.

Proposition 7.2. *Suppose (V, Ω) is a symplectic vector space and $U \subset (V, \Omega)$ a Lagrangian subspace. Let W be any vector space complement to U . Then W canonically determines a Lagrangian complement to U .*

Some context: Consider the space $(\mathbb{R}^4, \text{std})$ with basis e_1, e_2, f_1, f_2 . There are many Lagrangians, and many Lagrangian complements. For instance, the subspace $U = \text{span}(e_1, e_2)$ is Lagrangian, and naturally we see that $L = \text{span}(f_1, f_2)$ is a Lagrangian complement to U ; however, the space $L' = \text{span}(f_1 + e_2, f_2 + e_1)$ is also a Lagrangian complement to U , and there are clearly many others.

Proof. The moral of the argument is that we start with some space W and we “shift” until it becomes Lagrangian. We begin by noting that since U is Lagrangian, then Ω restricts to a non-degenerate pairing $\Omega : U \times W \rightarrow \mathbb{R}$, where $(u, w) \mapsto \Omega(u, w)$. Therefore, the natural map $\tilde{\Omega} : U \rightarrow W^*$, with $u \mapsto \Omega(u, \cdot)$ is an isomorphism. And so, on one hand we have that each $w \in W$ determines an element of W^* via $w' \mapsto \Omega(w, w')$. On the other hand, since the restriction Ω is non-degenerate we also have $U \simeq W^*$. Hence, there exists a unique $u_w \in U$ such that $\Omega(u_w, w') = \Omega(w, w')$ for all $w' \in W$. Now, we compute for $w, y \in W$

$$\begin{aligned} \Omega\left(w - \frac{u_w}{2}, y - \frac{u_y}{2}\right) &= \Omega(w, y) - \frac{1}{2}\Omega(u_w, y) + \Omega\left(w, -\frac{u_y}{2}\right) + \Omega\left(\frac{u_w}{2}, \frac{u_y}{2}\right) \\ &= \Omega(w, y) - \frac{1}{2}\Omega(u_w, y) - \frac{1}{2}\Omega(w, u_y) \\ &= \Omega(w, y) - \frac{1}{2}\Omega(w, y) - \frac{1}{2}\Omega(w, y) = 0, \end{aligned}$$

where the second equality follows from the fact that elements of U pair to 0 since U is Lagrangian. And so, we can take the complement to be $\{w - \frac{1}{2}u_w : w \in W\}$. $\square$

Another proposition that we need is the following, where in our context, the vector spaces V will be the corresponding tangent spaces.

Proposition 7.3. *Let V be a vector space and Ω_0, Ω_1 be linear symplectic forms on V . Let U be a subspace of V Lagrangian to both (V, Ω_0) and (V, Ω_1) . Let W be a complement to U in V . Then from W we can canonically construct a linear isomorphism $L : V \rightarrow V$ such that $L|_U = \text{id}$ and $L^*(\Omega_1) = \Omega_0$.*

Proof. The proof was not given in class, but it is in the textbook. $\square$

We will also use the following classical result, which we state without proof.

Theorem 7.4 (Whitney Extension Theorem). *Let M be a manifold, $\iota : X \hookrightarrow M$ a submanifold, and $\dim(X) < \dim(M)$. Suppose at each $p \in X$ we are given a linear isomorphism $L_p : T_p M \rightarrow T_p M$ such that $p \mapsto L_p$ is smooth and $L_p|_{T_p X} = \text{id}$ for all $p \in X$. Then there exists a neighborhood $\mathcal{N}$ of X and an embedding $h : \mathcal{N} \rightarrow M$ such that $h|_X = \text{id}$ and $dh_p = L_p$ for all $p \in X$.*

In our context, this theorem allows us to extend pointwise linear identifications near X to genuine local diffeomorphisms. The proof is standard and was skipped in class.

Proof. (Weinstein's Neighborhood Theorem) Pick for each $p \in X$ a complementary space W_p to $T_p X$ in $T_p M$ such that $p \mapsto W_p$ is smooth; e.g. we can take $W_p = (T_p X)^\perp$ using some Riemannian metric. By the two previous linear algebra propositions and Whitney's extension theorem, there exists a neighborhood $\mathcal{N}$ of X and $h : \mathcal{N} \rightarrow M$ with $h|_X = \text{id}$ and $dh_p = L_p$ for all $p \in X$, where L_p is produced as in the previous proposition. We then have

$$(h^* \omega_1)_p = (dh_p)^* \omega_1|_p = (L_p^*) \omega_1|_p = \omega_0|_p.$$

Now apply Relative Moser to $\omega_0, h^* \omega_1$ to get a neighborhood U_0 of X and a $f : U_0 \rightarrow \mathcal{N}$ such that $f|_X = \text{id}$ and $f^*(h^* \omega_1) = \omega_0$. Then $h \circ f$ is the desired symplectomorphism from $(U_0, \omega_0) \rightarrow (U_1, \omega_1)$. $\square$

Note that the heart of the argument is just Relative Moser, and the main difference was at the linear algebra level.

Remark: There is a slightly more general version of this result called the Coisotropic Embedding Theorem. It says that Weinstein is true under a weaker assumption that $T_p X$ is coisotropic for ω_0, ω_1 , for all p (i.e. there is no $v \in T_p M \setminus T_p X$ orthogonal to $T_p X$). One still needs equality of pullbacks.

One of the main corollaries of Weinstein is the following result, which will be discussed in the upcoming lectures.

Corollary 7.5. *If $L \subset (M, \omega)$ is Lagrangian, there exists a neighborhood $\mathcal{N}$ of L and a symplectomorphism from $(\mathcal{N}, \omega|_{\mathcal{N}})$ to a neighborhood of 0 in T^*L .*

7.2. Applications of Weinstein. We are interested in studying the tangent space (at the identity) to the group

$$\text{Symp}(M, \omega) = \{\text{Symplectomorphisms } M \rightarrow M\}.$$

It is clear that this is indeed a group, and we should think of it as being infinite-dimensional. Furthermore we also have to be careful about working with the right underlying topology; to that end, we begin by stating the definitions of C^0, C^1 topologies.

Definition 7.6. (1) A sequence of maps $f_i : X \rightarrow Y$ is said to be **C^0 -convergent** if it converges uniformly on compact sets.

(2) A sequence of maps $f_i : X \rightarrow Y$ is said to be **C^1 -convergent** if the sequence $df_i : TX \rightarrow TY$ C^0 -converges.

Let (M, ω) be a compact symplectic manifold and $f \in \text{Symp}(M, \omega)$. Previously, we have seen that $\text{Graph}(f)$ and $\Delta := \text{Graph}(\text{id})$ are Lagrangian in $(M \times M, \tilde{\omega})$. The idea now is to pick f close to id so that $\text{Graph}(f)$ is close to $\text{Graph}(\text{id})$ which is Lagrangian; the upshot being that we are then in situation we can handle via closed 1-forms. Indeed, by Corollary 7.5, there exists a neighborhood U of $\Delta \simeq M$ in $(M \times M, \tilde{\omega})$ which is symplectomorphic to a neighborhood U_0 of the zero section in T^*M via $\varphi : U \rightarrow U_0$ with $\varphi(p, p) = (p, 0)$.

In particular, suppose f is C^1 -close to id :

- (1) Then $\text{Graph}(f) \subset U$ and let $j : M \hookrightarrow U$, where $m \mapsto (p, f(p))$, be the embedding of the graph and $\iota : M \hookrightarrow U$ with $p \mapsto (p, p)$.
- (2) ι and j are C^1 -close.
- (3) Compose with φ to get $j_0 : M \hookrightarrow U_0$ and $\iota_0 : M \rightarrow U_0$, where $j_0 = \varphi \circ j$ and $\iota_0 = \varphi \circ \iota = \text{Zero Section}$.
- (4) j_0 being C^1 -close enough to ι_0 implies that $j_0(M)$ intersects each fiber of T_p^*M at one pt μ_p and $p \mapsto \mu_p$ is smooth.
- (5) $j_0(M)$ is the image of the smooth section $\mu : M \rightarrow T^*M$; i.e. it is a 1-form, so $\text{graph}(f) = \varphi^{-1}(\{(p, \mu_p)\})$.

And recall that f is a symplectomorphism if and only if μ is closed. Therefore, we conclude that a C^1 -neighborhood of id in the group $\text{Symp}(M, \omega)$ is homeomorphic to a C^1 -neighborhood of 0 in the vector space of closed 1-forms. Whence, it is natural to identify

$$T_{\text{id}}(\text{Symp}(M, \omega)) \simeq \{\text{closed 1-forms}\}.$$

8. WEINSTEIN TUBULAR NEIGHBORHOODS (02/03, SC, AW)

This lecture covers material from [CdS01, Chapter 9]. For today, let M be a submanifold of $\mathbb{R}^n$, so that M has a natural Riemannian metric that it inherits from that on $\mathbb{R}^n$.

Lemma 8.1. *If $X \subseteq (M, \omega)$ is Lagrangian, then $T^*X \cong NX$ as smooth vector bundles on X .*

Proof. There is a natural map

$$NX \longrightarrow T^*X \quad \text{given by} \quad (x, v) \longmapsto (x, \omega(v, \cdot)).$$

This is clearly a bundle map and is fibre-wise injective, for if there is a $v \in (NX)_x$ such that $\omega(v, \cdot)$ is identically zero, then it would be possible to enlarge the isotropic subspace $T_x X \subseteq T_x M$, which is absurd since $T_x X$ is Lagrangian in $T_x M$ with respect to $\omega(\cdot, \cdot)$. Finally, since both NX and T^*X are bundles of the same rank, this map must be an isomorphism. $\square$

Recall the Tubular Neighborhood Theorem:

Theorem 8.2 (Tubular Neighborhood Theorem). *Let $\iota : X \hookrightarrow M$ be a submanifold, and $\iota_0 : X \hookrightarrow NX$ be the zero-section. Then there exist neighborhoods U_0 of X in NX and U of X in M , and a diffeomorphism $\varphi : U_0 \rightarrow U$ with $\iota = \varphi \circ \iota_0$.*

Combining the last two results with Weinstein's theorem from last time, we have:

Theorem 8.3 (Weinstein Tubular Neighborhood Theorem). *Suppose (M, ω) is a symplectic manifold, $\iota: X \hookrightarrow M$ a compact Lagrangian submanifold, and $\iota_0: X \hookrightarrow T^*X$ the zero-section. Then there exists a neighborhood U_0 of X in T^*X , a neighborhood U of X in M , and a symplectomorphism $\varphi: (U_0, \omega_0) \rightarrow (U, \omega)$ with $\varphi \circ \iota_0 = \iota$, where ω_0 is the canonical symplectic form on T^*X .*

Proof. Since $T^*X \cong NX$, the Tubular Neighborhood Theorem gives neighborhoods $\mathcal{N}_0$ of X in T^*X and $\mathcal{N}$ of X in M and a diffeomorphism $\psi: \mathcal{N}_0 \rightarrow \mathcal{N}$ such that $\psi \circ \iota_0 = \iota$, where we use ι_0 to denote the zero-section of the cotangent bundle as well.

Let $\omega_1 = \psi^*\omega$. Using Weinstein's Lagrangian Neighborhood Theorem (7.1), we can find neighborhoods U, U_1 of X in $\mathcal{N}_0$, and a diffeomorphism $\theta: U \rightarrow U_1$ such that $\theta^*\omega_1 = \omega_0$. Finally, set $\varphi = \psi \circ \theta$, and shrink neighborhoods appropriately if necessary to conclude. $\square$

Remark 8.4. This theorem shows that cotangent bundles are important. Even if one is primarily interested in compact manifolds, they cannot escape the non-compact case!

Last time we saw the following application of Theorem 7.1:

Proposition 8.5. *A small C^1 neighborhood of id in $\text{Symp}(M, \omega)$ is homeomorphic to a C^1 neighborhood of 0 in the vector space of closed 1-forms.*

We take a look at another application involving fixed points today.

Theorem 8.6. *Let (M, ω) be a compact symplectic manifold with $H_{\text{dR}}^1(M) = 0$. Then any symplectomorphism of M that is C^1 -close to the identity has at least two fixed points.*

Proof. Let $f \in \text{Symp}(M, \omega)$ be C^1 -close to the identity. Then $\text{graph}(f)$ corresponds to a closed 1-form μ on M due to Proposition 8.5. Since $H_{\text{dR}}^1(M) = 0$, μ is exact, that is, $\mu = dh$ for some $h \in C^\infty(M)$. Further, since M is compact, h has at least two critical points corresponding to the point of global maximum and the point of global minimum. Consequently, $\mu = dh$ is zero at at least two points. Hence, $\text{graph}(f)$ intersects the diagonal in $M \times M$ at least twice, so that f has at least two fixed points. $\square$

Example 8.7. Any $f \in \text{Symp}(S^2)$ that is C^1 -close to the identity has at least two fixed points. On the other hand, consider $\text{Symp}(S^1 \times S^1)$, where $H_{\text{dR}}^1 \neq 0$. In this case a symplectomorphism inducing a tiny rotation need not have any fixed points.

Lagrangian Intersection Problem.

Definition 8.8. A submanifold $\iota_Y: Y \hookrightarrow M$ is C^1 -close to a submanifold $\iota_X: X \hookrightarrow M$ if there is a diffeomorphism $\varphi: X \rightarrow Y$ such that $\iota_Y \circ \varphi$ is C^1 -close to ι_X .

Theorem 8.9. *Let (M, ω) be a symplectic manifold. Suppose $X \hookrightarrow M$ is a compact Lagrangian submanifold with $H_{\text{dR}}^1(X) = 0$, then every Lagrangian submanifold of M which is C^1 -close to X intersects it in at least two points.*

Proof. Note that the following is the same proof as that of Theorem 8.6. X has a neighborhood that is a neighborhood of the zero-section in T^*X . Write the C^1 -close Lagrangian submanifold as the image of a closed 1-form μ on X . Again, $\mu = dh$ for

some $h \in C^\infty(X)$, which has at least two critical points, whence the section (of the cotangent bundle) corresponding to μ admits at least two intersection points with the zero-section. $\square$

Definition 8.10. $f \in \text{Symp}(M, \omega)$ is **exactly homotopic to the identity** if it is the time 1 map ρ_1 of an isotopy (ρ_t) generated by a time-dependent vector field v_t satisfying $\iota_{v_t}(\omega) = dh_t$ for some smooth family of functions h_t .

Remark 8.11. Most people would say (ρ_t) is a *Hamiltonian flow*, and ρ_1 is Hamiltonian. The “exact” in the name comes from the exactness of the form dh_t . These generate a “finite codimension” subgroup $\text{Ham}(M, \omega) \subseteq \text{Symp}(M, \omega)$

The Arnold Conjecture. Let (M, ω) be a compact symplectic manifold, $f: M \rightarrow M$ be exactly homotopic to the identity. Then

$\# \text{Fix}(f) \geq$ minimum number of critical points of a smooth function $h \in C^\infty(M)$.

If we assume the fixed points are non-degenerate, we get

$$\begin{aligned} \# \text{Fix}(f) &\geq \text{minimum number of critical points of a Morse function } h \in C^\infty(M) \\ &\geq \sum_i \dim H_{\text{dR}}^i(M). \end{aligned}$$

Special Case 1. Assuming f is C^1 -close to the identity, we have proved it above.

Special Case 2. Flow in solution to $\iota_{v_t}(\omega) = dh$ but with h not time-dependent, has also been proved above.

$$\begin{aligned} \{\text{critical points of } h\} &\longleftrightarrow \{\text{zeros of the vector field}\} \\ &\longleftrightarrow \{\text{fixed points of the flow}\} \\ &\subseteq \{\text{fixed points of time 1 map}\}. \end{aligned}$$

Contact Manifolds. Note that a symplectic manifold (M, ω) is even dimensional. What about odd-dimensional manifolds? In particular, if one has a symplectic manifold with boundary (whatever that means), what structure does its boundary inherit? This motivates the following

Definition 8.12. A **contact element** on a manifold M is a point $p \in M$, called the **contact point**, together with a hyperplane² $H_p \subseteq T_p M$.

Note that the set of hyperplanes in a vector space V corresponds to the projectivisation of the dual space, that is,

$$\mathbb{P}V^* = (V^* \setminus \{0\}) / \mathbb{R}^\times,$$

where an element $[\alpha] \in \mathbb{P}V^*$ corresponds to the hyperplane $\ker \alpha$.

Suppose H is a smooth field of contact elements on M , that is $H: p \mapsto H_p \subseteq T_p M$. Then H can be locally defined as the “kernel” of some smooth 1-form α , i.e., $H_p = \ker \alpha_p$. This smooth form α is called a **locally defining 1-form**. Note that α is not unique, but is defined up to multiplication by a smooth (nowhere-vanishing) function.

²a linear subspace of codimension 1

9. CONTACT STRUCTURES (02/05, RB, ZH)

From last time, we recall Definition 8.12: a contact element on a manifold is a pair (p, H_p) with $p \in M$, $H_p \subset T_p M$ a hyperplane. A smoothly varying family of contact elements $p \mapsto H_p \subset T_p M$ is given locally by a 1-form α with $H = \ker \alpha$ (the locally defining 1-form).

Definition 9.1. A **contact structure** on a manifold M is a smooth field of contact elements H such that for any locally defining 1-form α , $(d\alpha)_p|_{H_p}$ is a symplectic form on H_p . In this case α is called a (local) contact form. The pair (M, H) is called a **contact manifold**.

Remark 9.2.

- (1) At each point $p \in M$ we can choose a splitting

$$T_p M = H_p \oplus L$$

with $H_p = \ker \alpha_p$ and L one-dimensional. Since H_p admits a symplectic form, it is even dimensional, so $T_p M$ (and M) must be odd-dimensional. The bilinear form $(d\alpha)_p$ is degenerate since $\dim T_p M$ is odd, and by assumption on H_p we have that $\ker(d\alpha)_p = \{v \in T_p M \mid (d\alpha)_p(v, w) = 0 \forall w \in T_p M\}$ is one dimensional. So we can arrange the splitting to be given by $L = \ker(d\alpha)_p$ (here we use that $(d\alpha)_p|_{H_p}$ is non-degenerate since it is symplectic). A warning is that $\ker(d\alpha)_p$ is not canonical: it depends on the choice of α , and changes under replacing α with $f\alpha$ for f a nowhere vanishing function on M .

- (2) Note also that α must be non-zero on any 1-dimensional complement L above.
 (3) The condition that $(d\alpha)_p|_{H_p}$ is symplectic is equivalent to $(d\alpha)_p^n|_{H_p} \neq 0$ being a volume form on H_p , where $\dim M = 2n + 1$.
 (4) A global contact form exists if and only if the line bundle TM/H is trivial. The idea is that if the quotient has a nowhere vanishing section s , we can replace any locally defining form α with $\alpha_{\text{new}} := \alpha/\alpha(s)$. Since $\alpha_{\text{new}}(s) = 1$ these forms all glue.

For now we will assume that there is a global contact form.

Proposition 9.3. *Let H be a field of tangent hyperplanes defined by a 1-form α , so that $H_p = \ker \alpha_p$. Then H is a contact structure if and only if $\alpha \wedge (d\alpha)^n$ is a volume form.*

Remark 9.4. This condition should be viewed as the form α being “generic everywhere,” similar to how the non-degeneracy condition for a 2-form holds generically.

Proof. Since α defines a hyperplane field, it is everywhere non-zero. Both conditions are local so it suffices to assume there is a local trivialization $e_1, f_1, e_2, f_2, \dots, e_n, f_n, r$ of TM where $e_i, f_i, \dots, e_n, f_n$ forms a basis of $H = \ker \alpha$ pointwise, and $\dim M = 2n + 1$. Note that by definition we have $\alpha(r) \neq 0$ and $\alpha(e_i) = \alpha(f_i) = 0$, and so

$$\alpha \wedge (d\alpha)^n(e_1, f_1, \dots, e_n, f_n, r) = \pm \alpha(r)(d\alpha)^n(e_1, f_1, \dots, e_n, f_n).$$

Now $\alpha \wedge (d\alpha)^n$ is a volume form if and only if it is nonzero when evaluated on the above basis, which by the equality holds if and only if $(d\alpha)^n|_H \neq 0$, i.e. if and only if $(d\alpha)|_H$ is symplectic. $\square$

The following provides motivation for the definition of a contact structure.

Lemma 9.5. *Let N be a manifold of dimension $2n + 1$. Then a 1-form α on N is a contact form on N if and only if $d(e^t\alpha)$ is a symplectic form on $N \times \mathbb{R}$, where t is the coordinate on $\mathbb{R}$.*

Proof. We calculate

$$(3) \quad d(e^t\alpha) = e^td\alpha + e^tdt \wedge \alpha = e^t(d\alpha + dt \wedge \alpha).$$

As we will see below, we would actually like to work with $d\alpha + dt \wedge \alpha$, but it isn't clear this form is closed, which is why we multiply by e^t . Using the fact that $\wedge$ is commutative for even forms, and that $dt \wedge dt = 0$, we have

$$(d(e^t\alpha))^{n+1} = e^{(n+1)t} \sum_{i+j=n+1} \binom{n+1}{i} (d\alpha)^i \wedge (dt \wedge \alpha)^j = (n+1)e^{(n+1)t} (d\alpha)^n \wedge dt \wedge \alpha.$$

This is non-zero (i.e. $d(e^t\alpha)$ is non-degenerate) if and only if $\alpha \wedge (d\alpha)^n$ is nonzero, which is exactly the condition for a contact structure (note that since $d(e^t\alpha)$ is exact, $d(e^t\alpha)$ is closed). $\square$

Say (N, α) is a contact manifold with defining 1-form α and $\dim N = 2n + 1$. If $L_0 \subset N$ is a submanifold, we set

$$L = L_0 \times \mathbb{R} \subseteq (N \times \mathbb{R}, d(e^t\alpha)).$$

Motivated by the Lagrangian Creed, we are led to the following

Question: When is L isotropic in $N \times \mathbb{R}$?

Answer: Note that $\frac{\partial}{\partial t} \in T_{(p,t)}L$ for any $p \in L_0$ and $t \in \mathbb{R}$. Thus for any $v \in T_pL_0$, we need $d(e^t\alpha)(\frac{\partial}{\partial t}, v) = 0$. Using (3) and $dt(\frac{\partial}{\partial t}) = 1$ we compute that

$$d(e^t\alpha) \left(\frac{\partial}{\partial t}, v \right) = e^t\alpha(v).$$

So we require $T_pL_0 \subseteq H_p = \ker(\alpha_p)$ for all $p \in L_0$. Also, if $v, w \in T_pL_0$, we require

$$0 = d(e^t\alpha)(v, w) = e^td\alpha(v, w),$$

and so we must have $d\alpha$ is zero on L_0 . To sum up, if $\iota : L_0 \rightarrow N$ is the embedding, a necessary and sufficient condition for L to be an isotropic subspace is (a) $\iota^*\alpha = 0$ and (b) $\iota^*d\alpha = 0$ (for the converse, just trace the equalities backwards). Note that in addition L is Lagrangian if additionally $\dim L_0 = n$ (equivalently if $\dim L = \frac{1}{2}(\dim N \times \mathbb{R}) = n + 1$).

Remark 9.6.

- (1) The naturality of d gives (a) $\implies$ (b) above. So it suffices to assume that $T_pL_0 \subseteq H_p$ for all $p \in N$.
- (2) We define $L_0 \subseteq (N, H)$ to be **Legendrian** if L defined above is Lagrangian, i.e. if $T_pL_0 \subseteq H_p$ for all $p \in L_0$ and $\dim L_0 = n$.
- (3) The above discussion shows that any submanifold L_0 of N everywhere tangent to H (meaning $T_pL_0 \subseteq H_p$) has $\dim L_0 \leq n$. Since $\dim H_p = 2n$, this might be surprising! This means that H is “maximally non-integrable”.

In what follows we will describe some contact manifolds without proof.

Example 9.7. Set $N = \mathbb{R}^{2n+1}$ with coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z)$, and defining 1-form $\alpha = dz - \sum_{i=1}^n y_i dx_i$. This is known as the **standard contact structure**. The below figure displays the hyperplane field $H = \ker \alpha$ on the plane $z = 0$ in the case $n = 1$.

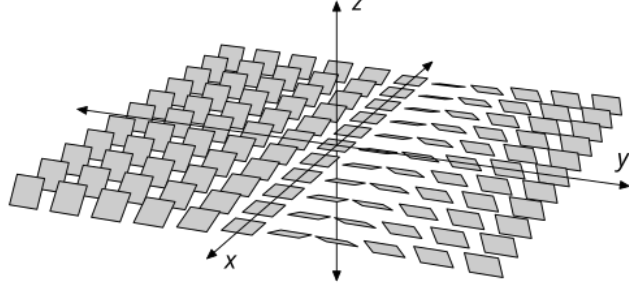


FIGURE 6. The standard contact plane field on $\mathbb{R}^3$.

Note that in this case one can see the fact that H is non-integrable: a small counter-clockwise loop starting at the origin initially moving in the positive x -direction would arrive back at the origin with negative z -coordinate.

Example 9.8. We can take $N = \mathbb{S}^{2n-1}$ with defining 1-form α to be the restriction of the Liouville form $\sum_{i=1}^n x_i dy_i - y_i dx_i$ where $\mathbb{R}^{2n} \supset \mathbb{S}^{2n-1}$ has coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$.

Example 9.9. For any M , the projectivized cotangent bundle $\mathbb{P}(T^*M)$ has a contact structure.

Example 9.10. Every compact orientable 3-manifold has a contact structure (this is not an easy theorem).

In general, it is interesting to classify contact structures on a given manifold.

From now, we will study Example 9.7 in the case $n = 1$ (also the case depicted in Figure 6). Explicitly, this is $(\mathbb{R}^3, \alpha = dz - ydx)$. A calculation shows that

$$H_{(x,y,z)} = \text{span} \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y \frac{\partial}{\partial z} \right).$$

Remark 9.11. Note that $H_{(x,y,z)}$ is invariant under translation in the x and z directions. On lines parallel to the y -axis, the hyperplanes H “spin”. Compare with Figure 6.

A Legendrian submanifold in $(\mathbb{R}^3, dz - ydx)$ has dimension 1. If it is compact and connected, we call it a **Legendrian knot**.

If $\phi : S^1 \rightarrow \mathbb{R}^3$ is a submanifold given by $\phi(t) = (x(t), y(t), z(t))$, being Legendrian amounts to the condition

$$0 = (dz - ydx)(\phi') = z'(t) - y(t)x'(t),$$

or equivalently (if $x'(t) \neq 0$)

$$(4) \quad y(t) = \frac{z'(t)}{x'(t)}.$$

In words, this says that “the y -coordinate is the slope in the x - z plane.”

Example 9.12. In this example and the next, we replace S^1 with $\mathbb{R}$ for convenience. Define $(x(t), y(t), z(t)) = (\frac{1}{2}t^2, t, \frac{1}{3}t^3)$. One can verify that Equation (4) is satisfied by differentiating. If we project onto the xz plane, we get the cuspidal cubic curve $8x^3 = 9z^2$ depicted below. Note that we have a cusp in the projection but not in $\mathbb{R}^3$: one can check that $y'(0) = 1$ and so we have an embedded submanifold.

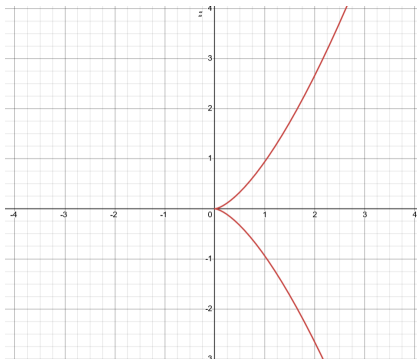


FIGURE 7. The equation $8x^3 = 9z^2$ in the xz -plane

Example 9.13. Fix a parameter $a \in \mathbb{R}$. We define

$$\begin{aligned} x(t) &= t + a(t^3 - 3t) \\ y(t) &= t \\ z(t) &= \frac{1}{2}t^2 + a\left(\frac{3}{4}t^4 - \frac{3}{2}t^2\right). \end{aligned}$$

We leave it to the reader to verify condition (4). We don’t know why this is cool yet, but we will next time. As we vary a , we get a Reidemeister move! This will tell us about Legendrian knots.

10. LEGENDRIAN KNOTS (02/10, MN, AK)

A good reference for this lecture is [Etn05, Sections 1, 2.1-2.3, 2.4-2.7].

The goal is to study closed 1-dimensional Legendrian submanifolds (i.e. Legendrian knots) in $\mathbb{R}^3$, equipped with its standard contact structure. We will see that Legendrian knots are simultaneously flexible and rigid: flexible enough that there exist lots of examples, yet rigid enough to support invariants that go beyond those of smooth knot theory.

Legendrian knots can be viewed as the contact analogues of Lagrangian submanifolds in symplectic geometry. A major advantage of working in the 1-dimensional setting is

that one can write down explicit equations and directly visualize many of the phenomena that arise.

10.1. Setup. The standard contact structure on $\mathbb{R}^3$ is given by the contact form

$$\alpha = dz - y dx,$$

which satisfies

$$\alpha \wedge d\alpha = dz \wedge dx \wedge dy \neq 0.$$

The associated completely non-integrable contact plane field is

$$H_{(x,y,z)} = (\ker \alpha)_{(x,y,z)} = \text{span} \left\{ \frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y \frac{\partial}{\partial z} \right\}.$$

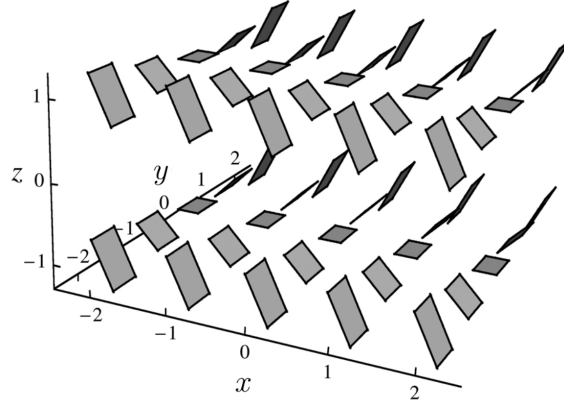


FIGURE 8. The standard contact plane field on $\mathbb{R}^3$. Image reproduced from [Sab09].

Definition 10.1.

- (1) A **Legendrian knot** in $\mathbb{R}^3$ is an embedding

$$\phi: S^1 \rightarrow \mathbb{R}^3$$

such that ϕ is everywhere tangent to the contact plane field H , i.e.

$$\phi'(t) \in H_{\phi(t)} \quad \forall t \in S^1.$$

- (2) Two Legendrian knots L_0 and L_1 are said to be **Legendrian isotopic** if there is a continuous family L_t of Legendrian knots starting at L_0 and ending at L_1 .³

Unlike smooth knots in $\mathbb{R}^3$, Legendrian knots are constrained to lie inside the twisting, non-integrable plane field H , and therefore exhibit additional geometric rigidity. However, as in the smooth category, reparametrizations of Legendrian knots remain Legendrian.

³By the Contact Isotopy Extension Theorem (see Theorem 2.12 of [Etn05]), classifying Legendrian knots up to Legendrian isotopy is equivalent to classifying them up to ambient contact isotopy, despite the latter being *a priori* a stronger equivalence.

⁴3D visualization available at: <https://public.websites.umich.edu/~alexmw/LU.html>

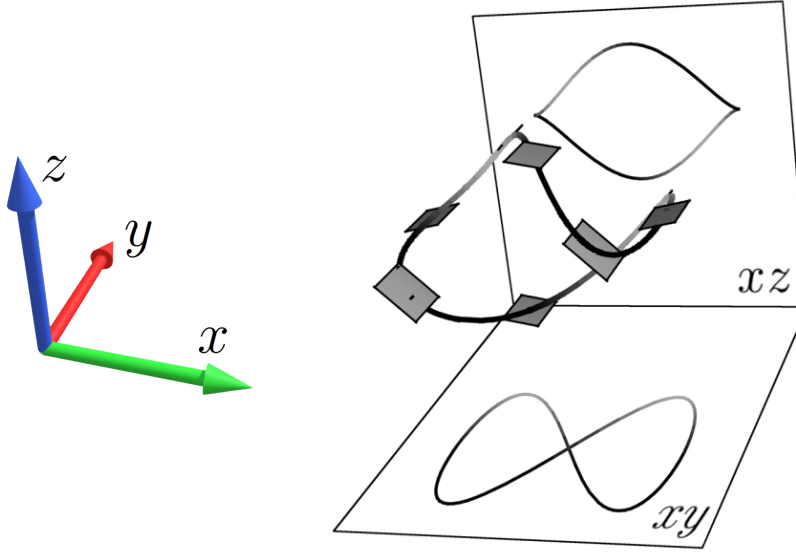


FIGURE 9. A Legendrian unknot,⁴ with its front projection in back and its Lagrangian projection below. The explicit parametrization is $(x(t), y(t), z(t)) = (\cos(t), \sin(2t), -\frac{2}{3}\sin^3(t))$. Image reproduced from [Sab09].

To write down explicit equations of a Legendrian knot, we parametrize the knot as

$$\phi(t) = (x(t), y(t), z(t)),$$

and observe the condition:

$$\begin{aligned} \phi \text{ is Legendrian} &\iff \alpha(\phi'(t)) = 0 \quad \forall t \\ (5) \qquad \qquad \qquad &\iff z'(t) - y(t)x'(t) = 0 \quad \forall t. \end{aligned}$$

Assuming a mild genericity condition⁵, the y -coordinate is determined by the slope of the projection onto the (x, z) -plane:

$$(6) \qquad y(t) = \frac{z'(t)}{x'(t)} \quad \text{if } x'(t) \neq 0, \quad \text{and} \quad y(t) = \lim_{s \rightarrow t} \frac{z'(s)}{x'(s)} \quad \text{if } x'(t) = 0.$$

Definition 10.2. The projection of a Legendrian knot L to the (x, z) -plane is called the **front projection** of L . Explicitly, if L is parametrized by ϕ , the front projection is the map

$$\begin{aligned} \phi_{\Pi}: S^1 &\rightarrow \mathbb{R}_{x,z}^2 \\ t &\mapsto (x(t), z(t)). \end{aligned}$$

Lemma 10.3 (Properties of front projection). *Suppose L is a Legendrian knot satisfying the genericity condition in Footnote 5. Then its front projection ϕ_{Π} satisfies:*

- (1) ϕ_{Π} is a smooth immersion except at finitely many points (generalized cusps);
- (2) The image of ϕ_{Π} has a well-defined tangent line everywhere;

⁵Namely, that $x'(t)$ vanishes only at finitely many points of S^1 .

- (3) The image of ϕ_Π has no vertical tangent lines;
- (4) At each self-intersection of ϕ_Π , the strand with the higher slope has the higher y -coordinate (“under strand”).

Proof. The properties follow from the smoothness of the original knot ϕ :

- (1) By the genericity assumption, $x'(t)$ vanishes finitely often. At points where $x'(t) \neq 0$, the map is an immersion. Where $x'(t) = 0$, equation (6) implies $z'(t) = y(t)x'(t) = 0$. Thus, the derivative of the projection vanishes only at these finitely many points (singularities).
- (2) Since L is a smooth knot, $y(t)$ is a continuous function. By equation (6), the slope of the curve in the (x, z) -plane corresponds exactly to $y(t)$. Since $y(t)$ exists everywhere, the geometric tangent line is well-defined everywhere.
- (3) The slope of ϕ_Π at t is given by $y(t) \in \mathbb{R}$. Since $y(t)$ is finite everywhere, the slope is never infinite. Hence, there are no vertical tangent lines.
- (4) Let the self-intersection occur at parameter values $t_1 \neq t_2$. Since the knot is embedded in $\mathbb{R}^3$, we must have $y(t_1) \neq y(t_2)$. Since $y(t)$ is the slope of the front projection, the strand with the larger slope corresponds to the larger y -value.

□

10.2. Flexibility of Legendrian Knots. The goal of this section is to show that every smooth knot in $\mathbb{R}^3$ can be isotoped to a Legendrian knot. Recall that smooth knots in $\mathbb{R}^3$ can be faithfully studied through their projections to 2-planes, called *knot diagrams*. Furthermore, two smooth knots are smoothly isotopic in $\mathbb{R}^3$ if and only if their knot diagrams can be related by a sequence of the three *Reidemeister moves*.

There is a parallel theory for Legendrian knots:

Theorem 10.4.

- (1) Legendrian knots can be faithfully studied through **front diagrams**.
- (2) Two Legendrian knots are Legendrian isotopic if and only if their front diagrams can be related by a sequence of the three **Legendrian Reidemeister moves**.

Moreover, any knot diagram can be perturbed into a front diagram, establishing:

Corollary 10.5. Every smooth knot in $\mathbb{R}^3$ can be isotoped to a Legendrian knot.

Definition 10.6.

- (1) A **front diagram** is a knot diagram in the (x, z) -plane with the following modifications, motivated by Lemma 10.3:
 - (a) The diagram is smooth everywhere except at finitely many points (at least two). At these non-smooth points, the diagram has a generalized cusp.⁶

⁶Formally, a generalized cusp is a singularity that is locally diffeomorphic to the map $t \mapsto (t^2, t^3)$ around $t = 0$.

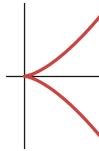




FIGURE 10. Knot diagram of smooth unknot and trefoil knot.

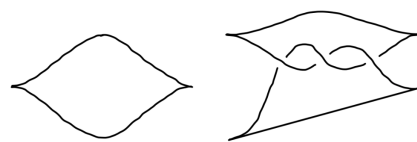


FIGURE 11. Front diagram of Legendrian unknot and trefoil knot.

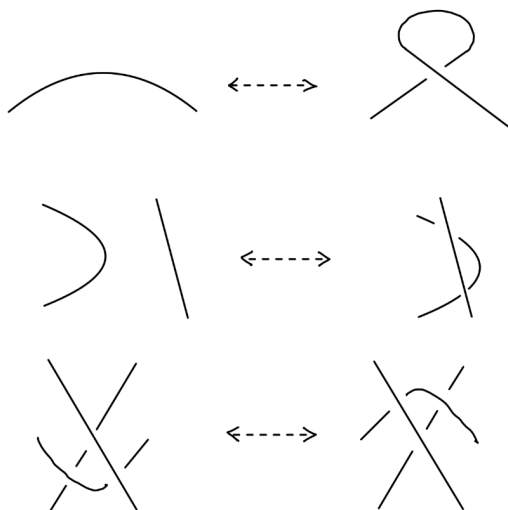


FIGURE 12. Smooth Reidemeister moves.

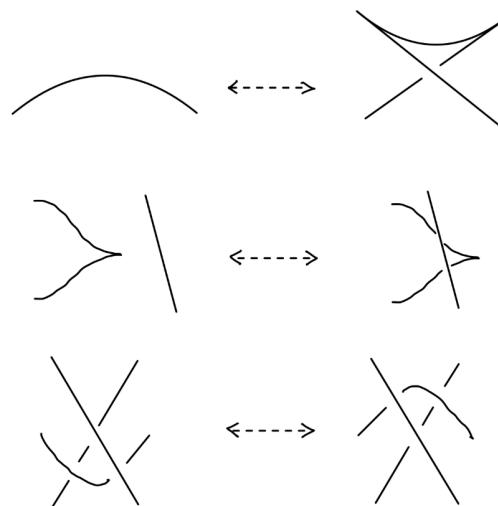


FIGURE 13. Legendrian Reidemeister moves.

- (b) There are no vertical tangencies.
 - (c) At each crossing, the strand with the *higher* slope goes *under* the strand with the *lower* slope.
- (2) The three **Legendrian Reidemeister moves** are depicted in Figure 13.

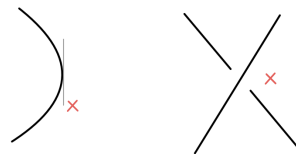
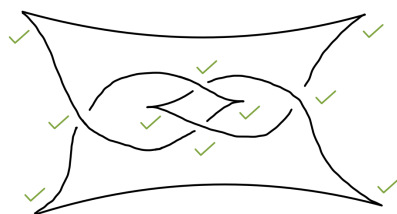


FIGURE 14. Left: The front diagram of the Legendrian figure eight knot. Right: Disallowed configurations in a front diagram.

Remark 10.7 (Understanding the crossing rule, i.e. condition (c)). From Lemma 10.3, we know that the strand with the higher slope has the higher y -coordinate. However, we must consider the orientation of our axes. As shown in Figure 15, we view the (x, z) -plane such that the positive y -axis points *away* from the viewer (into the page). Therefore:

Higher Slope $\implies$ Higher $y \implies$ Further Back $\implies$ Under Crossing.

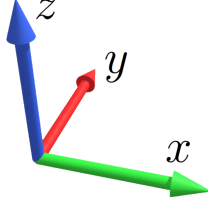


FIGURE 15. Orientation of the axes: the y -axis goes *under* the (x, z) -plane.

Proof of Theorem 10.4.

- (1) Lemma 10.3 shows that the front projection of any Legendrian knot is a front diagram. In the other direction, given a front diagram, we can parametrize it as $(x(t), z(t))$ and recover the y -coordinate by defining it to be the slope:

$$y(t) := \begin{cases} \frac{z'(t)}{x'(t)} & \text{if } x'(t) \neq 0, \\ \lim_{s \rightarrow t} \frac{z'(s)}{x'(s)} & \text{if } x'(t) = 0. \end{cases}$$

We assume that front diagrams are regular enough that the limit above always exists and $y(t)$ is continuous. Appealing to equation (6), we see that

$$\phi: t \mapsto (x(t), y(t), z(t))$$

is a Legendrian knot whose front projection gives the required front diagram.

- (2) Neither direction of this part is easy (see [Swi92]). The most complex move is the one involving the creation/cancellation of cusps. To aid intuition, Alex Wright, with the help of ChatGPT, produced Example 9.13 and an animation visualizing how these cusps are formed under Legendrian isotopy.⁷

□

Proof of Corollary 10.5. Any smooth knot diagram can be modified to a front diagram by performing the following local perturbations:

- (1) Perturb vertical tangencies into “zig-zags” (adding one cusps).
- (2) At crossings where the slope rule is violated (i.e. higher slope goes over), perform a local twist to restore the condition.

These modifications are depicted in the following figure.

□

⁷Animation available at: <https://public.websites.umich.edu/~alexmw/LR.html>

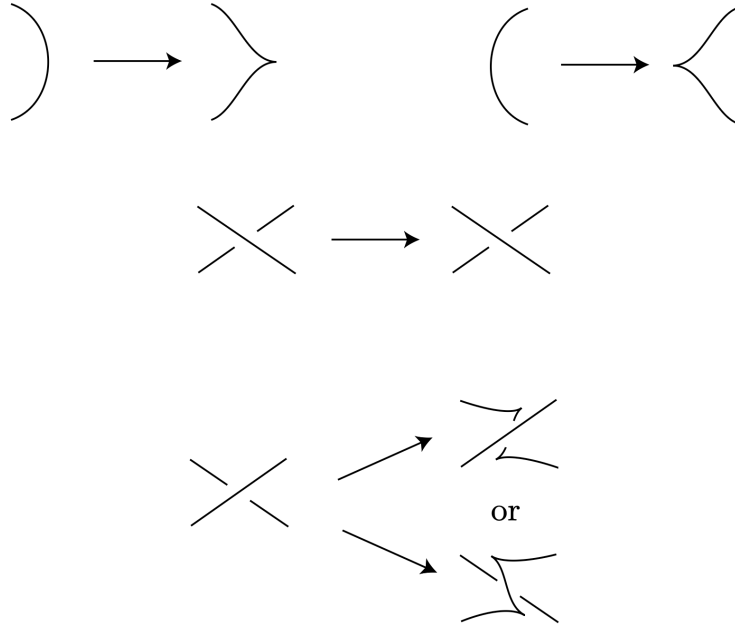


FIGURE 16. Modifying smooth knot diagram to a front diagram. Image reproduced from [Etn05].

Remark 10.8. Note that there is a choice involved when modifying a smooth knot diagram to a front diagram (specifically, the choice of where to add zig-zags). Different choices may lead to inequivalent Legendrian knots starting from a single smooth knot type (See rotation number in Definition 10.9).

10.3. Rigidity of Legendrian knots. By the definition of Legendrian knots, every Legendrian knot is a smooth knot. This gives us a natural “forgetful” map:

$$\left\{ \begin{array}{c} \text{Legendrian knots} \\ \text{up to Legendrian isotopy} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Smooth knots} \\ \text{up to smooth isotopy} \end{array} \right\}.$$

In Subsection 10.2, we established that the above map is a surjection. We did this by constructing a partial inverse:

$$\left\{ \begin{array}{c} \text{Legendrian knots} \\ \text{up to Legendrian isotopy} \end{array} \right\} \longleftarrow \left\{ \begin{array}{c} \text{Smooth knot} \\ \text{diagrams} \end{array} \right\}.$$

Furthermore, the three standard Reidemeister moves have Legendrian counterparts. Despite this, the partial inverse does not descend to a well-defined map on smooth isotopy classes. That is, we do *not* have an inverse:

$$\left\{ \begin{array}{c} \text{Legendrian knots} \\ \text{up to Legendrian isotopy} \end{array} \right\} \not\longleftarrow \left\{ \begin{array}{c} \text{Smooth knot diagrams} \\ \text{up to equivalence} \end{array} \right\}.$$

The fundamental reason for this failure is depicted in Figure 17. A smooth isotopy (like pulling a string tight) can remove the “zig-zag” shown in the figure, but in the Legendrian world, this zig-zag is rigid.



FIGURE 17. The two segments on the left are smoothly isotopic (one can be pulled straight), but on the right, the two are not Legendrian isotopic.

To understand why the two segments in the right of Figure 17 are distinct, recall that a Legendrian knot is tangent to the contact plane field H . Consequently, as shown in the figure below, we can define a canonical non-vanishing normal vector field to a Legendrian knot $\phi(t)$ by picking $v(t)$ in $H_{\phi(t)}$ orthogonal to $\phi'(t)$. This is called the Thurston–Bennequin framing of the knot.

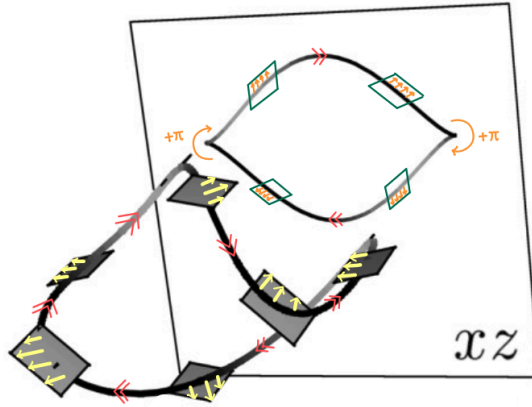


FIGURE 18. Oriented Legendrian unknot and front projection along with contact field and canonical non-vanishing normal vector field (Thurston–Bennequin framing).

Drawing the contact plane field and normal vector field to the knot segments in Figure 17, we see in the following figure that the two Legendrian knot segments are not isotopic because they are not isomorphic as “ribbons,” i.e., the normal vector field for the zig-zag knot segment twists around the knot by 2π while the normal vector field for the straight knot segment has no twisting. Such rigidity of Legendrian knots is captured by numerical invariants.

Definition 10.9 (Invariants of Legendrian Knots). A Legendrian knot L possesses classical smooth invariants (like knot type). Further, it has many strictly Legendrian invariants such as:

- (1) The **Thurston–Bennequin number**, denoted $\text{tb}(L)$, measures the “twisting of H around L ”. It is defined as:

$$\text{tb}(L) := \text{lk}(L, L')$$

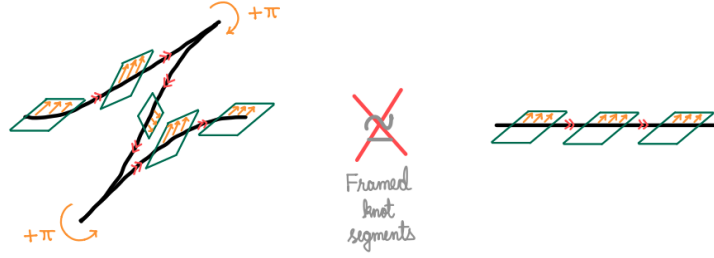
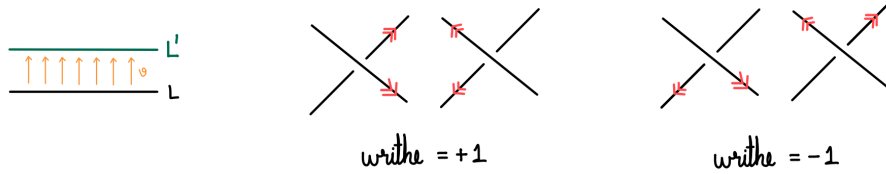


FIGURE 19. Two knot segments along with contact field and canonical normal vector field (Thurston–Bennequin framing).

where L' is the push-off of L along the canonical normal vector field. Combinatorially, for a front diagram:

$$\text{tb}(L) := w(L) - \frac{1}{2}(\# \text{ of cusps}),$$

where $w(L)$ is the *writhe* (the signed crossing number).



- (2) For oriented Legendrian knots that are topologically null-homologous, the **rotation number**, denoted $\text{rot}(L)$, is the winding number of the tangent vector field $\phi'(t)$ (relative to a trivialization). Combinatorially, for an oriented front diagram:

$$\text{rot}(L) := \frac{1}{2}((\# \text{ of down cusps}) - (\# \text{ of up cusps})).$$



Remark 10.10. To get a feeling for these invariants, observe that the above combinatorial counts do not change under Legendrian Reidemeister moves depicted in Figure 13.

To see another form of rigidity of Legendrian knots, we turn to the projection of a Legendrian knot in the (x, y) -plane (See Figure 9).

Definition 10.11. The projection of a Legendrian knot L to the (x, y) -plane is called the **Lagrangian projection**⁸ of L . Explicitly, if L is parametrized by ϕ , the Lagrangian

⁸The terminology “Lagrangian projection” comes from the fact that $d\alpha|_{xy\text{-plane}}$, which is a symplectic form, vanishes when restricted to $\pi_{xy}(L)$. This is more notable in higher dimensions.

projection is the map

$$\begin{aligned}\pi_{xy}: S^1 &\rightarrow \mathbb{R}_{x,y}^2 \\ t &\mapsto (x(t), y(t)).\end{aligned}$$

Lemma 10.12. *For any oriented Legendrian knot, the signed area enclosed by the Lagrangian projection in the (x, y) -plane is exactly zero. In particular, the Lagrangian projection cannot be a loop without self-intersections.*

Proof. Let L be an oriented closed Legendrian knot parametrized by $\phi(t) = (x(t), y(t), z(t))$ for $t \in S^1$. Since L is a closed curve, the total change in the z -coordinate as we traverse the entire knot must be zero. Therefore,

$$\int_{S^1} z'(t) dt = 0.$$

Recall that Equation (5) for Legendrian knots gives:

$$z'(t) = y(t)x'(t).$$

Next, Green's Theorem says the signed area A enclosed by the Lagrangian projection $\pi_{xy}(L)$ in the (x, y) -plane is given by the line integral:

$$A = - \oint_{\pi_{xy}(L)} y dx = - \int_{S^1} y(t)x'(t) dt = - \int_{S^1} z'(t) dt = 0.$$

□

Remark 10.13.

- (1) The Lagrangian projection of the Legendrian unknot in Figure 9 has two loops enclosing equal but opposite area.
- (2) Using a similar calculation as above, note the Legendrian knot can be recovered from its Lagrangian projection, up to translation along the z -axis⁹, using the formula:

$$z(t) := z_0 + \int_0^t y(s)x'(s) ds.$$

- (3) Unlike the front projection, the Lagrangian projection is an immersion into the (x, y) -plane. This is because $x'(t)$ and $y'(t)$ cannot simultaneously be zero; if they were, Equation (5) would force $z'(t)$ to also be zero, violating the condition that the Legendrian knot is embedded in $\mathbb{R}^3$.

11. ALMOST COMPLEX STRUCTURE (02/12, YW, YD)

Reference: da Silva V.12-13.

⁹Notice that translation along the z -axis is a contactomorphism of $(\mathbb{R}^3, \alpha)$.

11.1. Almost complex structure.

Definition 11.1. Let V be a finite dimensional vector space over $\mathbf{R}$. A *complex structure* on V is a linear map $J: V \rightarrow V$ with $J^2 = -\text{Id}$. We call (V, J) a complex vector space.

If V has a symplectic form Ω , then a complex structure J is *compatible with Ω* if $G(u, v) := \Omega(u, Jv)$ on $V \times V$ is an inner product.

Remark 11.2. (1) G is symmetric if and only if $\Omega(Ju, Jv) = \Omega(u, v)$. Indeed, note

$$G(Ju, v) = \Omega(Ju, Jv) \text{ and } G(v, Ju) = \Omega(v, -u) = \Omega(u, v).$$

(2) If G is positive definite, then $\Omega(u, Ju) \geq 0$ for all u . If for some u we have $\Omega(u, Ju) = 0$ then $u = 0$.

Example 11.3. Consider $(\mathbf{R}^2, \Omega)$ with $\Omega(e, f) = 1, \Omega(e, e) = \Omega(f, f) = 0$. Define a complex structure by

$$J(e) = f, J(f) = -e.$$

One can check

- (1) $J^2 = -\text{Id}$,
- (2) $\Omega(v, w) = \Omega(Jv, Jw)$, so G is symmetric,
- (3) $\Omega(xe + yf, J(xe + yf)) = \Omega(xe + yf, -ye + xf) = x^2 + y^2$. This is the standard inner product on $\mathbf{R}^2$.

Remark 11.4. (1) Generalizing the example above, any symplectic vector space (V, Ω) has a compatible complex structure.

(2) The textbook shows that, given (V, Ω) and any inner product G_0 , we can canonically get a complex structure J compatible with Ω . However, $\Omega(\cdot, J\cdot)$ may not recover G_0 . This is because, by Riesz representation theorem, we get a unique linear operator $I: V \rightarrow V$ such that $G_0(\cdot, \cdot) = \Omega(\cdot, I\cdot)$, but we don't necessarily have $I^2 = -\text{Id}$, and if this is the case, then $I \neq J$.

Definition 11.5. A *Riemannian manifold* is a pair (M, g) , where M is a manifold, $x \mapsto g_x$ is a smoothly varying family of inner products on $T_x M$.

Definition 11.6. An *almost complex structure* on a manifold M is a smoothly varying family $x \mapsto J_x \in \text{End}(T_x M)$, where $J_x^2 = -\text{Id}$ for all $x \in M$. The pair (M, J) is called an almost complex manifold.

Note that in the two definitions above, there is no symplectic form involved.

Example 11.7. Let M be a complex manifold. An almost complex structure J_x is given by multiplication by i on $T_x M$ for any $x \in M$. If an almost complex structure arises in this way, we say it is *integrable*.

Example 11.8. When the real dimension of a compact manifold M is two, then by uniformization, any almost complex structure is integrable. This is not true in higher dimension.

Definition 11.9. Let (M, ω) be a symplectic manifold. An almost complex structure J is *compatible with ω* if $x \mapsto g_x(u, v) := \omega_x(u, Jv)$ is a Riemannian metric. We call (ω, g, J) a *compatible triple*.

An important insight due to Gromov is that, among all almost complex structures, although integrability is rare, we can always find compatible ones, which suffice for many purposes.

Proposition 11.10. *Let (M, ω) be a symplectic manifold. Then there are (many) compatible almost complex structures.*

Proof. Skipped. (Put many Riemannian metrics on M and use Remark 11.4(2).) $\square$

Next, we return to Moser's equation (cf. Theorem 5.8). Recall that if we have cohomologous symplectic forms $[\omega_t] = [\omega_0]$ for, say, $t \in [0, 1]$, then $[\frac{d}{dt}\omega_t] = 0$, and so there are smooth 1-forms μ_t such that $d\mu_t = \frac{d}{dt}\omega_t$. Moser's equation seeks a solution v_t in

$$\iota_{v_t}\omega_t = -\mu_t.$$

Whereas before we wanted ω_t to vary, so we could relate different symplectic forms, now we want to fix $\omega_t = \omega$. Let us consider some special situations.

- (1) The case when $\mu_t = dh_t$ is exact is important. If this is the case, then the flow generated by v_t is called a *Hamiltonian flow*. (In textbook, this is called *an isotopy exactly homotopic to the identity*.) The function h_t is called the *Hamiltonian*.
- (2) If h_t is independent of t , then so is v_t . This special case is called *autonomous*.

Let (ω, g, J) be a compatible triple. For any u , we have

$$dh_t(u) = g(\text{grad}(h_t), u) = \omega(\text{grad}(h_t), Ju) = \omega(J\text{grad}(h_t), -u).$$

Then Moser's equation translates to

$$\omega(v_t, u) + \omega(J\text{grad}(h_t), -u) = 0.$$

Hence the solution can be explicitly written as

$$v_t = J\text{grad}(h_t).$$

Remark 11.11. Since

$$g(u, Ju) = \Omega(u, J^2u) = \Omega(u, -u) = 0,$$

geometrically J rotates vectors by 90° .

Example 11.12. In the autonomous case, we can consider level sets of $h_t = h$. Then $v_t = J\text{grad}(h)$ is tangent to level sets of h . That is to say, the flow generated by v_t preserves h . This fact admits a shorter proof, without knowing that the solution to Moser's equation v can be written as $J\text{grad}(h)$. This is because $D_v h = dh(v) = -\iota_v \omega(v) = -\omega(v, v) = 0$, hence h doesn't change in the v -direction.

11.2. J-holomorphic maps.

Definition 11.13. Let $(M_1, J_1), (M_2, J_2)$ be two almost complex manifolds. A smooth map $\phi: M_1 \rightarrow M_2$ is called *J-holomorphic* if for any $x \in M_1$, we have

$$(D\phi)_x \circ (J_1)_x = (J_2)_{\phi(x)} \circ (D\phi)_x.$$

In particular, holomorphic maps are J-holomorphic.

- (1) If $\dim(M_1) = 2$, this is called a J-holomorphic curve.
- (2) A submanifold X of (M, J) is *almost complex* if for all $x \in X$, the subspace $T_x X \subseteq T_x M$ is J_x -invariant.

Lemma 11.14. *If (M, ω, J) is compatible and X is an almost complex submanifold, then X is symplectic.*

Proof. We need to show $\omega_x|_{T_x X \times T_x X}$ is non-degenerating at any $x \in X$. For any $0 \neq u \in T_x X$, we have $\omega_x(u, Ju) > 0$. Since $Ju \in T_x X$ by assumption, we are done. $\square$

Definition 11.15. Let (M, ω) be a symplectic manifold. If M is also a complex manifold, and J is its associated integrable almost complex structure, and J is compatible, then (M, J, ω) is a *Kähler manifold*.

Example 11.16. The projective space $\mathbf{CP}^n$ with ω being the Fubini–Study metric is Kähler. Any complex submanifold of a Kähler manifold is also Kähler.

11.3. More on the group Symp . By the correspondence between C^1 neighborhood of $\text{Id} \in \text{Symp}(X, \omega)$ and a neighborhood of the zero section in the vector space of closed 1-forms (cf. Lecture 7), and using the group structure to translate, we know:

Corollary 11.17. *The topological group $\text{Symp}(X, \omega)$ is locally path connected.*

We define two subgroups of $\text{Symp}(X, \omega)$.

Definition 11.18. The subgroup $\text{Symp}_0(X, \omega)$ is the identity component of $\text{Symp}(X, \omega)$. Moreover, let $\text{Ham}(X, \omega)$ be the set of all $\phi \in \text{Symp}_0(X, \omega)$ which admit a Hamiltonian flow from the identity.

Remark 11.19. The set $\text{Ham}(X, \omega)$ is a subgroup, and is normal in $\text{Symp}_0(X, \omega)$.

We will also use the universal cover $\widetilde{\text{Symp}}_0(X, \omega)$, which is also a topological group.

12. FLUX (02/17, YD, JS)

Reference: [MS17, Chapter 10.2, Chapter 12.1 and Chapter 12.4] and [Eva23, Chapter 1.1].

12.1. From Symplectomorphism groups to flux. Recall the symplectomorphism groups defined in the last lecture:

$$\text{Ham}(X, \omega) \triangleleft \text{Symp}_0(X, \omega) \subseteq \text{Symp}(X, \omega)$$

We now introduce the flux homomorphism, which measures the difference between $\text{Ham}(X, \omega)$ and $\text{Symp}_0(X, \omega)$. However, before that, there might be a well-definedness problem if we just consider the map from $\text{Symp}_0(X, \omega)$. So we need to consider the universal covering of it. Intuitively, different paths will result in different flux, even if they have same start and end points. That is why we need to record not only the endpoint, but also the homotopy class of the path.

Definition 12.1. The universal cover $\widetilde{\text{Symp}}_0(X, \omega)$ is defined as the space of paths starting at the identity, modulo homotopy with fixed endpoints:

$$\widetilde{\text{Symp}}_0(X, \omega) = \{\psi : [0, 1] \rightarrow \text{Symp}_0(X, \omega) \mid \psi_0 = \text{id}\} / \sim$$

where $\psi \sim \phi$ if they are homotopic relative to their endpoints.

Before defining the flux homomorphism, we must formally ensure that $\widetilde{\text{Symp}}_0(X, \omega)$ is actually a group, so that we can talk about homomorphisms from it.

Theorem 12.2. $\widetilde{\text{Symp}}_0(X, \omega)$ is a topological group. For any two classes $[\{g_t\}]$ and $[\{h_t\}]$, the group multiplication is defined geometrically via the concatenation and translation of paths.

$$(g \cdot h)_t = \begin{cases} g_{2t}, & 0 \leq t \leq \frac{1}{2} \\ g_1 \circ h_{2t-1}, & \frac{1}{2} \leq t \leq 1 \end{cases}$$

This traces the path g_t from id to g_1 , followed by the path h_t translated by g_1 , arriving at the final endpoint $g_1 \circ h_1$.

Proof. Let $F_s(t)$ and $G_s(t)$ be homotopies such that $F_0 = \phi$, $F_1 = \phi'$, $G_0 = \psi$, $G_1 = \psi'$, with fixed endpoints $F_s(1) = \phi_1$ and $G_s(1) = \psi_1$, $F_s(0) = G_s(0) = \text{id}$. Define the concatenated homotopy $H_s : [0, 1] \times [0, 1] \rightarrow \text{Symp}_0(X, \omega)$ as

$$H_s(t) = \begin{cases} F_s(2t), & 0 \leq t \leq \frac{1}{2} \\ \phi_1 \circ G_s(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

$H_s(t)$ is continuous at the boundary $t = \frac{1}{2}$

$$F_s(2 \times \frac{1}{2}) = F_s(1) = \phi_1 = \phi_1 \circ \text{id} = \phi_1 \circ G_s(0)$$

And

$$\begin{aligned} H_0(t) &= \phi\psi(t) \\ H_1(t) &= \phi'\psi'(t) \end{aligned}$$

So, $H_s(t)$ is a homotopy from $\phi\psi$ to $\phi'\psi'$, they are in the same homotopy class of path. Then we check the existence of inverse. For $g_t \in \widetilde{\text{Symp}}_0(X, \omega)$, $\bar{g}_t = g_1^{-1} \circ g_{1-t}$ is the inverse. We can check this by:

$$g_0^{-1} = g_1^{-1} \circ g_1 = \text{id}$$

And

$$(g\bar{g})_t = \begin{cases} g_{2t}, & 0 \leq t \leq \frac{1}{2} \\ g_1 g_{2t-1}^{-1} = g_1 g_1^{-1} g_{2-2t} = g_{2-2t}, & \frac{1}{2} \leq t \leq 1 \end{cases}$$

The path first goes from id to g_1 , then reverse back via the same path. Hence it is homotopic to id .

$$(\bar{g}g)_t = \begin{cases} g_{2t}^{-1} = g_1^{-1} g_{1-2t}, & 0 \leq t \leq \frac{1}{2} \\ g_1^{-1} g_{2t-1} = g_1^{-1} g_{2t-1}, & \frac{1}{2} \leq t \leq 1 \end{cases}$$

The path first goes from id to $g_1^{-1}g_0 = g_1^{-1}$, then reverse back via the same path. For the same reason it is homotopic to id . Thus, $\bar{g}$ is indeed the inverse of g . $\square$

Now, we can get into flux.

Definition 12.3. The flux homomorphism is defined as

$$\begin{aligned} \text{Flux} : \widetilde{\text{Symp}}_0(X, \omega) &\rightarrow H_{dR}^1(X; \mathbb{R}) \\ \text{Flux}([\{\psi_t\}]) &= \int_0^1 [\iota_{X_t} \omega] dt \end{aligned}$$

where $H_{dR}^1(X; \mathbb{R})$ denotes the first de Rham cohomology group of X and $\frac{d}{dt}\psi_t = X_t \circ \psi_t$ (so X_t is the time depending vector field generating the isotopy $\{\psi_t\}$).

Theorem 12.4. Flux is well defined and

$$\ker(\text{Flux}) = \text{Ham}(X, \omega)$$

i.e.

$$\phi \in \text{Ham}(X, \omega) \iff \exists [\{\psi_t\}] \in \ker(\text{Flux}) \quad \text{s.t.} \quad \psi_1 = \phi.$$

The equality comes from taking the time-1 maps.

Proof. We first verify that flux is well defined. Let $\{\psi_t\}$ and $\{\phi_t\}$ be two homotopic symplectic isotopies with fixed endpoints. We want to show

$$\text{Flux}([\{\phi_t\}]) = \text{Flux}([\{\psi_t\}]).$$

Since the de Rham pairing

$$H^1(M; \mathbb{R}) \times H_1(M; \mathbb{R}) \rightarrow \mathbb{R}$$

detects cohomology classes, it suffices to prove that for every closed curve γ representing a class $[\gamma] \in H_1(M; \mathbb{R})$ we have

$$\langle \text{Flux}([\{\phi_t\}]), [\gamma] \rangle = \langle \text{Flux}([\{\psi_t\}]), [\gamma] \rangle.$$

The de-Rham pairing is given by

$$\begin{aligned} \langle \text{Flux}([\{\phi_t\}]), [\gamma] \rangle &= \left\langle \int_0^1 [\iota_{X_t} \omega] dt, [\gamma] \right\rangle \\ &= \int_0^1 \langle [\iota_{X_t} \omega], [\gamma] \rangle dt \\ &= \int_0^1 \left(\int_{\gamma} \iota_{X_t} \omega \right) dt. \\ &= \int_0^1 \left(\int_0^1 \omega(X_t(\gamma(s)), \dot{\gamma}(s)) ds \right) dt \end{aligned}$$

We claim the last integral is only dependent on $[\gamma]$ and $[\{\phi_t\}]$. Indeed, define β as follows:

$$\beta : \mathbb{R}/\mathbb{Z} \times [0, 1] \rightarrow M, \quad \beta(s, t) = \phi_t^{-1} \gamma(s)$$

Differentiating $\phi \beta(s, t)$ we obtain both

- $d\phi_t(\beta)\frac{\partial\beta}{\partial s} = \dot{\gamma}(s)$
- $-d\phi_t(\beta)\frac{\partial\beta}{\partial t} = X_t \circ \phi_t(\beta(s, t)) = X_t\gamma(s)$

Thus, we can rewrite the integral above as

$$\int_0^1 \left(\int_0^1 \omega(X_t(\gamma(s)), \dot{\gamma}(s)) ds \right) dt = \int_0^1 \left(\int_0^1 \omega(-d\phi_t(\beta)\frac{\partial\beta}{\partial t}, d\phi_t(\beta)\frac{\partial\beta}{\partial s}) ds \right) dt,$$

and since $\phi^*\omega = \omega$, the right hand side is given by

$$\int_0^1 \int_0^1 \omega(-d\phi_t(\beta)\frac{\partial\beta}{\partial t}, d\phi_t(\beta)\frac{\partial\beta}{\partial s}) ds dt = \int_0^1 \int_0^1 \omega(-\frac{\partial\beta}{\partial t}, \frac{\partial\beta}{\partial s}) ds dt = \int_0^1 \int_0^1 \omega(\frac{\partial\beta}{\partial s}, \frac{\partial\beta}{\partial t}) ds dt = \int_{\mathbb{R}/\mathbb{Z} \times [0,1]} \beta^*\omega.$$

Moreover, as $d\omega = 0$, by Stokes' theorem it follows that $\int_{\mathbb{R}/\mathbb{Z} \times [0,1]} \beta^*\omega$ is invariant under homotopies of β with fixed boundary. But the homotopies of β only depend on the homotopies of γ and homotopies of ϕ_t with fixed boundary.

Now, to show “ $\ker(\text{Flux}) = \text{Ham}(X, \omega)$ ”, begin by taking $\phi \in \text{Ham}(X, \omega)$. Then there exist a Hamiltonian isotopy ψ such that $\psi_1 = \phi$, $\psi_0 = \text{id}$ and $dH_t = \iota_{X_t}\omega$, which means $\iota_{X_t}\omega$ is exact; thus $[\iota_{X_t}\omega] = 0$ and we see

$$\text{Flux}([\{\psi_t\}]) = \int_0^1 [\iota_{X_t}\omega] dt = 0$$

Conversely, suppose $[\{\psi_t\}] \in \ker(\text{Flux})$, $\int_0^1 [\iota_{X_t}\omega] dt = 0$ so the integral is exact, and thus

$$\int_0^1 \iota_{X_t}\omega dt = dF$$

for some function F . We can construct a new isotopy ψ' , where

$$\psi'_t = \begin{cases} \psi_{2t}, & 0 \leq t \leq \frac{1}{2}, \\ \phi_F^{1-2t} \circ \psi_1, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

satisfies

$$\int_0^1 \iota_{X'_t}\omega dt = dF - dF = 0$$

since ω is non-degenerate

$$\int_0^1 X'_t dt = 0$$

We define:

$$Y_t = - \int_0^t X'_s ds$$

Let θ_t^s be the flow generated by Y_t ; we then have

$$\partial_s \theta_t^s = Y_t \circ \theta_t^s \quad \theta_t^0 = \text{id}.$$

Also, as $Y_0 = Y_1 = 0$, then $\theta_0^s = \theta_1^s = \text{id}$. Let $\psi_t'' = \theta_t^1 \circ \psi_t'$. We can see that $\psi_0'' = \text{id}$, and $\psi_1'' = \phi_F^{-1} \circ \psi_1$. Furthermore, notice

$$\begin{aligned} \text{Flux}(\{\psi_t''\}_{0 \leq t \leq T}) &= \text{Flux}(\{\theta_t^1\}_{0 \leq t \leq T}) + \text{Flux}(\{\psi_t'\}_{0 \leq t \leq T}) \\ &= \text{Flux}(\{\theta_T^s\}_{0 \leq s \leq 1}) + \int_0^T [\iota_{X_t'} \omega] dt \\ &= [\iota_{Y_T} \omega] + \int_0^T [\iota_{X_t'} \omega] dt \\ &= 0. \end{aligned}$$

This means from 0 to any time Flux is 0, which also means ψ_t'' is an Hamiltonian isotopy, we have that $\psi_1'' = \phi_F^{-1} \circ \psi_1$ is Hamiltonian. Because ϕ_F is Hamiltonian, then ψ_1 is also Hamiltonian, which completes our proof. $\square$

This proof also gives us a picture of what flux looks like. The double integral measures the symplectic area swept out by that loop under the isotopy. In this sense, the symplectic form ω serves as a natural tool for measuring how much a symplectic isotopy moves a curve.

The flux through a loop γ

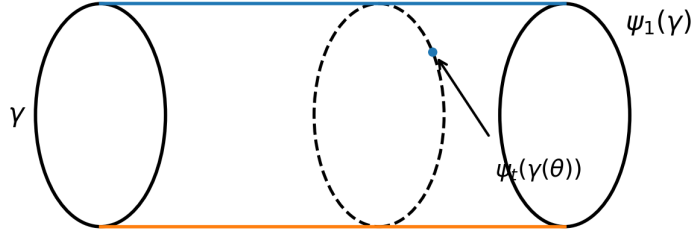


FIGURE 20. The Flux cylinder through a loop γ .

Example 12.5. Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with $\omega = dx \wedge dy$, and consider

$$\psi_t(x, y) = (x, y) + (a_t, b_t).$$

Then the generating vector field is

$$X_t = \dot{a}_t \partial_x + \dot{b}_t \partial_y,$$

and hence

$$\iota_{X_t} \omega = \dot{a}_t dy - \dot{b}_t dx.$$

Therefore

$$\text{Flux}([\psi_t]) = (a_1 - a_0)[dy] - (b_1 - b_0)[dx].$$

Since dx and dy are not exact on T^2 , if the isotopy has nonzero net displacement then its flux is nonzero; thus it is not Hamiltonian.

This example highlights that in general $\text{Symp}_0(X, \omega) \neq \text{Ham}(X, \omega)$.

12.2. Symplectic rigidity. One obvious observation is that for a symplectic manifold (M^{2n}, ω) , the symplectic form ω itself measures 2-dimensional quantities, while ω^n measures $2n$ -dimensional volume. We want to see which is more rigid.

Definition 12.6. In the symplectic space (M^{2n}, ω) , we define the ball and the cylinder

- $B^{2n}(R) := \{(x, y) \in \mathbb{R}^{2n} \mid |x|^2 + |y|^2 < R^2\}$
- $Z^{2n}(R) := \{(x, y) \in \mathbb{R}^{2n} \mid |x_1|^2 + |y_1|^2 < R^2\}$.

Note that $Z^{2n}(R)$ only restricts the first symplectic coordinate pair (x_1, y_1) , and is unbounded in the remaining directions.

Throughout the course so far, we have seen several results that show how symplectic manifolds can be quite similar, for instance at the local level. The upcoming theorem is a result that goes in the other direction.

Theorem 12.7 (Gromov’s non-squeezing theorem). *If there exists a symplectic embedding*

$$(B^{2n}(r), \omega_0) \hookrightarrow (Z^{2n}(R), \omega_0),$$

then

$$r \leq R.$$

Remark 12.8. We omit the proof here as it requires techniques beyond the scope of this text. The proof refers to [MS17, Chapter 12.1 and Chapter 12.4]

Remark 12.9. This is trivial when $n = 1$. The theorem is important for higher dimensions.

Remark 12.10. The statement will be invalid if we use “volume-preserving embedding” to replace “symplectic embedding” when $n > 1$. For example,

$$\begin{aligned} (x_1, y_1, \dots, x_n, y_n) \mapsto & \left(\frac{x_1}{r/R}, \frac{y_1}{r/R}, \right. \\ & \left(\frac{r}{R} \right)^{\frac{1}{(n-1)}} x_2, \left(\frac{r}{R} \right)^{\frac{1}{(n-1)}} y_2, \\ & \dots, \\ & \left. \left(\frac{r}{R} \right)^{\frac{1}{(n-1)}} x_n, \left(\frac{r}{R} \right)^{\frac{1}{(n-1)}} y_n \right) \end{aligned}$$

is a volume-preserving embedding map for arbitrary $r, R > 0$ so the theorem statement doesn’t hold. This implies symplectic forms are more rigid than volume forms.

Remark 12.11. The statement is false for $Z_{all}^{2n}(R) := \{(x, y) \in \mathbb{R}^{2n} \mid |x_1|^2 + |x_2|^2 < R^2\}$. For example,

$$(x_1, y_1, \dots, x_n, y_n) \mapsto \begin{pmatrix} \frac{x_1}{r/R}, y_1(r/R), \\ \frac{x_2}{r/R}, y_2(r/R), \\ \dots, \\ x_n, y_n \end{pmatrix}$$

is a symplectomorphism since it preserves all $dx_i \wedge dy_i$. This holds for arbitrary $r, R > 0$, which again contradicts the statement.

Definition 12.12. A symplectic capacity is a function

$$c : \{\text{symplectic manifolds}\} \rightarrow [0, \infty]$$

such that:

- (1) **Monotonicity:** if there exists a symplectic embedding

$$(M_1, \omega_1) \hookrightarrow (M_2, \omega_2),$$

then

$$c(M_1, \omega_1) \leq c(M_2, \omega_2).$$

- (2) **Conformality:** For $\lambda > 0$,

$$c(M, \lambda\omega) = \lambda c(M, \omega).$$

- (3) **Non-triviality:**

$$c(B^{2n}(1), \omega_0) > 0, \quad c(Z^{2n}(1), \omega_0) < \infty.$$

Remark 12.13. The third axiom (non-trivial) is essential. If one removes it, then trivial examples such as the zero function and $\text{Vol}(M, \omega)^{1/n}$ would satisfy the remaining axioms. Thus the third condition rules out invariants that are too coarse to detect symplectic rigidity.

Remark 12.14. The non-squeezing theorem is equivalent to the existence of capacity such that:

$$c(B^{2n}(1), \omega_0) = c(Z^{2n}(1), \omega_0)$$

Remark 12.15. A modified capacity can be used to prove:

Theorem 12.16 (Eliashberg's C^0 -rigidity theorem).

$$\overline{\text{Symp}(M, \omega)}^{C^0} \cap \text{Diff}(M) = \text{Symp}(M, \omega).$$

This says that any diffeomorphism which is a C^0 limit of symplectomorphisms is a symplectomorphism.

12.3. Mechanics. We now move to mechanics. Let (p, q) be the coordinates of $\mathbb{R}^2$ and $H(p, q)$ be a smooth function.

Definition 12.17. We say a smooth path $(p(t), q(t))$ satisfies Hamilton's equation for a Hamiltonian H if:

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p} \quad (*)$$

Let H be the energy, m be the mass, v be the velocity, p be the momentum and q be the position. We know $H = \frac{p^2}{2m}$ is kinetic energy. Then $(*)$ becomes:

$$\dot{p} = 0, \quad \dot{q} = p/m = v$$

This is exactly Newton's first law. If we add some external force on it, for example, the potential energy then,

$$H = \frac{p^2}{2m} + U(q)$$

and $(*)$ becomes:

$$\dot{p} = -\frac{\partial U}{\partial q}, \quad \dot{q} = v$$

Notice the term $-\frac{\partial U}{\partial q}$ is the negative gradient of U , which is force. So actually this is Newton's second law. Then we check the energy

$$\dot{H} = \frac{\partial H}{\partial q} \dot{q} + \frac{\partial H}{\partial p} \dot{p} = -\dot{p}\dot{q} + \dot{q}\dot{p} = 0$$

Hence, the energy is conserved.

13. BASICS OF HAMILTONIAN FLOWS (02/19, AK, JS)

13.1. Orbits/periods of Hamiltonian flows. Reference: Evans Chapter 1.1, 1.2.

Recall that (p, q) form a coordinate system on $\mathbb{R}^2$ with Hamiltonian $H : \mathbb{R}^2 \rightarrow \mathbb{R}$. **Hamilton's equations** for $p = p(t)$, $q = q(t)$ are

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p},$$

where the dot denotes time derivative. By the chain rule, we can show that the Hamiltonian is constant along the orbit of $(p(t), q(t))$:

$$\dot{H} = \dot{p} \frac{\partial H}{\partial p} + \dot{q} \frac{\partial H}{\partial q} = 0.$$

Example 13.1. [Harmonic oscillator] For a simple harmonic oscillator, the Hamiltonian is given by

$$H_1 = \frac{1}{2}(p^2 + q^2).$$

Then Hamilton's equations are

$$\dot{p} = -q, \quad \dot{q} = p.$$

The solution is

$$\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} p(0) \\ q(0) \end{pmatrix}$$

which describes rotational motion with constant angular speed and period 2π .

Example 13.2. Consider the Hamiltonian

$$H_2 = \sqrt{p^2 + q^2} = \sqrt{2H_1}.$$

Then Hamilton's equations are

$$\dot{p} = -\frac{q}{H_2}, \quad \dot{q} = \frac{p}{H_2}.$$

We know that H_2 is constant along the orbit of $(p(t), q(t))$. The key point is that $(\dot{p}, \dot{q})$ is the same as the example above with an additional factor of $1/H_2$. Hence the solution becomes

$$\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} \cos(t/H_2) & -\sin(t/H_2) \\ \sin(t/H_2) & \cos(t/H_2) \end{pmatrix} \begin{pmatrix} p(0) \\ q(0) \end{pmatrix}.$$

Note that the orbits in this example have period $2\pi H_2$.

Theorem 13.3. *If all level sets of H are periodic orbits (i.e. circles), then there exists $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ such that for the Hamiltonian $\alpha \circ H$, the orbits have period 2π .*

From the point of view of this theorem, the two examples above are related by $H_1 = \frac{1}{2}H_2^2$, so $\alpha(x) = \frac{x^2}{2}$.

Proof. Hamilton's equation for $\alpha \circ H$ can be derived as

$$\begin{aligned} \dot{p} &= -\frac{\partial(\alpha \circ H)}{\partial q} = -\alpha'(H) \frac{\partial H}{\partial q}; \\ \dot{q} &= \frac{\partial(\alpha \circ H)}{\partial p} = \alpha'(H) \frac{\partial H}{\partial p}. \end{aligned}$$

Replacing H with $\alpha \circ H$ leads to replacing $(\dot{p}, \dot{q})$ with $\alpha'(H)(\dot{p}, \dot{q})$, i.e. 'speed up' the velocity of the flow by $\alpha'(H)$.

Let the period for the periodic orbit $H^{-1}(b)$ of the original flow be $T(b)$. For the new flow its period is $T(b)/\alpha'(b)$. Setting

$$\alpha(b) = \frac{1}{2\pi} \int_0^b T(c) dc$$

gives us $\alpha'(b) = \frac{1}{2\pi} T(b)$, which implies that the new flow has period 2π . $\square$

In general, periods can be tricky to compute explicitly. For instance, calculating the period of a simple pendulum requires elliptic integrals. Here's one useful expression for the period that leads to a geometric interpretation of the period as symplectic area:

Theorem 13.4. *In a one-parameter family of periodic orbits (circles) $\Omega_b := H^{-1}(b)$ of a Hamiltonian flow, the period $T(b)$ of Ω_b is*

$$\frac{d}{db} \int_{\Omega_b} p \, dq.$$

Proof. For each orbit Ω_b , we pick a new coordinate system so that q is valued in $\mathbb{R}/2\pi\mathbb{Z}$, which allows us to write

$$\Omega_b := \{(p_b(q), q) : q \in \mathbb{R}/2\pi\mathbb{Z}\},$$

where $p = p_b(q) : \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}$ is a family of functions smoothly varying in b . The period can be written as

$$T(b) = \int_{\Omega_b} dt = \int_0^{2\pi} \frac{dq}{\dot{q}} = \int_0^{2\pi} \frac{dq}{\partial H / \partial p}.$$

Note that since $H(p_b(q), q) = b$, differentiating each side by b gives us

$$\frac{\partial H}{\partial p} \frac{\partial p_b}{\partial b} = 1$$

by the chain rule. Hence we have

$$T(b) = \int_0^{2\pi} \frac{\partial p_b}{\partial b} dq = \int_{\Omega_b} \frac{dp}{db} dq = \frac{d}{db} \int_{\Omega_b} p dq.$$

□

Remark 13.5. In the last result, we had

$$\alpha(b) = \frac{1}{2\pi} \int_0^b T(c) dc$$

with $T(b) = 2\pi\alpha'(b)$. We replaced H with $\alpha \circ H$ to get all periods of the new flow to be equal to 2π . Note that the last result implies that

$$\alpha(b) = \frac{1}{2\pi} \int_{\Omega_b} p dq$$

up to a constant. Hence we have

$$\alpha(b_1) - \alpha(b_0) = \frac{1}{2\pi} \int_{\Omega_{b_1} \cup \overline{\Omega}_{b_0}} p dq = \frac{1}{2\pi} \int_{C_{b_0, b_1}} dp \wedge dq$$

by Stokes' theorem, where C_{b_0, b_1} is the cylinder bounded by Ω_{b_0} and Ω_{b_1} . If $\alpha(b_0) = 0$, then $\alpha(b_1)$ is $(1/(2\pi))$ times the symplectic area of the cylinder. It follows that $T(b) = 2\pi\alpha'(b)$, i.e. the period is proportional to the derivative of the area.

13.2. Back to symplectic forms. Given the 2-form $\omega = dp \wedge dq$ and vector field $V = (\dot{p}, \dot{q})$, we have that

$$\begin{aligned} dH &= \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial q} dq; \\ i_V(\omega) &= i_V(dp \otimes dq - dq \otimes dp) = \dot{p}dq - \dot{q}dp. \end{aligned}$$

Hence we have

$$i_V(\omega) = -dH \iff \dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}.$$

Recall that $i_V\omega = -dH$ is how Hamiltonian flow is defined in Evans' book.

Definition 13.6. The solution V_H to the equation $i_{V_H}\omega = -dH$ is called a **Hamiltonian vector field** with Hamiltonian H . The flow ϕ_t^H generated by the solution V_H is called the **Hamiltonian flow** generated by H .

The Hamiltonian flow leaves both the symplectic form and the Hamiltonian invariant:

$$(\phi_t^H)^*\omega = \omega, \quad (\phi_t^H)^*H = H.$$

This is a simple check that can be done using Cartan's magic formula:

$$\begin{aligned} \frac{d}{dt}(\phi_t^H)^*\omega &= (\phi_t^H)^*\mathcal{L}_{V_H}\omega \\ &= (\phi_t^H)^*(i_{V_H}d\omega + di_{V_H}\omega) = (\phi_t^H)^*(-ddH) = 0. \end{aligned}$$

One can similarly check that $\frac{d}{dt}(\phi_t^H)^*H = 0$ via

$$\mathcal{L}_{V_H}H = i_{V_H}dH = -\omega(V_H, V_H) = 0.$$

Lemma 13.7. *The Lie bracket of two Hamiltonian vector fields is another Hamiltonian vector field given by*

$$[V_F, V_H] = V_{\omega(V_F, V_H)}.$$

Proof. We recall Cartan's second magic formula:

$$\mathcal{L}_V \circ i_W - i_W \circ \mathcal{L}_V = i_{[V, W]}.$$

Plugging in our vector fields, we obtain

$$i_{[V_F, V_H]}\omega = \mathcal{L}_{V_F}i_{V_H}\omega - i_{V_H}\mathcal{L}_{V_F}\omega = \mathcal{L}_{V_F}i_{V_H}\omega,$$

where the second term vanishes by $\mathcal{L}_{V_F}\omega = 0$ as ω is preserved. Now we are done by applying Cartan's first magic formula:

$$\begin{aligned} i_{[V_F, V_H]}\omega &= \mathcal{L}_{V_F}i_{V_H}\omega = i_{V_F} \circ d(i_{V_H}\omega) + d \circ i_{V_F}(i_{V_H}\omega) \\ &= -i_{V_F} \circ d(dH) + d \circ i_{V_F}i_{V_H}\omega \\ &= 0 + d\omega(V_H, V_F) = -d\omega(V_F, V_H). \end{aligned}$$

□

Definition 13.8. The **Poisson bracket** of functions F, H is defined by

$$\{F, H\} = \omega(V_F, V_H).$$

Corollary 13.9. *Two Hamiltonian flows ϕ_t^F and ϕ_t^H commute if and only if $d\{F, H\}$ is zero, i.e. $\{F, H\}$ is locally constant.*

Recall that two flows commute if and only if the Lie bracket of the two generating vector fields vanishes. The corollary above is a result in a similar vein, showing us that two Hamiltonian flows generated by F and H commute if and only if their Poisson bracket is locally constant.

But does ϕ_t^F preserve H ? For the flows to not only commute but also preserve each other's level sets, we must indeed have that the Poisson bracket is zero. This is justified by the following calculation:

Lemma 13.10.

$$\frac{d}{dt}(F \circ \phi_t^G) = \{G, F \circ \phi_t^G\}.$$

Proof. Note

$$\left. \frac{d}{dt} \right|_{t=0} (F \circ \phi_t^G) = dF(V_G) = -\omega(V_F, V_G) = -\{F, G\} = \{G, F\}.$$

□

14. INTEGRABLE HAMILTONIAN SYSTEMS (02/24, YW, AW)

Reference: [Eva23, Chapter 1.3–1.4].

14.1. Integrable Hamiltonian systems.

Definition 14.1 (Hamiltonian $\mathbf{R}^n$ -action). Suppose (X, ω) is symplectic, with a map

$$H = (H_1, \dots, H_n): X \rightarrow \mathbf{R}^n$$

satisfying $\{H_i, H_j\} = 0$ for all i, j . Let V_{H_i} the Hamiltonian vector field associated to H_i . Assume that V_{H_i} can be integrated for all t . We have an $\mathbf{R}^n$ -action on X by

$$t = (t_1, \dots, t_n) \in \mathbf{R}^n \mapsto \phi_t^H = \phi_{t_1}^{H_1} \circ \dots \circ \phi_{t_n}^{H_n}.$$

This is called a *Hamiltonian $\mathbf{R}^n$ -action*.

For any $x \in X$, we use the following notation to denote the orbit of x under this action

$$\Omega(x) = \{\phi_t^H(x) : t \in \mathbf{R}^n\}.$$

Intuitively, X is foliated by level sets of H and the $\mathbf{R}^n$ action preserves all the leaves.

Example 14.2 (Non-Example). Let $H = (H_1, H_2): \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be given by

$$H_1(x, y) = x, \quad H_2(x, y) = y,$$

and let $\omega = dx \wedge dy$ be the standard symplectic form on $\mathbf{R}^2$. From

$$\iota_{V_{H_1}}(dx \wedge dy) = -dx$$

it follows that

$$V_{H_1} = \frac{\partial}{\partial y} \quad \text{and} \quad \phi_t^{H_1}(x, y) = (x, y + t),$$

which does not preserve level sets of H_2 . Similarly,

$$V_{H_2} = -\frac{\partial}{\partial x} \quad \text{and} \quad \phi_t^{H_2}(x, y) = (x - t, y),$$

which does not preserve level sets of H_1 . This is because

$$\{x, y\} = \omega(V_{H_1}, V_{H_2}) = (dx \wedge dy) \left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x} \right) \neq 0.$$

Lemma 14.3. Suppose $H: X \rightarrow \mathbf{R}^n$ generates a Hamiltonian $\mathbf{R}^n$ -action. Then the orbits are isotropic. Moreover, if H has a regular value, then $n \leq \dim(X)/2$.

Proof. For any $x \in X$, the tangent space $T_x\Omega(x)$ is spanned by $\{V_{H_1}, \dots, V_{H_n}\}$, and by the assumption on the action, $\omega(V_{H_i}, V_{H_j}) = \{H_i, H_j\} = 0$. So $\Omega(x)$ is isotropic.

If $H(x)$ is a regular value for some $x \in X$, then $(dH_1)_x, \dots, (dH_n)_x$ are all linearly independent. Since the map $H \mapsto V_H$ is linear, the vectors $(V_{H_1})_x, \dots, (V_{H_n})_x$ are linearly independent. So $\dim T_x\Omega(x) = n$. Since $\Omega(x)$ is isotropic, we have $n \leq \dim(X)/2$. $\square$

Corollary 14.4. *If $\dim(X) = 2n$, all fibers of $H: X \rightarrow \mathbf{R}^n$ are connected, and $\{H_i, H_j\} = 0$ for all i, j , then each regular fiber of H is a Lagrangian and is a single orbit.*

Here, a fiber F of H is regular if H is regular at all points in F .

Proof. If b is a regular value of H , then $H^{-1}(b)$ is an n -dimensional submanifold of X . Moreover, for any $x \in H^{-1}(b)$, the orbit $\Omega(x)$ is also an n -dimensional submanifold of X . Because all H_i are constant on $\Omega(x)$, we further have $\Omega(x) \subseteq H^{-1}(b)$ and is open. Since $H^{-1}(b)$ is connected, equality $\Omega(x) = H^{-1}(b)$ must hold. $\square$

Definition 14.5. Let (X^{2n}, ω) be a symplectic manifold. We say that a smooth map $H: X \rightarrow \mathbf{R}^n$ is a *complete commuting Hamiltonian system* if

$$\{H_i, H_j\} = 0 \quad \text{for all } i, j.$$

We say that a complete commuting Hamiltonian system H is an *integrable Hamiltonian system* if moreover:

- (1) $H(X)$ contains a dense open set of regular values,
- (2) H is proper ($\implies$ the flows are complete),
- (3) H has connected fibers.

Remark 14.6. In light of Lemma 14.3, an integrable Hamiltonian system has as many commuting conserved quantities as possible. Condition (1) rules out trivial examples, e.g. when all H_i are constant, and condition (2) implies that the flows are complete.

Definition 14.7. Let $H: X \rightarrow \mathbf{R}^n$ generate a Hamiltonian $\mathbf{R}^n$ -action, and pick an open subset $B \subseteq H(X)$. A *local section over B* is a map $\sigma: B \rightarrow X$ such that $H \circ \sigma = \text{Id}_B$.



Remark 14.8. If X is a family of genus one tori, then a choice of σ might be compared to picking an origin to turn a family of genus 1 Riemann surfaces into a family of elliptic curves.

Remark 14.9. Because $H \circ \sigma = \text{Id}_B$, the differential dH is surjective on $\sigma(B)$, and hence $\sigma(B)$ consists entirely of regular points.

Remark 14.10. Because the set of regular points is invariant under $\mathbf{R}^n$ -action, if each fiber is a single orbit, then B consists entirely of regular values.

Definition 14.11. The period lattice at $b \in B$ is

$$\Lambda_b^H := \{t \in \mathbf{R}^n : \phi_t^H(\sigma(b)) = \sigma(b)\} = \text{Stab}(\sigma(b)) \subseteq \mathbf{R}^n.$$

The period lattice is the bundle

$$\Lambda^H = \{(b, t) \in B \times \mathbb{R}^n : t \in \Lambda_b^H\}.$$

The period lattice is *standard* if

$$\Lambda^H = B \times (2\pi\mathbb{Z})^n.$$

The goal for the next few lectures is to make the period lattice for H standard by post-composing it with a change of coordinates. In physics, this corresponds to the existence of action-angle coordinates. In math, this is related to an integral affine structure on B .

14.2. Translation surfaces. Lastly, we discussed an interlude, which is not related to the content before. Let X be a Riemann surface, and ω be a holomorphic 1-form on X , which locally can be written as $\omega = f(z)dz$. Let Σ be the set of zeros of ω . By integration, for any $p \notin \Sigma$ there is a local coordinate w near p such that $\omega = dw$ near p .

Any two such local coordinates w_1, w_2 are equivalent if, on the overlap of their domains of definition, we have $dw_2 = T^*(dw_1)$, where T is the transition map. Then $dT \equiv 1$ and so $T(w_2) = w_2 + c$ for some constant $c \in \mathbb{C}$. This means $X \setminus \Sigma$ is a translation surface. In particular, it admits a flat metric.

15. LIOUVILLE COORDINATES (02/26, JZ, AK)

Reference: [Eva23, Chapter 1.4-1.6].

Recall from last lecture:

Definition 15.1. Let (X^{2n}, ω) be a symplectic manifold. Then $H: X \rightarrow \mathbb{R}^n$ is a *complete commuting Hamiltonian system* if $\{H_i, H_j\} = 0$ for all i, j . It is an *integrable Hamiltonian system* if moreover,

- $H(X)$ contains a dense open set of regular values.
- H is proper and has connected fibers.

Remark 15.2. Recall from our discussion at the end of Lecture 13 that $\{H_i, H_j\} = 0$ is equivalent to $H_1, \dots, H_n$ having commuting flows and these flows preserving each other's level sets.

The setup for what follows: Take open subset $B \subset H(X)$ in $\mathbb{R}^n$ and a local section $\sigma: B \rightarrow X$ to H . Define

$$\Lambda_b^H = \{t \in \mathbb{R}^n : \phi_t(\sigma(b)) = \sigma(b)\},$$

$$\Lambda^H = \{(b, t) \in B \times \mathbb{R}^n : t \in \Lambda_b^H\}.$$

The set Λ_b^H is called *the period lattice at b* . Note that this is just the stabilizer $\text{Stab}(\sigma(b))$ of the action. The set Λ^H is called *the period lattice*, or the *isotropy subgroup*.

It makes more sense to think of Λ^H as a bundle of lattices, with fiber Λ_b^H over the point $b \in B$. But Evans calls Λ^H the period lattice, so this is the terminology we will use. If $\Lambda^H = B \times (2\pi\mathbb{Z})^n$, we say Λ^H is *standard*. In the next lecture, we will show that we can always alter H , at least locally, so that Λ^H is standard. The first step in

showing this is proving the existence of *Liouville Coordinates* (known as *action-angle coordinates* in physics).

Before we prove the existence of Liouville Coordinates in Theorem 15.10, we establish the following facts about the period lattice at b :

Lemma 15.3. Λ_b^H consists of $t \in \mathbb{R}^n$ such that ϕ_t^H fixes every point of the orbit $\Omega(\sigma(b))$.

Proof. Since $\Omega(\sigma(b)) = \{\phi_s^H(\sigma(b)) : s \in \mathbb{R}^n\}$, we want to show $\phi_t^H(\sigma(b)) = \sigma(b)$ if and only if $\phi_t^H(\phi_s^H(\sigma(b))) = \phi_s^H(\sigma(b))$ for all s . One direction is trivial. Assuming $\phi_t^H(\sigma(b)) = \sigma(b)$, observe that

$$\phi_t^H(\phi_s^H(\sigma(b))) = \phi_s^H(\phi_t^H(\sigma(b))) = \phi_s^H(\sigma(b))$$

where we use $\phi_t^H \circ \phi_s^H = \phi_{t+s}^H$ to swap ϕ_s^H and ϕ_t^H . $\square$

Remark 15.4. If $\Omega(\sigma(b))$ is dense in $H^{-1}(b)$, then $\Lambda_b^H = \{t : \phi_t^H|_{H^{-1}(b)} = \text{id}\}$.

The following lemma ensures our terminology makes sense.

Lemma 15.5. For all regular values $b \in B$, Λ_b^H is a discrete subgroup in $\mathbb{R}^n$.

Proof sketch. Since $\Lambda_b^H = \text{Stab}(\sigma(b))$, it is automatically a subgroup of $(\mathbb{R}^n, +)$. Since the flows $\phi_t^{H_i}$ are smooth, it is a closed subgroup. So by general Lie group theory, $\Lambda_b^H \cong \mathbb{R}^k \times \mathbb{Z}^l$. It suffices to show there is no $\mathbb{R}^k$ part. Crucially, we need b to be a regular value so that the V_{H_i} are linearly independent. It follows that for small enough values of t , but not $t = 0$, each flow $\phi_t^{H_i}$ moves $\sigma(b)$ in an independent direction. This implies that $t = 0$ is an isolated point in Λ_b^H . Therefore $\Lambda_b^H \cong \mathbb{R}^k \times \mathbb{Z}^l$ for $k > 0$ is impossible. $\square$

See Evans for a proof without Lie group theory, although the main idea is still to show $\Lambda_b^H \cap B_\epsilon(0) = \{0\}$ using that the V_{H_i} are linearly independent.

Example 15.6. Recall Example 13.1, with $H_1: \mathbb{R}^2 \rightarrow \mathbb{R}$, $H_1(p, q) = \frac{1}{2}(p^2 + q^2)$. Let $B = \mathbb{R}_{>0}$ and define the section $\sigma(b) = (\sqrt{2b}, 0)$. From the computation in Example 13.1, $\phi_t^{H_1}(\sigma(b)) = (\sqrt{2b} \cos(t), \sqrt{2b} \sin(t))$. Thus $\Lambda^{H_1} = \{(b, 2\pi k) : b \in B, k \in \mathbb{Z}\} = B \times 2\pi\mathbb{Z}$ — the period lattice is standard!

Example 15.7. Recall Example 13.2, with $H_2: \mathbb{R}^2 \rightarrow \mathbb{R}$, $H_2(p, q) = \sqrt{p^2 + q^2}$. This time, let $B = \mathbb{R}_{>0}$ and $\sigma(b) = (b, 0)$. We have $\phi_t^{H_2}(\sigma(b)) = (b \cos(t/b), b \sin(t/b))$. Thus, $\Lambda^{H_2} = \{(b, 2\pi bk) : b \in B, k \in \mathbb{Z}\}$ which is not standard.

Example 15.8. Here is a more involved example:

Consider some $H: \mathbb{R}^2 \rightarrow \mathbb{R}$ whose level sets are shown in the picture above. Let b_0 be the value such that $H^{-1}(b_0)$ are the two isolated points, and b_1 the value such that $H^{-1}(b_1)$ is shaped like the ∞ symbol. Neither b_0 nor b_1 are regular values, but all other values of b are regular since the picture implies one of $dH(\partial x)$ or $dH(\partial y)$ will be nonzero.

For a specific example of this setup, let $H(x, y) = y^2 + (x^2 - 1)^2$, $B = \mathbb{R}_{>0}$, $b_0 = 0$ and $b_1 = 1$. The fiber $H^{-1}(b_1)$ is also called a *separatrix*, because it is the boundary that “separates” the two different types of fibers. Namely, $H^{-1}(b)$ is either diffeomorphic to

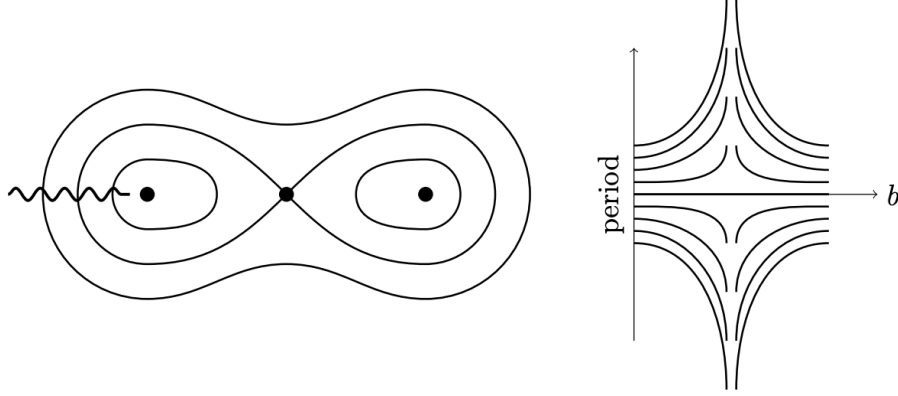


FIGURE 21. Level sets of a Hamiltonian system (left) and its corresponding period lattice (right). Image reproduced from [Eva23, Figure 1.3].

one circle or two disjoint circles for all $b \neq b_0, b_1$. The fiber $H^{-1}(b_1)$ is the case that occurs “in between”.

There are three types of orbits in this example. The first type are fixed points (marked in a $\bullet$ in the diagram). Then there are the two parts of the separatrix, i.e., the figure eight minus the center fixed point. One can check that the separatrices have infinite orbits, i.e., by solving a differential equation, we find it takes infinite time to go around one side of the ∞ symbol. Finally, the closed loops inside or outside of the separatrix are closed orbits.

15.1. Liouville Coordinates. Now we consider the special case where $\sigma: B \rightarrow X$ is a *Lagrangian Section*, meaning that the image $\sigma(B)$ is a Lagrangian submanifold in X . As seen by the next theorem, this extra assumption gets us lots of mileage, ensuring the existence of *Liouville Coordinates*, also known as Action-Angle coordinates.

First, note that the assumption that σ is Lagrangian can always be achieved, at least locally:

Lemma 15.9. *Let $H: X \rightarrow \mathbb{R}^n$ be an integrable Hamiltonian system. Then for any regular point $x \in X$, there exists a local Lagrangian section passing through x .*

Theorem 15.10 (Liouville Coordinates). *Let $H: X \rightarrow \mathbb{R}^n$ be an integrable Hamiltonian system, $B \subset \mathbb{R}^n$ be an open set, and $\sigma: B \rightarrow X$ be a local Lagrangian section. Define*

$$\Psi: B \times \mathbb{R}^n \rightarrow X, \quad \Psi(b, t) = \phi_t^H(\sigma(b)).$$

Then Ψ is both an immersion and a submersion and

$$\Psi^*\omega = \sum db_i \wedge dt_i,$$

i.e., $(b_i, t_i)_{i=1}^n$ are standard symplectic coordinates on a neighborhood of $\sigma(B)$.

Proof. We just need to compute that $\Psi^*\omega = \sum db_i \wedge dt_i$. From this, it follows that Ψ is an immersion and a submersion at all points p , as otherwise $\Psi^*\omega$ will be degenerate at p .

In order to show the 2-forms $\Psi^*\omega$ and $\sum db_i \wedge dt_i$ are equal, we just need to show they are equal as functions on pairs of tangent vectors. In other words, we must check that the two 2-forms evaluate the same on the three different types of pairings, i.e., on $(\partial t_i, \partial t_j)$, on $(\partial b_i, \partial b_j)$, and on $(\partial b_i, \partial t_j)$. This is a good exercise in understanding the definitions. Argue as follows:

- On $\partial t_i, \partial t_j$: We have $(\Psi^*\omega)(\partial t_i, \partial t_j) = \omega(\Psi_*(\partial t_i), \Psi_*(\partial t_j))$. By definition of the flows, $\Psi_*(\partial t_j) = V_{H_j}$. So then

$$\omega(\Psi_*(\partial t_i), \Psi_*(\partial t_j)) = \omega(V_{H_i}, V_{H_j}) \circ \Psi = \{H_i, H_j\} \circ \Psi = 0.$$

This matches how $\sum db_i \wedge dt_i$ evaluates on $\partial t_i, \partial t_j$.

- On $\partial b_i, \partial b_j$: We have $(\Psi^*\omega)(\partial b_i, \partial b_j) = \omega(\Psi_*(\partial b_i), \Psi_*(\partial b_j))$. By definition of the flows, $\Psi_*(\partial b_i) = (d(\phi_t^H) \circ d\sigma)(\partial b_i)$. Note that this vector is tangent to $\phi_t^H(\sigma(B))$. By assumption, $\sigma(B) \subset X$ is Lagrangian and since the H_i flows preserve the symplectic form, it follows that $\phi_t^H(\sigma(B))$ is also Lagrangian. So ω evaluated at pairs of tangent vectors of $\phi_t^H(\sigma(B))$ must vanish, i.e., $\omega(\Psi_*(\partial b_i), \Psi_*(\partial b_j)) = 0$. Again, this matches how $\sum db_i \wedge dt_i$ evaluates on $\partial b_i, \partial b_j$.
- On $\partial b_i, \partial t_j$: We have

$$(\Psi^*\omega)(\partial b_i, \partial t_j) = \omega(\Psi_*(\partial b_i), \Psi_*(\partial t_j)) = \omega(\Psi_*(\partial b_i), V_{H_j}) = dH_j(\Psi_*(\partial b_i)).$$

Unwrapping the chain rule, this is the same as $d(H_j \circ \Psi)(\partial b_i)$. Since ϕ_t^H preserves the level sets, we have $(H_j \circ \Psi)(b, t) = b_j$, so this is just $d(b_j)(\partial b_i) = \delta_{ij}$. In total, $(\Psi^*\omega)(\partial b_i, \partial t_j) = \delta_{ij}$ which is exactly how $\sum db_i \wedge dt_i$ evaluates on $\partial b_i, \partial t_j$.

This finishes the proof. $\square$

Remark 15.11. The assumption that $\sigma(B)$ is Lagrangian is quite important in the proof! At least locally, we can always find some σ satisfies this property. This is also an exercise in Evans.

In total, this shows $\Psi: (B \times \mathbb{R}^n, \omega_{std}) \rightarrow (X, \omega)$ is a local symplectomorphism. It follows from the theorem that

$$\Lambda^H = \Psi^{-1}(\sigma(B)) \subset B \times \mathbb{R}^n$$

is Lagrangian because Ψ preserves the symplectic forms and $\sigma(B)$ is Lagrangian.

Theorem 15.12 (Little Arnold-Liouville). *Let H be an complete commuting Hamiltonian system and $\sigma: B \rightarrow X$ be a local section. Then each orbit $\Omega(\sigma(b))$ is diffeomorphic to $\mathbb{R}^k/\mathbb{Z}^k \times \mathbb{R}^{n-k}$ for some k .*

Proof. Use the orbit-stabilizer theorem. The stabilizer is Λ_b^H and the orbit is $\Omega(\sigma(b))$, and working locally we see that this bijection is actually a diffeomorphism. $\square$

In particular, if H is a Hamiltonian integrable system, then every regular fiber over B is a torus by compactness.

Remark 15.13. Since integrable Hamiltonian systems are proper, it follows that the fibers are compact. In particular, if H is a Hamiltonian integrable system, then every regular fiber over B is a torus by compactness. Note also that we did not need σ to be a Lagrangian section here.

Our next goal is to mimic Theorem 13.3, showing (possibly after shrinking B) that there exists $\alpha: B \rightarrow C \subset \mathbb{R}^n$ such that $\alpha \circ H$ has the standard period lattice. We will discuss a more conceptual reason for this fact, but one can also just write out the equations and show some differential equation has a solution (this is what Evans does). In the next lecture, we will show:

Theorem 15.14 (Action-Angle Coordinates). *Let H be an integrable Hamiltonian system over a disc with only regular fibers and let $\sigma: B \rightarrow X$ be a Lagrangian section. Then each point in the disc has a neighborhood B such that there exists a diffeomorphism $\alpha: B \rightarrow C$ with standard $\Lambda^{\alpha \circ H}$.*

Setting $G = \alpha \circ H$, this says that $(c, t) \mapsto \phi_t^G(\sigma(\alpha^{-1}(c)))$ descends to a symplectomorphism

$$C \times \mathbb{R}^n / (2\pi\mathbb{Z})^n \rightarrow G^{-1}(C) = H^{-1}(B).$$

Later, we will say that G generates a *Hamiltonian torus action*. The Liouville (or standard) coordinates associated to G are called *action-angle coordinates*, where $c_i = \alpha_i \circ H \in C$ are the action coordinates and $t_i \in \mathbb{R}/2\pi\mathbb{Z}$ are the angle coordinates.

Corollary 15.15 (Big Arnold-Liouville). *If $H: M \rightarrow \mathbb{R}^n$ is an integrable Hamiltonian system, then any regular fiber is a torus and has a neighborhood symplectomorphic to*

$$(B \times T^n, \sum_{i=1}^n db_i \wedge dt_i),$$

where B is an open disc in $\mathbb{R}^n$. Under this symplectomorphism, the orbits of the original system correspond to the tori $\{b\} \times T^n$.

Conceptually, this can be understood as a version of the Weinstein tubular neighborhood theorem that also keeps track of the group action on M . This will make more sense after discussing how to naturally view $B \times \mathbb{R}^n$ as T^*B in Lecture 17.

16. (G, X) STRUCTURES ON MANIFOLDS (03/10, MN, YL)

A good reference for this lecture is [Gol22, Chapters 1 and 5].

Affine Structures on Manifolds.

Definition 16.1.

- (1) The **affine group of $\mathbb{R}^n$** is defined as:

$$\text{Aff}(\mathbb{R}^n) = \{x \mapsto Ax + b \mid A \in \text{GL}(n, \mathbb{R}), b \in \mathbb{R}^n\}.$$

- (2) For $U \subseteq \mathbb{R}^n$ open and connected, $\varphi: U \rightarrow \mathbb{R}^n$ is called an **affine map** if any one of the two equivalent conditions hold:
- (a) φ is a restriction of an affine map, or
 - (b) $(D\varphi)_x$ is constant for all $\forall x \in U$ and $(D\varphi)_x$ is invertible.
- (3) An **affine structure on a manifold M^n** is a (maximal) atlas of charts all of whose transition functions are affine maps.¹⁰

¹⁰Some care must be taken if one is requiring maximality, since if one works with disconnected charts one should use locally affine maps. This same comment applies later when we discuss (G, X) structures.

- (4) A **constant 1-form on** $U \subseteq \mathbb{R}^n$ is the exterior derivative, d , of a linear function. Equivalently, in coordinates, constant 1-forms are those that can be expressed as

$$\sum C_i dx_i$$

where the C_i 's are constant.

Lemma 16.2. *Suppose $\varphi : U \rightarrow V$ is a diffeomorphism between connected open subsets of $\mathbb{R}^n$. Then φ is affine if and only if φ^* takes constant 1-forms to constant 1-forms.*

Proof. Recall that

$$\varphi^*(dx_i) = \sum_j \frac{\partial \varphi_i}{\partial x_j} dx_j.$$

Lastly,

$$\begin{aligned} \varphi \text{ is affine} &\iff D\varphi \text{ is constant} \\ &\iff \varphi^*(dx_i) \text{ is a constant 1-form} \\ &\iff \varphi \text{ takes constant 1-forms to constant 1-forms} \end{aligned}$$

□

Corollary 16.3. *Suppose M has an atlas of charts whose transition functions pull back constant 1-forms to constant 1-forms. Then, this atlas gives an affine structure on M .*

□

Example 16.4. A flat (0-sectional curvature) Riemannian manifold has an affine structure because the transition maps of a flat Riemannian manifold lies in $\text{Isom}(\mathbb{R}^n)$ and

$$\text{Isom}(\mathbb{R}^n) \subset \text{Aff}(\mathbb{R}^n).$$

Remark 16.5. The previous example suggests that affine structures are a natural generalization of flat Riemannian structures. We can make this precise by noting that an affine structure on M provides a globally consistent notion of “straightness” by modeling the manifold locally on $\mathbb{R}^m$. More concretely, an embedded submanifold $N^n \subset M^m$ is considered “straight” (or an affine submanifold) if, within any local affine coordinate chart, N coincides with a union of affine subspaces of $\mathbb{R}^m$.

Definition 16.6. Analogous to Definition 16.1,

- (1) The **integral affine group of** $\mathbb{R}^n$ is defined as:

$$\text{Aff}_{\mathbb{Z}}(\mathbb{R}^n) = \{x \mapsto Ax + b \mid A \in \text{GL}(n, \mathbb{Z}), b \in \mathbb{R}^n\}.$$

- (2) An **integral 1-form on** $U \subseteq \mathbb{R}^n$ is the exterior derivative, d , of a linear function with integral coefficients. Equivalently, in coordinates, integral 1-forms are those that can be expressed as

$$\sum m_i dx_i$$

where the m_i 's are integers.

- (3) An **integral affine structure on a manifold** M^n is an atlas of charts all of whose transition functions are integral affine maps. Alternatively, the transition functions pullback integral 1-forms to integral 1-forms.

Remark 16.7. An integral affine structure on a manifold M gives

- (1) a canonical volume form $d \text{vol}$ on M (assuming M is oriented¹¹) by taking

$$d \text{vol} := dx_1 \wedge \cdots \wedge dx_n,$$

in “oriented integral affine charts” of M . This is well-defined because the derivatives of the transition maps take values in

$$GL(n, \mathbb{R})^+ \cap GL(n, \mathbb{Z}) \subseteq SL(n, \mathbb{R}).$$

- (2) a notion of “rational directions” on M by taking $v \in T_x M$ to be rational if

$$\alpha(v) \in \mathbb{Q} \quad \forall \text{ integral 1-forms } \alpha.$$

- (3) a notion of integral affine submanifolds $N \subset M$. Geometrically, N is an integral affine submanifold if it looks locally “straight” and aligns with the integral structure of M ; that is, within any integral affine coordinate chart, N coincides with a union of integral affine subspaces¹² of $\mathbb{R}^n$.

Exercise: What are the integral affine submanifolds of $\mathbb{R}^n/\mathbb{Z}^n$?

Example 16.8. Given $A \in GL(n, \mathbb{Z})$ and an invariant open subset $\Omega \subset \mathbb{R}^n$ on which $\langle A \rangle$ acts properly discontinuously such that the quotient is Hausdorff¹³, we have:

$$\Omega/\langle A \rangle$$

is a manifold with an integral affine structure. In the case $n = 2$ we have three possibilities:

- (1) A is elliptic. For example,

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

has order 6 and $\langle A \rangle/\langle A^6 \rangle$ acts freely properly discontinuously on

$$\Omega := \mathbb{R}^2 \setminus \{0\}.$$

- (2) A is parabolic. For example,

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

¹¹When M is not oriented, we only get a canonical density or a measure on M as non-oriented manifolds do not admit global volume forms.

¹²A subspace $W \subset \mathbb{R}^n$ is integral if W is the span of integral vectors. An affine subspace $H \subset \mathbb{R}^n$ is integral if H is a translation of an integral subspace by a real vector $b \in \mathbb{R}^n$. Alternatively, H is an integral affine subspace if H can be defined by linear equations whose coefficients are all integral except possibly the constant terms.

¹³To ensure the quotient space X/Γ of a properly discontinuous action is Hausdorff, the action must satisfy an additional separation condition (proper action). Specifically, for any two points $x, y \in X$ that do not lie in the same orbit, there must exist neighborhoods U_x of x and U_y of y such that $g(U_x) \cap U_y = \emptyset$ for all $g \in \Gamma$. In the third example above, $\langle A \rangle$ acts properly discontinuously on $\mathbb{R}^2 \setminus \{(0, 0)\}$, but the resulting quotient is not Hausdorff. Because of the hyperbolic dynamics of the matrix A , if we pick one point on the expanding eigenline and another on the contracting eigenline, any open neighborhoods around them will eventually be stretched and compressed into each other by the group action.

acts freely properly discontinuously on

$$\Omega := \{(x, y) \in \mathbb{R}^2 : y > 0\}.$$

(3) A is hyperbolic. For example,

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

acts freely properly discontinuously on

$$\Omega := \mathbb{R}^2 \setminus (E_s \cup E_u),$$

where

E_u = unstable generalized eigen-space

$$= \text{span} \left\{ \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} \right\}$$

E_s = stable generalized eigen-space

$$= \text{span} \left\{ \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix} \right\}.$$

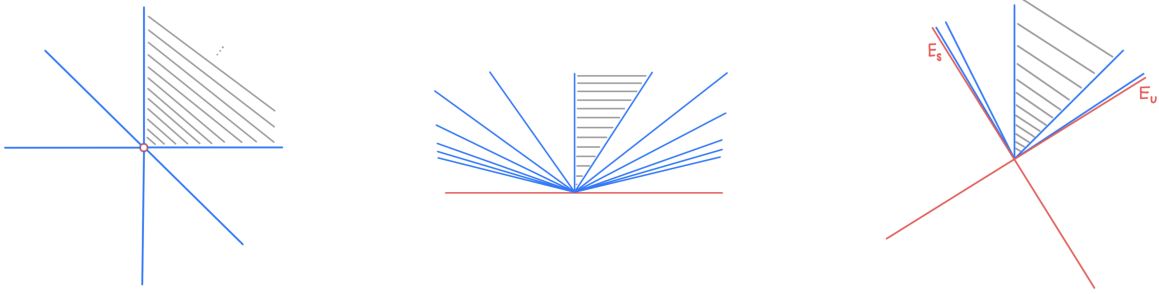


FIGURE 22. Fundamental domains under the action of an elliptic, parabolic, and hyperbolic A on $\mathbb{R}^2$.

(G, X) -structures on manifolds. An affine structure on a manifold M is a special kind of geometry where the transition maps take values in $\text{Aff}(\mathbb{R}^n)$. We can consider other geometries on a manifold where the transition maps take values in a specified Lie group G . Towards this, consider a manifold X and a Lie group G with an action

$$G \curvearrowright X$$

that is faithful¹⁴ and transitive. Pick a basepoint $x_0 \in X$ and consider the orbit map:

$$\phi: G \rightarrow X$$

$$g \mapsto g \cdot x_0.$$

By the transitivity of the action, it follows that

$$G/\text{Stab}(x_0) \simeq X$$

¹⁴Recall that G acts on X faithfully if $g = e$ is the only element that acts as the identity on X .

that is, X is a homogeneous space.

Example 16.9. Examples of (G, X) :

- $(\text{Aff}(\mathbb{R}^n), \mathbb{R}^n)$
- $(\text{Isom}(\mathbb{R}^n), \mathbb{R}^n)$
- $(\text{Isom}(S^n), S^n)$
- $(\text{Isom}(\mathbb{H}^n), \mathbb{H}^n)$
- $(\text{PGL}(n+1, \mathbb{R}), \mathbb{R}P^n)$
- $(\text{PGL}(n+1, \mathbb{C}), \mathbb{C}P^n)$
- $(\text{PSp}(2g, \mathbb{R}), \mathfrak{h}_g)$

Given a pair (G, X) , we call X the **model space** and G the **group of symmetries**.

Remark 16.10. Notice that in all the examples above, G is a finite-dimensional Lie group. One could also consider the pair $(\text{Diffeo}(M), M)$, but that would be quite different.

Definition 16.11. A (G, X) -**structure on a manifold** M is an atlas of charts $\{(U_i, \varphi_i)\}$ where the charts $\varphi_i : U_i \rightarrow X$ take values in the model space X and the transition maps $\varphi_i \circ \varphi_j^{-1}$ are restrictions of action by elements of the group of symmetries G . The charts of such an atlas are called (G, X) -charts.

Example 16.12.

- (1) An affine structure is an $(\text{Aff}(\mathbb{R}^n), \mathbb{R}^n)$ -structure.
- (2) An integral affine structure is a $(\text{GL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n, \mathbb{R}^n)$ -structure.
- (3) A Riemannian manifold with constant curvature -1 is an $(\text{Isom}(\mathbb{H}^n), \mathbb{H}^n)$ -structure.

A (G, X) -structure on M gives rise to a developing map and a holonomy representation.

Lemma 16.13. *Suppose M has a (G, X) -structure.*

- (1) *There exists a global **developing map***

$$\text{dev} : \widetilde{M} \rightarrow X$$

that is compatible with the (G, X) -charts of M . That is, for any chart (φ, U) where U has a lift $\widetilde{U} \subset \widetilde{M}$, the two maps:

$$\varphi \circ \pi : \widetilde{U} \rightarrow X$$

$$\text{dev} : \widetilde{U} \rightarrow X$$

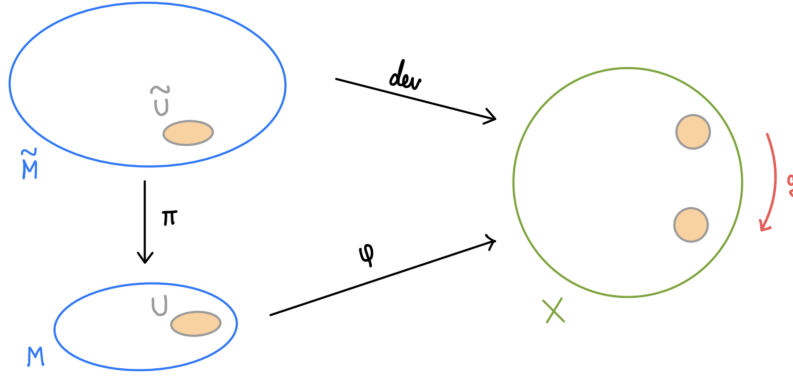
are equal up to postcomposition by the action of G . Pictorially, see Figure 23.

- (2) *For any choice of $\text{dev} : \widetilde{M} \rightarrow X$, there exists a unique group homomorphism called the **holonomy representation**:*

$$\rho : \pi_1(M) \rightarrow G$$

such that $\text{dev}(\gamma \cdot x) = \rho(\gamma) \cdot \text{dev}(x)$ for all $\gamma \in \pi_1(M)$ and $x \in \widetilde{M}$. In other words, the developing map

$$\text{dev} : \widetilde{M} \rightarrow X$$

FIGURE 23. Developing map and compatibility with (G, X) -charts.

is equivariant with respect to the action of $\pi_1(M)$ on $\tilde{M}$ by deck transformations and the action of $\pi_1(M)$ on X via the holonomy ρ .

Reference: Goldman Theorem 5.2.2.

Recall, in Riemannian geometry, a Riemannian manifold M is complete if geodesics can be extended for all time.

Definition 16.14. A (G, X) -structure is **complete** if the developing map $\text{dev}: \tilde{M} \rightarrow X$ is a covering map (i.e. a global diffeomorphism if X is simply connected).

Example 16.15. Three examples of incomplete (G, X) -structures along with their developing maps and holonomy representations.

- (1) The punctured plane, $\mathbb{R}^2 \setminus \{0\}$, with an $(\text{Isom}(\mathbb{R}^2), \mathbb{R}^2)$ -structure:
 - Let $\pi: \widetilde{\mathbb{R}^2 \setminus \{0\}} \rightarrow \mathbb{R}^2 \setminus \{0\}$ be the universal cover.
 - $\text{dev}: \widetilde{\mathbb{R}^2 \setminus \{0\}} \rightarrow \mathbb{R}^2$ is the same as π .
 - $\rho: \pi_1(\mathbb{R}^2 \setminus \{0\}) \rightarrow \text{Isom}(\mathbb{R}^2)$ is the trivial homomorphism.
- (2) An open cylinder realized as the quotient

$$\{x > 0, y > 0\} / \langle A \rangle$$

where A is the doubling map which in terms of matrices is

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

This open cylinder has an $(\text{Aff}(\mathbb{R}^2), \mathbb{R}^2)$ -structure:

- Let $\pi: \{x > 0, y > 0\} \rightarrow \{x > 0, y > 0\} / \langle A \rangle$ be the universal cover.
 - $\text{dev}: \{x > 0, y > 0\} \rightarrow \mathbb{R}^2$ is just the open inclusion.
 - $\rho: \pi_1(\{x > 0, y > 0\} / \langle A \rangle) \simeq \langle A \rangle \rightarrow \text{Aff}(\mathbb{R}^2)$ is just the inclusion of $A \in \text{GL}(2, \mathbb{R})$ into $A \in \text{Aff}(\mathbb{R}^2)$.
- (3) A torus realized as the quotient

$$(\mathbb{R}^2 \setminus \{0\}) / \langle A \rangle$$

where A is again the doubling map. This torus has an $(\text{Aff}(\mathbb{R}^2), \mathbb{R}^2)$ -structure:

- Let $\pi: \widetilde{\mathbb{R}^2 \setminus \{0\}} \rightarrow (\mathbb{R}^2 \setminus \{0\}) / \langle A \rangle$ be the universal cover.

- $\text{dev}: \widetilde{\mathbb{R}^2 \setminus \{0\}} \rightarrow \mathbb{R}^2$ is the same as the developing map in the first example.
- $\rho: \pi_1((\mathbb{R}^2 \setminus \{0\})/\langle A \rangle) \rightarrow \text{Aff}(\mathbb{R}^2)$ is given by

$$\begin{aligned} \pi_1((\mathbb{R}^2 \setminus \{0\})/\langle A \rangle) &\simeq \langle A, \gamma \rangle \rightarrow \text{Aff}(\mathbb{R}^2) \\ A &\mapsto A \\ \gamma &\mapsto \text{Id} \end{aligned}$$

Here, γ is the element $\theta \mapsto e^{i\theta} \in (\mathbb{R}^2 \setminus \{0\})/\langle A \rangle$.



FIGURE 24. Affine structures on open cylinder and torus obtained as a quotient of action of A on quadrant and punctured plane.

Remark 16.16.

- (1) Unlike Riemannian geometry, where compact Riemannian manifolds are always complete, the third example above shows that compact affine manifolds are not necessarily complete. To get a feeling for this, notice that the simple closed loop $\gamma \in (\mathbb{R}^2 \setminus \{0\})/\langle A \rangle$ cannot be represented by a “straight” curve (affine submanifold).
- (2) If the holonomy representation ρ is trivial, the equivariant map $\text{dev}: \widetilde{M} \rightarrow X$ becomes invariant under the action by deck transformations and descends to $\text{dev}: M \rightarrow X$.
- (3) If $\widetilde{M}$ is compact and X is non-compact, then M does not admit any (G, X) -structure because the developing map $\text{dev}: \widetilde{M} \rightarrow X$ would necessarily have critical points and fail to be a local isomorphism. For example, S^n does not have an affine structure.

Fact 16.17. *A closed surface Σ_g of genus $g \geq 2$ does not have an affine structure.*

To get a sense of this, notice that the following polygon in $\mathbb{R}^2$ can be used to define an (in fact, many) affine structure on $\Sigma_2 \setminus \{p\}$. But somehow, this affine structure cannot be extended over the corner point p .

17. ACTION-ANGLE COORDINATES (03/12, YL, MN)

This lecture covers material from [Eva23, Section 1.6].

Book-keeping Fact: Recall that there is a bijection between flat connections on a vector bundle $E \rightarrow B$ and atlases of local trivializations $\phi_i: E|_{U_i} \rightarrow U_i \times \mathbb{R}^n$ where $\phi_i \circ \phi_j^{-1}(x, v) = (x, A_{ij}v)$ for $x \in U_i \cap U_j$, and A_{ij} is a constant matrix.

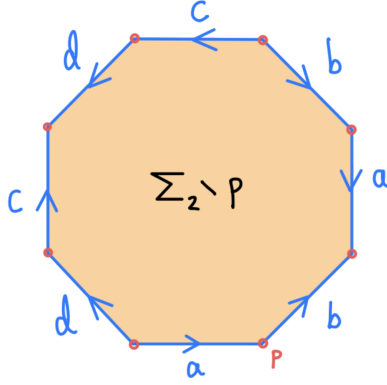


FIGURE 25. The regular octagon along with a choice of affine identification of the edges defines an affine structure on $\Sigma_2 \setminus \{p\}$.

Remark 17.1. There is a map $(*)$ from affine structures on M to flat connections on TM , defined by mapping an atlas of charts $\{(U_i, \phi_i)\}$ where $\phi_j \circ \phi_i^{-1} \in \text{Aff}(\mathbb{R}^n)$ to an atlas of charts $\{(U_i \times \mathbb{R}^n, (\phi_i, D\phi_i))\}$. Note that the transition maps are:

$$\begin{aligned} (\phi_j, D\phi_j) \circ (\phi_i, D\phi_i)^{-1} &= (\phi_j \circ \phi_i^{-1}, D\phi_j \circ (D\phi_i)^{-1}) \\ &= (\phi_j \circ \phi_i^{-1}, D(\phi_j \circ \phi_i^{-1})) \end{aligned}$$

where $D(\phi_j \circ \phi_i^{-1})$ is a constant matrix due to the affine nature of $\phi_j \circ \phi_i^{-1}$.

Furthermore, the map $(*)$ can be shown to be a bijection onto the torsion-free flat connections.

Auslander Conjecture: Let M be a complete compact affine manifold. Then $\pi_1(M)$ has a finite-index subgroup which is solvable.

Example 17.2. (Affine structures on the 2-torus) Take a quadrilateral in $\mathbb{R}^2$ where at least one pair of opposite sides are parallel, and pick two affine transformations ϕ_1, ϕ_2 that map edges to their opposite edges while leaving the lines containing the adjacent edges invariant. Identifying the edges of the quadrilateral using ϕ_1 and ϕ_2 defines an affine manifold homeomorphic to the 2-torus. (The identification of the two parallel edges has derivative that is a scalar multiple of the identity.)

Back to Evans Chapter 1:

Let (X^{2n}, ω) be a symplectic manifold. A map $H : X \rightarrow \mathbb{R}$ is a **complete commuting Hamiltonian system** if $\{H_i, H_j\} = 0$ for all $i, j \in \{1, \dots, n\}$.

If both of the following additional conditions are met, it is called an **integrable Hamiltonian system**:

- $H(X)$ has a dense open subset of regular values.
- H is proper (so that fibers are compact) and has connected fibers.

Given an integrable Hamiltonian system, fix an open subset $B \subset H(X)$ with a Lagrangian section $\sigma : B \rightarrow X$.

The **period lattice** at $b \in B$ is defined as:

$$\Lambda_b^H := \{t \in \mathbb{R}^n \mid \phi_t^H(\sigma(b)) = \sigma(b)\}$$

Note that this is a discrete subgroup in $\mathbb{R}^n$.

Define the **global period lattice** to be:

$$\Lambda^H := \{(b, t) \in B \times \mathbb{R}^n \mid t \in \Lambda_b^H\}$$

The “Liouville local coordinate” is the map $\Psi : B \times \mathbb{R}^n \rightarrow X$ given by $(b, t) \mapsto \phi_t^H(\sigma(b))$. We have the relation:

$$\Psi^*\omega = \sum_{i=1}^n db_i \wedge dt_i$$

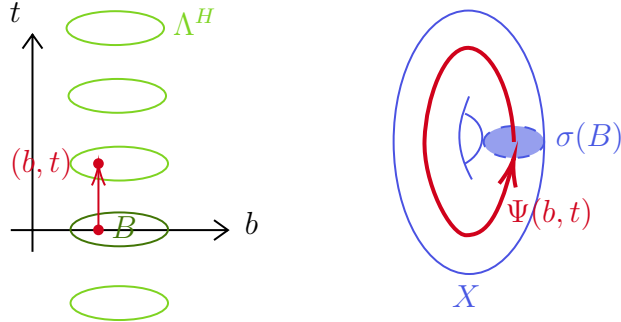


FIGURE 26. X viewed as quotient of $B \times \mathbb{R}^n$ by Λ^H

Theorem 17.3. (Little Arnold-Liouville) *Given an integrable Hamiltonian system, each orbit $\Omega(\sigma(b)) := \{\phi_t^H(\sigma(b)) \mid t \in \mathbb{R}^n\}$, for b a regular value, is a torus, i.e., it is diffeomorphic to $\mathbb{R}^n/\mathbb{Z}^n$.*

Theorem 17.4. (Action-Angle Coordinates) *Let $H : X \rightarrow B \subset \mathbb{R}^n$ be an integrable Hamiltonian system over a disc B with only regular fibers and with a Lagrangian section σ . Then, for any $b_0 \in B$, there is a subdisc $B_0 \subset B$ and a diffeomorphism $\alpha : B_0 \rightarrow C \subset \mathbb{R}^n$ such that if $G = \alpha \circ H$, then $\Lambda^G = C \times (2\pi\mathbb{Z})^n$.*

Corollary 17.5. *The map $(c, t) \mapsto \phi_t^G(\sigma(\alpha^{-1}(c)))$ from $C \times (\mathbb{R}^n/(2\pi\mathbb{Z})^n)$ to $G^{-1}(C) = H^{-1}(B_0)$ is a symplectomorphism.*

Proof. By definition, this map is a bijection, and as previously established, it is a local symplectomorphism. $\square$

Remark 17.6. The flows ϕ_t^G and ϕ_t^H have the same fibers and hence share the same orbits. We can think of ϕ_t^G as a reparametrization of ϕ_t^H .

Proof of Theorem 17.4.

Pick sections $\tau_i : B \rightarrow B \times \mathbb{R}^n$ such that $2\pi\tau_1(b), \dots, 2\pi\tau_n(b)$ form a $\mathbb{Z}$ -basis for $\Lambda_b^H \subset B \times \mathbb{R}^n$ at each $b \in B$. This global frame exists because the disc B is contractible.

Note that each image $2\pi\tau_i(B)$ is locally the preimage of the Lagrangian section $\sigma(B)$ via the local symplectomorphism Ψ . Consequently, each $\tau_i(B)$ is a Lagrangian submanifold of $B \times \mathbb{R}^n$.

Under the standard identification $B \times \mathbb{R}^n \cong T^*B$, each section τ_i naturally corresponds to a 1-form $\eta_i \in \Omega^1(B)$, given in local coordinates by

$$(\eta_i)_b := \sum_{j=1}^n (\tau_i)_j(b) db_j.$$

The condition that $\tau_i(B)$ is a Lagrangian submanifold is exactly equivalent to the condition that η_i is a closed 1-form.

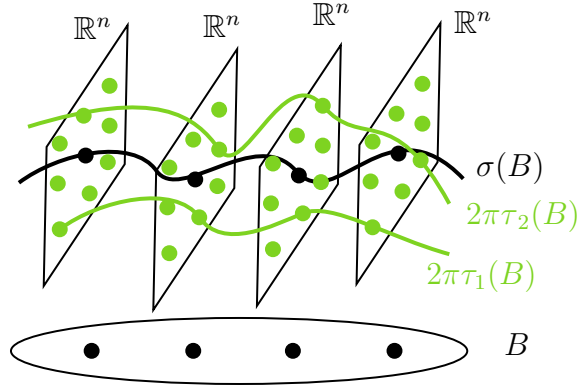


FIGURE 27. The n linearly independent Lagrangian sections (or images of closed 1-forms in T^*B) spanning the period Lattice over B .

Because B is a simply connected disc, these closed forms are exact. We can therefore define the map $\alpha : B \rightarrow \mathbb{R}^n$ using:

$$d\alpha_i := \eta_i,$$

or equivalently as

$$\alpha(b) := \left(\int_{b_0}^b \eta_1, \dots, \int_{b_0}^b \eta_n \right).$$

Because the sections $\{\tau_i\}_{i=1}^n$ span the fibers of $B \times \mathbb{R}^n$, the 1-forms $\{(\eta_i)_{b_0}\}_{i=1}^n$ form a basis for $(T^*B)_{b_0}$. Thus, the components of $d\alpha$ are linearly independent at b_0 , meaning there is a disc neighborhood $B_0 \subset B$ of b_0 where α is a diffeomorphism from B_0 to $\alpha(B_0) =: C$.

Without loss of generality, we can set $B_0 = B$ and let $G = \alpha \circ H$. We now want to show that $\Lambda^G = C \times (2\pi\mathbb{Z})^n$.

Key Claim: If we view (c, t) as an element of T^*C , then $\Psi^G(c, t) = \Psi^H(\alpha^*(c, t))$. Assuming the claim holds, we can compute the new period lattice directly:

$$\begin{aligned}
\Lambda^G &= \{(c, t) \mid \Psi^G(c, t) = \Psi^G(c, 0)\} \\
&= \{(c, t) \mid \Psi^H(\alpha^*(c, t)) = \Psi^H(\alpha^*(c, 0))\} \\
&= \{(c, t) \mid \alpha^*(c, t) \in \Lambda^H\} \\
&= \{(c, t) \mid \alpha^*(c, t) \in \text{span}_{\mathbb{Z}}\{2\pi\eta_1, \dots, 2\pi\eta_n\}\} \\
&= \{(c, t) \mid \alpha^*\left(\sum_i t_i dc_i\right) \in \text{span}_{\mathbb{Z}}\{2\pi\eta_1, \dots, 2\pi\eta_n\}\} \\
&= \{(c, t) \mid \sum_i t_i \cdot \alpha^*(dc_i) \in \text{span}_{\mathbb{Z}}\{2\pi\eta_1, \dots, 2\pi\eta_n\}\} \\
&= \{(c, t) \mid \sum_i t_i d\alpha_i \in \text{span}_{\mathbb{Z}}\{2\pi\eta_1, \dots, 2\pi\eta_n\}\} \\
&= \{(c, t) \mid \sum_i t_i \eta_i \in \text{span}_{\mathbb{Z}}\{2\pi\eta_1, \dots, 2\pi\eta_n\}\} \\
&= C \times (2\pi\mathbb{Z})^n \quad \square
\end{aligned}$$

Remark 17.7. The key idea in the preceding proof was to shift focus from the Hamiltonian H and the Lagrangian section σ to the cotangent bundle T^*B and the period lattice $\Lambda \subseteq T^*B$ spanned by closed 1-forms. Over a simply connected base, this manifests as an equivalence of categories:

$$(B, X, H, \sigma) \longleftrightarrow (B, \Lambda^B),$$

where the dictionary between them is given by:

$ \begin{array}{ccc} \Lambda_b^H & \longrightarrow & (X, \omega) \\ \sigma \uparrow & & \downarrow H \\ & & B \end{array} $	$ \begin{array}{ccc} \Lambda_b^B & \longrightarrow & (T^*B/\Lambda^B, \omega_{\text{std}}) \\ 0 \uparrow & & \downarrow \pi \\ & & B \end{array} $
σ : Lagrangian section	0 : zero section (closed 1-form)
$B \times \mathbb{R}^n \rightarrow X$: Liouville local coordinates	$B \times \mathbb{R}^n \cong T^*B \rightarrow T^*B/\Lambda^B$: quotient by lattice Λ^B
$\Lambda^H = \text{span}_{\mathbb{Z}}\{\mathbb{Z}\text{-basis}\}$ $= \text{span}_{\mathbb{Z}}\{\sigma_i : B \rightarrow B \times \mathbb{R}^n \mid n \text{ Lag sec}\}$	$\Lambda^B = \text{span}_{\mathbb{Z}}\{\mathbb{Z}\text{-basis}\}$ $= \text{span}_{\mathbb{Z}}\{2\pi\tau_i \mid n \text{ closed 1-forms}\}$
ϕ_t^H : flow on X induced by ω and $H^*db_1, \dots, H^*db_n$	ϕ_t^B : flow on T^*B/Λ^B induced by ω_{std} and $\pi^*db_1, \dots, \pi^*db_n$
Integral affine structure on B induced by H	Flat torsion-free connection on T^*B induced by the local system Λ^B

18. LAGRANGIAN FIBRATIONS (03/17, JS, YW)

Reference: [Eva23, Chapter 1.6-2.2].

Last time, in the discussion of action-angle coordinates, we had the following.

Theorem 18.1 (Action–Angle coordinates). *Let $H : X \rightarrow B \subseteq \mathbb{R}^n$ be an integral Hamiltonian system with a Lagrangian section. Then, locally near a regular fiber, we can find $\alpha : B \rightarrow C \subseteq \mathbb{R}^n$ such that*

$$\Lambda^{\alpha \circ H} = C \times 2\pi\mathbb{Z}^n.$$

Set $G = \alpha \circ H$.

Recall we have $\Phi^H(b, t) = \phi_t^H(\sigma(b))$ and $\Phi^G(c, t) = \phi_t^G(\sigma(\alpha^{-1}(c)))$.

Key claim. If we view $(c, t) \in C \times \mathbb{R}^n$ as an element of T^*C , then

$$\Phi^G(c, t) = \Phi^H(\alpha^*(c, t)).$$

So it is natural to think of $C \times \mathbb{R}^n$ as T^*C .

Proof of claim. It suffices to note that the usual definition

$$\Phi^H(b, t) = \phi_t^H(\sigma(b))$$

really only depends on $\sum t_i dh_i$ as a 1-form, instead of the map H .

One way to write ϕ_t^H is

$$\phi_{t_1}^{H_1} \circ \phi_{t_2}^{H_2} \circ \dots \circ \phi_{t_n}^{H_n},$$

with $H_i = b_i \circ H$; another way is (exercise in Evans) $\phi_1^{\sum t_i H_i}$. By definition, this is the flow at $t = 1$ in the vector field X solving

$$\begin{aligned} \iota_X \omega &= -d\left(\sum t_i H_i\right) \\ &= -H^*\left(\sum t_i db_i\right) \quad (\text{by naturality}) \end{aligned}$$

Note that this only depends on $\sum t_i dh_i$. □

The following theorem is an analog of the orbit-stabilizer theorem.

Corollary 18.2 (Big Arnold–Liouville Theorem). *For any integral Hamiltonian system, any regular fiber has a neighborhood isomorphic to $B \times T^n$, where $T^n = \mathbb{R}^n / (2\pi\mathbb{Z})^n$.*

More on the coordinate-free point of view: For any closed 1-form df on B (a disc), we get a closed 1-form $H^*(df) = d(f \circ H)$ on X . There is a map

$$\{\text{closed 1-forms on } B\} \longrightarrow \text{Ham}(X, \omega) \quad (*)$$

$$df \longmapsto \phi_1^{f \circ H}.$$

Remark 18.3. Some people call this an “action of T^*B on X .” That is, solve $\iota_X(\omega) = -H^*(df)$ and then flow to time 1.

The kernel of $(*)$ corresponds to

$$\Lambda^H \subset B \times \mathbb{R}^n = T^*B.$$

Key fact:

The period lattice Λ_H is standard $\iff$ kernel of $(*)$ is $2\pi \operatorname{span}_{\mathbb{Z}}(db_i)$ $(**)$.

(This is why we get integral affine structures.)

18.1. Lagrangian Fibrations.

Definition 18.4. A stratification of a topological space B is a filtration

$$\emptyset = B_{-1} \subseteq B_0 \subseteq B_1 \subseteq \cdots$$

where

$$B = \bigcup_i B_i,$$

and

$$S_d(B) := B_d - B_{d-1};$$

this is a d -dimensional manifold (possibly $\emptyset$). We say B is n -dimensional if

$$B_n \neq \emptyset, \quad S_d(B) = \emptyset \text{ for } d > n.$$

The S_i are called strata, and S_n is the top stratum.

Definition 18.5. Let (X^{2n}, ω) be a symplectic manifold, and B be an n -dimensional stratified space. A proper continuous map $f : X \rightarrow B$ is a Lagrangian (torus)¹⁵ fibration if

- (1) f restricted to the preimage of the top stratum is a smooth submersion with connected Lagrangian fibers.
- (2) The other fibers are connected stratified spaces with isotropic strata.

We call $B^{\operatorname{reg}} := S_n(B)$ the regular locus ($n = \dim B$).

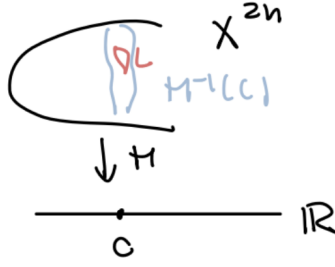
Warnings.

- (1) B is now very general (not always $\subseteq \mathbb{R}^n$).
- (2) Lagrangian torus fibrations need not be fibrations.

Definition 18.6. A Lagrangian fibration is regular if $B = B^{\operatorname{reg}}$ (Therefore, f is a proper submersion with connected Lagrangian fibers).

Lemma 18.7. Let (X, ω) be a symplectic manifold. Suppose $H : X \rightarrow \mathbb{R}$ is a (Hamiltonian) function and L is a Lagrangian submanifold contained in a fiber $H^{-1}(c)$. Then L is invariant under ϕ_t^H .

¹⁵We will see that the torus part follows from the conditions, so the two names are the same.



Proof. H is constant on $L \subseteq H^{-1}(c)$, so $dH(w) = 0$ for all $w \in TL$. By definition,

$$\iota_{V_H}(\omega) = -dH.$$

So, if $w \in T_x L$,

$$\omega(V_H, w) = (\iota_{V_H} \omega)(w) = -dH(w) = 0.$$

Thus, V_H is symplectically orthogonal to TL . Because L is a Lagrangian, we have $V_H \in TL$. Since V_H is tangent to L , the flow in V_H preserves L . $\square$

Lemma 18.8. *Let (X^{2n}, ω) be symplectic manifold and $f : X \rightarrow B^n$ be a regular Lagrangian fibration. Let $b_1, \dots, b_n$ be local coordinates for B . Then*

$$\{b_i \circ f, b_j \circ f\} = 0 \quad \forall i, j.$$

Proof. We have

$$\{b_i \circ f, b_j \circ f\} = \omega(V_{b_i \circ f}, V_{b_j \circ f}).$$

This is 0 since the fibers of f are Lagrangian and the $V_{b_i \circ f}$ are tangent to the fibers by the last lemma. $\square$

We have thus obtained:

Corollary 18.9. *The $b_i \circ f$ are an integral Hamiltonian system.*

So “ f restricted to a coordinate chart” is an integral Hamiltonian system.

Corollary 18.10. *The fibers are Lagrangian tori.*

Next, we are going to see that there is an integral affine structure on the base.

Lemma 18.11. *Suppose $G : X \rightarrow \mathbb{R}^n$, $H : X \rightarrow \mathbb{R}^n$ are submersions defining integral Hamiltonian systems with standard period lattices Λ^H, Λ^G , respectively.*

Suppose $\varphi : H(X) \rightarrow G(X)$ is a diffeomorphism (change of coordinates on the base) and $G = \varphi \circ H$. Then φ is integral affine

$$\text{i.e. } D\varphi \in GL(n, \mathbb{Z}).$$

Proof. **Proof 1.** See Evans’ text.

Proof 2. From (**), the standard period lattice is related to

$$2\pi \text{span}_{\mathbb{Z}}(db_i).$$

Therefore, φ^* must pull back $2\pi \text{span}_{\mathbb{Z}}(dx_i)$ to $2\pi \text{span}_{\mathbb{Z}}(dx_i)$.

Hence it pulls back integral 1-forms to integral 1-forms; whence φ is affine. $\square$

Corollary 18.12. *If $f : X \rightarrow B$ is a regular Lagrangian fibration, then B has a natural integral affine structure.*

Proof. We cover B by charts where f is an integral Hamiltonian system. Hence, we can cover by charts where additionally the period lattice is standard. $\square$

19. TORUS ACTIONS AND MOMENT IMAGES (03/19, AK, JZ)

Reference: [Eva23, Chapter 3.1]

Recall that given a regular Lagrangian fibration $f : X \rightarrow B$, the base space B inherits an integral affine structure. One question we should think about is whether this structure depends on the choice of Lagrangian section $\sigma : B \rightarrow X$.

Theorem 19.1. *Suppose we have regular Lagrangian fibrations $f : X \rightarrow B$, $g : Y \rightarrow B$ that give the same integral affine structure on B . If both fibrations have global Lagrangian sections, σ and τ , respectively, then there is a unique fibered symplectomorphism $\Phi : X \rightarrow Y$ such that $\Phi \circ \sigma = \tau$.*

Proof. See [Eva23, Theorem 2.26]. $\square$

Let $T^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ be the n -torus. The Arnold-Liouville theorem states that under suitable assumptions and after a suitable change of coordinates, the $\mathbb{R}^n$ -action generated by the Hamiltonian H factors through a T^n -action. From here on, we will work backwards, assuming that there is a global T^n -action and analyze the critical points of the Hamiltonian. From this viewpoint the Hamiltonian will also be called the moment map.

We note that every T^n -action gives a $\mathbb{R}^n$ -action, and conversely every $\mathbb{R}^n$ -action with $(2\pi\mathbb{Z})^n$ contained in its kernel gives a T^n -action. The stabilizer of a point by a Hamiltonian torus action can either refer to a subgroup of T^n or its lift to $\mathbb{R}^n$. We will call both of them ‘stabilizers’ interchangeably.

Definition 19.2. Let $H : X \rightarrow \mathbb{R}^n$ be an integrable Hamiltonian system whose associated $\mathbb{R}^n$ -action factors through T^n (also known as a **Hamiltonian torus action**). Then we call H the **moment map** for the torus action and denote it μ instead of H .

We also know from Theorem 19.1 that the image of the Hamiltonian sometimes determines the integrable Hamiltonian system up to fibered symplectomorphism. This motivates us to focus on the image $\mu(X)$ of the moment map.

Definition 19.3. A symplectic $2n$ -manifold is a **toric manifold** if it admits a Hamiltonian T^n -action. Moreover, $\mu(X)$ is called the **moment image** or **moment polytope**.

One also requires the action to be effective (no non-identity element acts trivially).

Example 19.4. Let us analyze the Hamiltonian torus action on the fibers of critical points. Suppose the moment image $\mu(X)$ is a closed (round) two-dimensional disc. (This will lead to a contradiction.)

- Take a point $p \in \partial\mu(X)$ with vertical tangency, which implies that $\mu_1 = x_1 \circ \mu$ has a critical point at every point in the fiber $\mu^{-1}(p)$. Hence $d\mu_1 = 0$ and the flow $\phi_t^{\mu_1}$ acts trivially on that fiber. On the other hand, since $d\mu_2$ is nonzero at points in $\mu^{-1}(p)$, the action of $\phi_t^{\mu_2}$ on the fiber is not trivial. Therefore, for points in the fiber, their stabilizer contains the subgroup $\mathbb{R}e_1$ but is not all of $\mathbb{R}^2$. Hence, the fiber is $\mathbb{R}^2/(\mathbb{R} \times 2\pi\mathbb{Z}) \cong S^1$, i.e., $\mu^{-1}(p)$ is a circle.

Compare this to what happens to interior points of the moment image, where the fibers are 2-tori with discrete stabilizers.

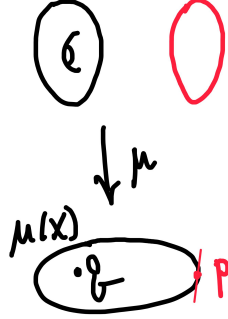


FIGURE 28. The fiber of $p \in \partial\mu(X)$ with vertical tangency is S^1 .

- Now suppose that $q \in \partial\mu(X)$ has an irrational tangent slope, say orthogonal to $(1, \sqrt{2})$. So $\mu_1 + \sqrt{2}\mu_2$ has a critical point at every point of the fiber $\mu^{-1}(q)$. Hence $d\mu_1 + \sqrt{2}d\mu_2 = 0$ and the stabilizer of a point on the fiber contains $\mathbb{R} \cdot (e_1 + \sqrt{2}e_2)$. Its image under the quotient map $\mathbb{R}^2 \rightarrow T^2$ is dense, and since stabilizers are closed, the stabilizer under the T^2 -action must in fact be all of T^2 . Thus, the stabilizer under the $\mathbb{R}^2$ action is all of $\mathbb{R}^2$. It follows that the fiber $\mu^{-1}(q)$ is just a single point.

On the other hand, traveling parallel to the tangent slope should result in greatest change, i.e., $d(-\sqrt{2}\mu_1 + \mu_2)$ won't be zero! Therefore we should still expect the fiber to be a circle.

In reality, $\mu(X)$ cannot be a disc, which can be explained by the contradicting descriptions of the fiber $\mu^{-1}(q)$ above. The Atiyah-Guillemin-Sternberg convexity theorem, which we will precisely state next lecture, tells us that the moment image is always rational convex polytope. We won't directly prove this fact, but the following proposition provides some evidence for it.

Proposition 19.5. *Let (X^{2n}, ω) be a symplectic manifold and $\mu : X \rightarrow \mathbb{R}^n$ be a moment map for a Hamiltonian T^n -action. Assume*

$$\delta : (0, 1)^{n-1} \rightarrow \partial\mu(X)$$

is a smooth embedding with a smooth lift $\gamma : (0, 1)^{n-1} \rightarrow X$ such that $\delta = \mu \circ \gamma$. Then the image of δ is contained in an affine hyperplane of rational slope ($\alpha \in \mathbb{Q}^n$, $c \in \mathbb{R}$):

$$\Phi = \{x \in \mathbb{R}^n \mid \alpha \cdot x = c\}.$$

Moreover, if $z \in \mu^{-1}(\text{im}(\delta))$, then $\text{Stab}(z) = \mathbb{R}\alpha$.

Proof. The main claim is that a vector $\alpha = (\alpha_1, \dots, \alpha_n)$ normal to $\text{im}(\delta) \subset \partial\mu(X)$ is proportional to a rational vector. Suppose for contradiction that it is not.

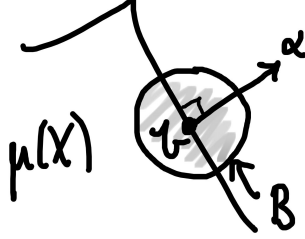


FIGURE 29. Boundary of moment map with normal vector α .

Fix a point $b = \delta(t)$ and ball B around b . Take an affine function $s : B \subset \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\ker s = T_b \partial\mu(X)$ and $H = s \circ \mu$ is locally minimized on $\mu^{-1}(b)$. So for $z \in \mu^{-1}(b)$ we have $0 = dH_z = ds_b \circ d\mu_z$.

Observe that H generates a $\mathbb{R}$ -action by $\{t \cdot (\alpha_1, \dots, \alpha_n) : t \in \mathbb{R}\} \subset T^n$. Note that this $\mathbb{R}$ -action fixes all points of the fiber $\mu^{-1}(b)$, for the same reason as demonstrated in the example above. The stabilizer of any $z \in \mu^{-1}(b)$ is a closed subgroup of T^n , so if α is not rational, we have $\dim \text{Stab}(z) \geq 2$. So we can pick two directions in the stabilizer, for example $\mathbb{R}\langle e_1, e_2 \rangle \subset \text{Stab}(z)$. Then $d\mu_1, d\mu_2 = 0$ on $\mu^{-1}(b)$ and so $\text{rk}(D\mu) \leq n - 2$.

Recall we assume the existence of a lift $\gamma : (0, 1)^{n-1} \rightarrow X$. So $\text{rk}(\delta) \leq n - 2$, which contradicts the assumption that δ is a smooth embedding. This proves the main claim. It follows that α is constant (because it is continuous and takes values in a countable set) and hence that $\text{im}(\delta)$ is a hyperplane. □

Definition 19.6. A **rational convex polytope** of $\mathbb{R}^n$ is a finite intersection of half spaces $S_{\alpha, b} = \{x \in \mathbb{R}^n \mid \alpha \cdot x \leq b\}$ with $\alpha \in \mathbb{Z}^n$ and $b \in \mathbb{R}$. We say that such a polytope is **Delzant** if every point has a neighborhood integral affine equivalent to a neighborhood of zero in $[0, \infty)^{n-k} \times \mathbb{R}^k$. Such a point is said to be in a **k -face**. A vertex is called **Delzant** if the germ of the polytope at that vertex is Delzant.

Here is a simple example that demonstrates a unique feature of Delzant polytopes. Let's temporarily drop the integrality and call such polytopes $\mathbb{R}$ -Delzant (this is non-standard terminology). In $n = 1$ and $n = 2$, all polytopes are $\mathbb{R}$ -Delzant. But for $n = 3$, take a cone where there are four 2-faces meeting at a cone point. Note that there is no affine equivalence between a neighborhood of that cone point and a neighborhood of the cone point of $[0, \infty)^3$, where three 2-faces meet at the cone point. Hence this 3-dimensional cone is not $\mathbb{R}$ -Delzant.

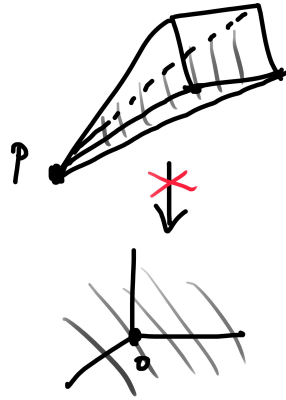


FIGURE 30. A three-dimensional polytope with a non-Delzant cone point.

In the next lecture, we will continue with our effort of better understanding the concept of Delzant polytopes by reintroducing the integrality condition.

20. DELZANT POLYTOPES AND EXAMPLES (03/26, JS, YW)

Reference: [Eva23, Chapter 3.2-3.3].

We begin by recalling the following definition from the previous lecture.

Definition 20.1. A **rational convex polytope** P is a subset of $\mathbb{R}^n$ defined as the intersection of a finite collection of half-spaces

$$S_{\alpha,b} = \{x \in \mathbb{R}^n \mid \alpha \cdot x \leq b\}$$

with $\alpha \in \mathbb{Z}^n$ and $b \in \mathbb{R}$.

We say such a P is **Delzant** if every point has a neighborhood isomorphic via an integral affine map to a neighborhood of the origin in

$$[0, \infty)^{n-k} \times \mathbb{R}^k.$$

A *vertex* is called Delzant if the germ at that vertex is Delzant.



FIGURE 31. Non-Delzant Polygon P

Example 20.2. The polygon P above is not Delzant, because the vertex $(0,0)$ is not Delzant. Indeed, the primitive integral vectors in the directions of the $\mathbb{Z}$ edges leaving $(0,0)$ are $(0,1)$ and $(2,1)$. Note that

$$\text{span}_{\mathbb{Z}} \langle (0,1), (2,1) \rangle = \{(a,b) \in \mathbb{Z}^2 \mid a \text{ even}\}$$

is of index 2 in $\mathbb{Z}^2$. This index is invariant under the $GL_2(\mathbb{Z})$ -action. This can also be seen from the fact that the fundamental domain at this vertex has area

$$|\det \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}| = 2.$$

This area is also invariant under the $GL_2(\mathbb{Z})$ -action. However, the vertex $(0, 1)$ is Delzant, as the primitive integral vectors in the two edge directions are $(0, 0), (0, 1)$. Similarly, we can also observe that $(2, 1)$ is Delzant as well.

A fundamental result is the following:

Theorem 20.3. *Let X be a compact toric manifold (a compact symplectic manifold X^{2n} with a Hamiltonian $\mathbb{T}^n$ action and a moment map $\mu : X \rightarrow \mathbb{R}^n$). Then*

- (1) *(Atiyah–Guillemin–Sternberg convexity Theorem) The moment image $\Delta := \mu(X)$ is a Delzant polytope. Moreover, if X is compact, Δ is the convex hull of $\mu(\text{Fix}_{\mathbb{T}^n})$.*
- (2) *(Delzant Existence Theorem) Every compact Delzant Δ arises in this way. Moreover, X can be taken to be a projective variety.*
- (3) *(Delzant Uniqueness Theorem) The moment polytope determines the pair (X, μ) up to a fibered symplectomorphism.*

We will not prove (1) or (3) although the arguments can be found in the standard texts, and the proof of (2) will be discussed later on in the course.

Next, we consider the example $(\mathbb{C}^n, \omega)$ with $\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$. As a warm up, let $H = \frac{1}{2}|z_1|^2 = \frac{1}{2}(x_1^2 + y_1^2)$ and $V = -y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1}$. Then $dH = x_1 dx_1 + y_1 dy_1$, and we can verify that $\iota_V \omega = -dH$. The flow of H corresponds to rotation in constant angular speed. The image $H(\mathbb{C}^n)$ is $[0, \infty)$.

More generally, we can work with $\mu : \mathbb{C}^n \rightarrow \mathbb{R}^n$, where

$$\mu(z_1, \dots, z_n) = \frac{1}{2}(|z_1|^2, \dots, |z_n|^2).$$

The Hamiltonian flow is $\phi_t^\mu(z_1, \dots, z_n) = (e^{it_1} z_1, \dots, e^{it_n} z_n)$. and the image is $\Delta = \mu(\mathbb{C}^n) = [0, \infty)^n$.

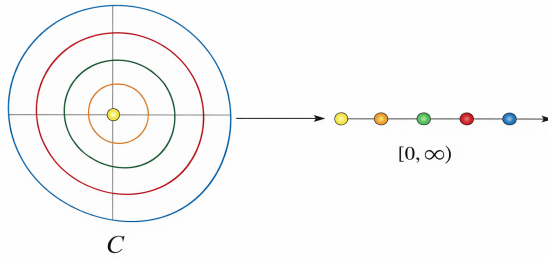


FIGURE 32. $n = 1$ Example

Understanding toy examples is key:

- $n = 1$: The moment map $\mu : \mathbb{C} \rightarrow [0, \infty)$ is in the sketch above, where the circles are 1 dimensional tori, and we have the degenerate circle corresponding to 0 in the image.
- $n = 2$: Here the preimages of the moment map $\mu : \mathbb{C}^2 \rightarrow [0, \infty) \times [0, \infty)$ behave in a similar manner. Write b_1, b_2 for the axes in the image of the map, and note that the preimage of a point on the positive b_1 -axis corresponds to a fixed radius circle. Recall that if we restrict to the b_1 -axis, then we are back in the $n = 1$ situation. Likewise, if we take a point in the interior of the plane, then one would obtain a torus, and if $(0, 0)$ in the image, this corresponds to the origin $\mathbb{C}^2$.



FIGURE 33. $n = 2$ Example

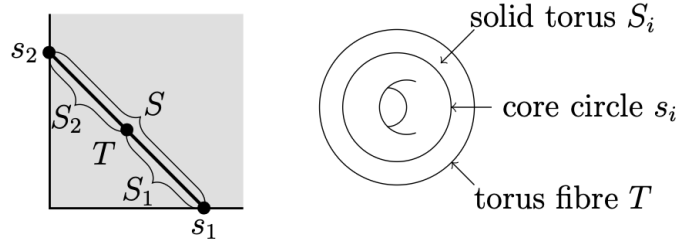
Remark 20.4. As we saw in these two initial examples, the critical values of μ are the boundary of the polytope; they are stratified: e.g. dimension 0 corresponds to points/vertices, dimension 1 corresponds to edges, and so on.

By definition, a Delzant polytope is locally isomorphic to $\mathbb{R}^k \times [0, \infty)^{n-k}$ in a neighborhood of a k -dimensional face. So, by Delzant uniqueness, this means the integral Hamiltonian system in a neighborhood of a critical point on a k -dimensional face is fibered isomorphic to

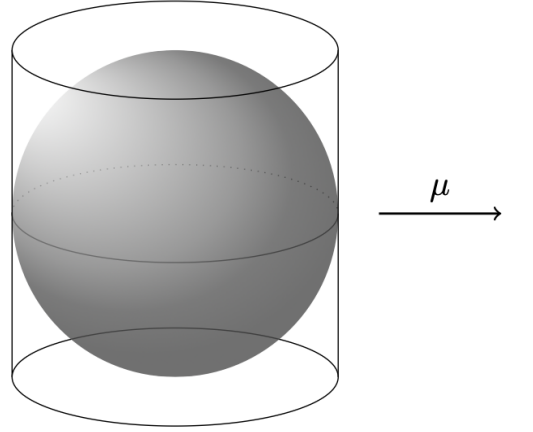
$$\mu : \mathbb{R}^k \times (S^1)^k \times \mathbb{C}^{n-k} \rightarrow \mathbb{R}^n, \quad \mu(p, q, z_{k+1}, \dots, z_n) = \left(p, \frac{1}{2}|z_{k+1}|^2, \dots, \frac{1}{2}|z_n|^2 \right).$$

The $\mathbb{R}^k \times (S^1)^k$ is the smooth part, whereas $\mathbb{C}^{n-k}$ is the singular part. If $k = n$, then this is precisely the action-angle coordinates. Additionally, in a neighborhood of 0 in $\mathbb{C}$, one obtains $\mathbb{C} \rightarrow \mathbb{R}$, where $z \mapsto \frac{1}{2}|z|^2$, and the corresponding Hamiltonian action is rotation.

Example 20.5. Back to $\mu : \mathbb{C}^2 \rightarrow \mathbb{R}^2$, where $\mu(z_1, z_2) = \frac{1}{2}(|z_1|^2, |z_2|^2)$. Let S^3 be $\{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\} \subseteq \mathbb{C}^2$. What is $\mu(S^3)$? As we can see in the figure below, it lies over the diagonal. The image also illustrates that the center point in the diagonal corresponds to a torus; moreover, the midpoint divides the diagonal into two segments, each corresponding to solid tori, and the endpoints correspond to core circles in the solid tori, as we had seen previously. The two solid tori glue to S^3 .

FIGURE 34. $\mu(S^3)$

Example 20.6. Consider (S^2, ω) , where ω is the area form on unit sphere $S^2 \subset \mathbb{R}^3$, and $\mu : S^2 \rightarrow \mathbb{R}$ with $\mu(x, y, z) = z$. Then the image is $[-1, 1]$. Below we have an illustration of the map.

FIGURE 35. (S^2, ω) example

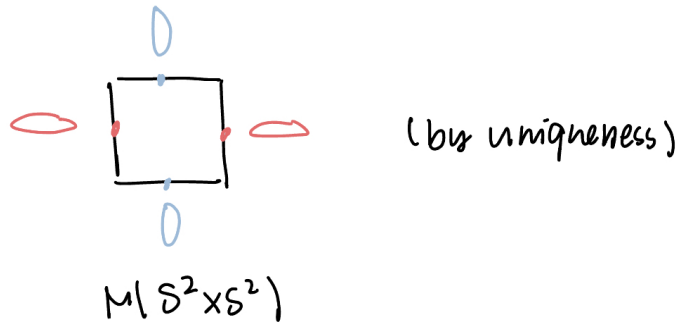
Two nice exercises based on this example are:

- (1) Exercise 1: μ is a moment map for a Hamiltonian S^1 action, acting by rotating the sphere horizontally.
- (2) (Archimedes) If $(a, b) \subset [-1, 1]$, then $\text{Area}(\mu^{-1}(a, b)) = 2\pi(b - a)$. In other words, there is an area preserving map from the sphere with two poles removed to a cylinder.

Example 20.7. A bit more generally, consider the generalization $(S^2, \lambda\omega)$, with $\mu(x, y, z) = \lambda z$. Note that $\text{Im}(\mu) = [-\lambda, \lambda]$, and this generates the same circle action on S^2 .

Even more generally, consider $(S^2 \times \cdots \times S^2, \sum \lambda_i pr_i^*(\omega_i))$, where pr_i is the natural projection. Then

$$\text{Im}(\mu) = \prod_i [-\lambda_i, \lambda_i].$$

FIGURE 36. $\mu(S^2 \times S^2)$ and the collapsing fibration

21. POLYTOPES AND INTEGRAL AFFINE MAP (03/31, YD, YD)

The source for this lecture is [Eva23, Sections 3.3, 3.4].

Let's recall from the last lecture, the Delzant polytopes correspond to compact toric symplectic manifolds. Recall also the action-angle coordinates, in which $\omega = \sum db_i \wedge dt_i$. Then we have:

Corollary 21.1. *If $S \subset \Delta = \mu(X)$, then $\text{Vol}_\omega(\mu^{-1}(S)) \sim \text{Vol}_{\mathbb{R}^n}(S)$.*

This corollary tells us the symplectic volume of the preimage of the set is a constant multiple of its Euclidean volume in the moment polytope

In this lecture, we will understand affine length and edge preimages.

Definition 21.2. If $\ell : [0, L] \rightarrow \mathbb{R}^n$ is a line segment of the form

$$\ell(t) = at + b \quad a \in \mathbb{Z}^n, b \in \mathbb{R}^n$$

and assume a is primitive. Then we say ℓ is a rational line segment and define its affine length to be L .

Example 21.3. Here the affine lengths of the edges are 1, 1, 2.

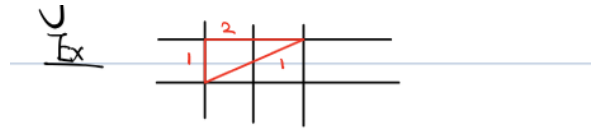


FIGURE 37

Remark 21.4. Thus the affine length is the length of the segment when it is parametrized by a primitive vector.

Equivalently, after applying an integral affine map, it is the euclidean length of the image segment when it starts at the origin and points in a coordinate direction.

Lemma 21.5. *Suppose ℓ is a rational line segment whose image is an edge of the moment polytope $\Delta = \mu(X)$. Then $\mu^{-1}(\ell)$ is a symplectic sphere, and its symplectic area is :*

$$\int_{\mu^{-1}(\ell)} \omega = 2\pi L$$

where L is the affine length.

Proof. By Delzant uniqueness, the preimage of the edge is equivariantly symplectomorphic to the toric 2-manifold associated to a line segment of affine length L . Along the interior of the edge, the fiber is S^1 , and this circle collapses at the two endpoints, so the full preimage is a 2-sphere. Hence it is symplectomorphic to $(S^2, \frac{L}{2}\omega_{S^2})$, where ω_{S^2} is the standard area form on S^2 normalized by

$$\int_{S^2} \omega_{S^2} = 4\pi.$$

Therefore,

$$\int_{\mu^{-1}(\ell)} \omega = \int_{S^2} \frac{L}{2} \omega_{S^2} = \frac{L}{2} \cdot 4\pi = 2\pi L.$$

□

Example 21.6. Consider $\mathbb{CP}^n$ and $\mu : \mathbb{CP}^n \rightarrow \mathbb{R}^n$ defined by

$$\mu([z_1 : z_2 : \cdots : z_{n+1}]) = \left(\frac{|z_1|^2}{|z|^2}, \frac{|z_2|^2}{|z|^2}, \dots, \frac{|z_n|^2}{|z|^2} \right)$$

where $|z|^2 = \sum_{i=1}^{n+1} |z_i|^2$. This has the Fubini-Study symplectic form. For $(t_1, t_2, \dots, t_n) \in T^n$, we have:

$$(t_1, t_2, \dots, t_n) \cdot [z_1 : z_2 : \cdots : z_{n+1}] = [e^{it_1} z_1 : \cdots : e^{it_n} z_n : z_{n+1}]$$

The moment image is $\{(b_1, \dots, b_n) \in \mathbb{R}^n | b_i \geq 0, \sum b_i \leq 1\}$. For $n = 1$, the moment polytope is one-dimensional, hence it is a line segment. For $n = 2$, it is two-dimensional, and the inequalities $b_1 \geq 0$, $b_2 \geq 0$, and $b_1 + b_2 \leq 1$ describe a triangle. The situation is similar for larger n .

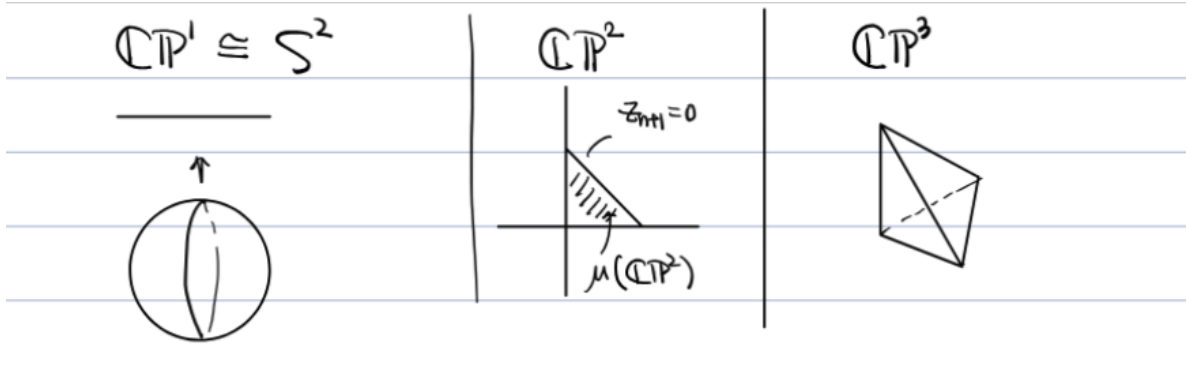


FIGURE 38

Example 21.7. The total space of the tautological bundle over $\mathbb{CP}^1$ is

$$O(-1) = \{(z_1, z_2, [z_3 : z_4]) \in \mathbb{C}^2 \times \mathbb{CP}^1 \mid z_1 z_4 = z_2 z_3\}.$$

It has a map

$$\pi : O(-1) \rightarrow \mathbb{CP}^1, \quad \pi(z_1, z_2, [z_3 : z_4]) = [z_3 : z_4].$$

The fiber of a point is the line corresponding to it. There is a symplectic form $\omega_{\mathbb{C}^2} \oplus \omega_{\mathbb{CP}^1}$ on $\mathbb{C}^2 \times \mathbb{CP}^1$. We can pull it back to $O(-1)$, and since $O(-1)$ is a complex submanifold, the pullback $i^*(\omega_{\mathbb{C}^2} \oplus \omega_{\mathbb{CP}^1})$ is symplectic. And there is a Hamiltonian torus action:

$$(t_1, t_2)(z_1, z_2, [z_3 : z_4]) = (e^{it_1} z_1, e^{it_2} z_2, [e^{it_1} z_3 : e^{it_2} z_4])$$

The associated moment map is

$$\mu(z_1, z_2, [z_3 : z_4]) = \left(\frac{1}{2}|z_1|^2 + \frac{|z_3|^2}{|z_3|^2 + |z_4|^2}, \frac{1}{2}|z_2|^2 + \frac{|z_4|^2}{|z_3|^2 + |z_4|^2} \right)$$

Then the image is (b_1, b_2) with $b_i \geq 0$ and $b_1 + b_2 \geq 1$. Notice here the zero section is $b_1 + b_2 = 1$.

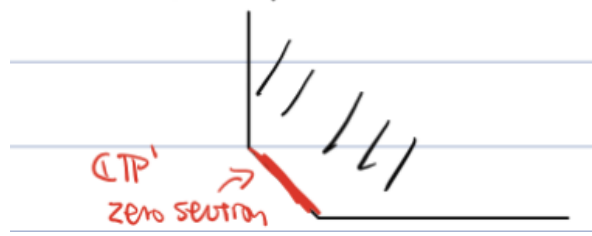


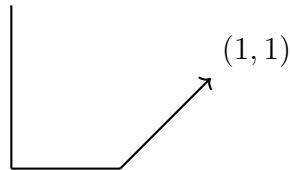
FIGURE 39

Remark 21.8. This example is important because the map

$$p : O(-1) \rightarrow \mathbb{C}^2 \quad p(z_1, z_2, [z_3 : z_4]) = (z_1, z_2)$$

is a blow down map. To see this, for any point except the origin we have $p^{-1}(z_1, z_2) = (z_1, z_2, [z_1 : z_2])$, but for the origin we have that $p^{-1}(0, 0)$ is the whole $\mathbb{CP}^1$.

Example 21.9. The moment polytope in 21.7 is integral affine equivalent to the moment polytope below.



Proof. Consider the matrix:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in GL(2, \mathbb{Z}).$$

Clearly this is an integral affine map and we have:

- $A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- $A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- $A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

This completes our proof. □

Example 21.10. For $n \geq 1$, we can consider:

$$O(-n) = \{(z_1, z_2, [z_3 : z_4]) \in \mathbb{C}^2 \times \mathbb{CP}^1 \mid z_1 z_4^n = z_2 z_3^n\}$$

It carries a Hamiltonian T^2 -action whose moment image is integrally affine equivalent to the polygonal region shown below. The compact horizontal edge corresponds to the zero section, and the outgoing noncompact edge has primitive direction $(n, 1)$.

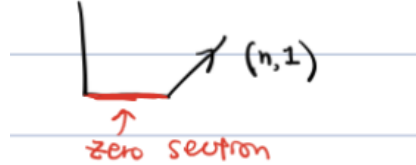


FIGURE 40

Example 21.11. For positive n , there is a picture of $O(n)$ that is similar, but the polytope is not compact and the map to the polytope is not proper (so does not give an integrable Hamiltonian system). It is best thought of as a local picture near the edge.

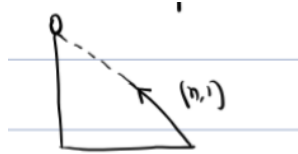


FIGURE 41

Lemma 21.12. Let $\Delta \subset \mathbb{R}^2$ be a moment polytope. Then for any edge e , there exists a neighborhood of $\mu^{-1}(e)$ fibered symplectomorphic to a neighborhood of the 0 section of $O(-n)$, where $n = \det([w, v])$

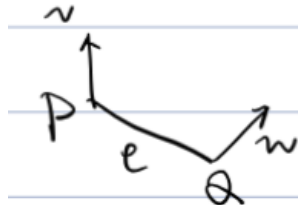


FIGURE 42

Proof. Suppose $v = (a, 1)$, $w = (b, 1)$, $e = (1, 0)$. Then $\det[w, v] = b - a$. We can consider the matrix:

$$A = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$$

It is an integral affine map, and we have

- $Av = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- $Aw = \begin{pmatrix} n \\ 1 \end{pmatrix}$
- $Ae = e$

This shows that the two local polytope neighborhoods are integrally affine equivalent. Therefore, by Delzant uniqueness, their preimages are equivariantly fibered symplectomorphic. Thus a neighborhood of $\mu^{-1}(e)$ is fibered symplectomorphic to a neighborhood of the zero section in $O(-n)$. □

Example 21.13. [Non-Delzant polytope] Let μ_n be the group of n -th roots of unity. Consider the action of μ_n on $\mathbb{C}^2$ given by

$$(z_1, z_2) \mapsto (\mu z_1, \mu^a z_2), \quad \gcd(a, n) = 1.$$

The quotient

$$X = \mathbb{C}^2 / \mu_n$$

is not a manifold, but a symplectic orbifold with cyclic quotient type

$$\frac{1}{n}(1, a).$$

Its associated cone is generated by $(0, 1)$ and (n, a) , which is non-Delzant when $n > 1$.

Remark 21.14. The cone generated by $(0, 1)$ and (n, a) is Delzant if and only if

$$\det \begin{pmatrix} 0 & n \\ 1 & a \end{pmatrix} = -n$$

has absolute value 1. Hence for $n > 1$ the cone is not Delzant.

22. SYMPLECTIC REDUCTION (04/02, YD, ZH) ☺

The sources for this lecture are [Eva23, Sections 4.1-4.3] and [Kar].

In this lecture, we study how a symplectic form can descend to a quotient after restricting to a level set of a regular value of a Hamiltonian. This construction is called symplectic reduction. We begin with the slice theorem (for a proof, see [Kar, page 3]).

Theorem 22.1 (The Slice Theorem). *Let $G \curvearrowright M$ be a proper action of a Lie group on a manifold. Let $m \in M$ and let $H := \text{Stab}(m)$. Consider*

$$V = T_m M / T_m(G \cdot m)$$

as an H -representation. Then there exists an H -invariant neighborhood D of the origin in V , a G -invariant neighborhood U of the orbit $G \cdot m$ in M , and a G -equivariant diffeomorphism

$$G \times_H D \rightarrow U$$

taking $[(e, 0)]$ to m .

Remark 22.2. Note that if $G = S^1$, then proper is automatic.

We have the following setup: let (X, ω) be a symplectic manifold, and let $H: X \rightarrow \mathbb{R}$ be a Hamiltonian. Suppose that $\phi_{2\pi}^H = \text{id}$. Then the flow gives rise to an action of the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ on X . Let $M_c := H^{-1}(c)$ denote the level set of c . The level set M_c is a submanifold of X when c is a regular value of H .

Definition 22.3. An action $G \curvearrowright X$ is called *free* if $\text{Stab}(x) = \{e\}$ for all $x \in X$. The action is called *effective* if $\bigcap_{x \in X} \text{Stab}(x) = \{e\}$.

Remark 22.4. Free is much stronger than effective. Free means that no element of G other than e can fix any point of M , whereas effective means that elements of G are allowed to fix *some* (but not all) points of M pointwise.

Corollary 22.5. Let $G \curvearrowright M$ be a proper action of a Lie group on a manifold.

- (1) If the action is free, then M/G is a manifold.
- (2) If all stabilizers are finite, then M/G is orbifold.

Lemma 22.6. Suppose that $H: X \rightarrow \mathbb{R}$ generates an S^1 -action.

- (1) The critical points of H are the fixed points.
- (2) The level sets of H are preserved by the action.
- (3) If $x \in M_c$ is a regular point, then

$$T_x(M_c) = \ker(dH_x).$$

- (4) If $v \in T_x(M_c)$ satisfies $\omega(v, u) = 0$ for all $u \in T_x(M_c)$, then $v \in \text{span}(V_H(x))$, where V_H is the Hamiltonian vector field at x .

Lemma 22.7. If c is a regular value of H , then $Q_c := M_c/S^1$ is an orbifold.

Proof. By Lemma 22.6, since the S^1 -action on M preserves the levels sets of H , the quotient Q_c is defined.

By Corollary 22.5, it suffices to show that the stabilizer of every point $x \in M_c$ is finite. If the stabilizer subgroup $\text{Stab}(x)$ of x were S^1 , then we would have $dH_x = 0$, but this would contradict the assumption that c is a regular value of H . Hence, $\text{Stab}(x)$ must be a proper subgroup of S^1 . We also have that $\text{Stab}(x)$ is a closed subgroup of S^1 . Since the only proper closed subgroups of S^1 are of the form $\mathbb{Z}/k\mathbb{Z}$, the result follows. $\square$

22.1. Symplectic Reduction. We have that S^1 acts on M_c . If this action is free, then $M_c/S^1 =: Q_c$ is a smooth manifold. We now show that this quotient inherits a symplectic form.

Lemma 22.8 (Symplectic Reduction). *Suppose that a Hamiltonian $H : X \rightarrow \mathbb{R}$ generates an S^1 -action, and that c is a regular value of H . For simplicity, assume that the S^1 -action on M_c is free.*

Let $i_c : M_c \hookrightarrow X$ and $p : M_c \rightarrow Q_c$ be the inclusion map and quotient map, respectively. There exists a unique symplectic form σ on Q_c such that

$$i_c^* \omega = p^* \sigma.$$

We call (Q_c, σ) the symplectic quotient or symplectic reduction of (X, ω) at level c .

Proof. We will prove that there exists a 2-form σ on Q_c such that $i_c^* \omega = p^* \sigma$. We leave it as an exercise to the reader to verify that σ is symplectic and that it is unique.

Suppose $u, v \in T_x M_c$. We want to show that $\omega(u, v)$ depends only on $p_*(u), p_*(v) \in T_{p(x)} Q_c$ (so that $\omega(u, v) = \sigma(p_* u, p_* v)$ for some 2-form on Q_c). We will show this by verifying that if $p(x) = p(x')$ and $u', v' \in T_{x'} M_c$ with $p_*(u') = p_*(u)$ and $p_*(v') = p_*(v)$, then $\omega(u, v) = \omega(u', v')$.

Since x and x' are in the same orbit, $x' = \phi_t^H(x)$ for some t . Since $p \circ \phi_t^H = p$, we have $p_*(\phi_t^H)_* = p_*$ and so

$$p_*(v' - (\phi_t^H)_* v) = p_*(v') - p_*(v) = 0.$$

Since $\ker(dp_{x'}) = T_{x'}(S^1 \cdot x') = \text{span}(V_H(x'))$ is spanned by $V_H(x')$ (here, we use that $T_x(S^1 \cdot x)$ is one-dimensional and that V_H is in $T_x(S^1 \cdot x)$ because the orbit $S^1 \cdot x$ comes from flowing x along ϕ_t^H), we have that

$$u' - (\phi_t^H)_* u = a V_H(x')$$

for some $a \in \mathbb{R}$. Similarly,

$$v' - (\phi_t^H)_* v = b V_H(x')$$

for some $b \in \mathbb{R}$. Therefore,

$$\begin{aligned} \omega(u', v') &= \omega((\phi_t^H)_* u + a V_H, (\phi_t^H)_* v + b V_H) \\ &= \omega((\phi_t^H)_* u, (\phi_t^H)_* v) + \omega((\phi_t^H)_* u, b V_H) + \omega(a V_H, (\phi_t^H)_* v) + \omega(a V_H, b V_H) \\ &= \omega(u, v) + b dH(u) - a dH(v) + 0. \end{aligned}$$

Since $u, v \in T_x M_c = \ker(dH_x)$, we have

$$b dH(u) = a dH(v) = 0.$$

Thus, $\omega(u', v') = \omega(u, v)$. □

Lemma 22.9. *Suppose that $G : X \rightarrow \mathbb{R}$ Poisson commutes with H .*

(1) *The restriction $G|_{M_c}$ descends to a function*

$$\overline{G} : Q_c \rightarrow \mathbb{R}.$$

(2) *The Hamiltonian vector field of $\overline{G}$ is*

$$V_{\overline{G}} = p_*(V_G).$$

(3) *If G generates a Hamiltonian circle action on M_c , then $\overline{G}$ generates a Hamiltonian circle action on the symplectic quotient Q_c .*

We now describe symplectic reduction on $\mathbb{CP}^{n-1}$ and weighted projective space.

Example 22.10. Let $X = \mathbb{R}^{2n} = \mathbb{C}^n$ with coordinates $(x_1, y_1, \dots, x_n, y_n)$ and the standard symplectic form $\omega = \sum_i dx_i \wedge dy_i$. Consider the Hamiltonian $H = \frac{1}{2} \sum_i x_i^2 + y_i^2$. The corresponding Hamiltonian vector field V_H is $V_H = (-y_1, x_1, \dots, -y_n, x_n)$. This generates an S^1 action on X by rotating each $x_i y_i$ -plane at constant angular speed. If $c > 0$, M_c is a sphere of radius $\sqrt{2c}$ and the symplectic quotient is

$$M_c/S^1 = S^{2n-1}/S^1 = \mathbb{CP}^{n-1}.$$

If $c = 1$, the symplectic form on $\mathbb{CP}^{n-1}$ is called the *Fubini-Study form* (the symplectic form is normalized so that $\mathbb{CP}^1 \subset \mathbb{CP}^{n-1}$ has area 2π).

Example 22.11 (Weighted Projective Space). Fix $(a_1, a_2, \dots, a_n) \in \mathbb{Z}_{>0}^n$. Consider the Hamiltonian

$$\begin{aligned} H: \mathbb{C}^n &\rightarrow \mathbb{R} \\ (z_1, z_2, \dots, z_n) &\mapsto \frac{1}{2} \sum_{k=1}^n a_k |z_k|^2 \end{aligned}$$

This generates the S^1 action

$$t \cdot (z_1, \dots, z_n) = (e^{ia_1 t} z_1, \dots, e^{ia_n t} z_n)$$

The symplectic quotient M_c/S^1 is an orbifold called the weighted projective space $\mathbb{P}(a_1, a_2, \dots, a_n)$. The point whose coordinates are non-zero only in the k -th coordinate has $\mathbb{Z}/a_k \mathbb{Z}$ as its stabilizer. If $\gcd(a_i, a_j) = 1$ for all $i \neq j$, then these points are the only orbifold points.

22.2. Symplectic Cut. We illustrate the idea of the (naive) definition of the symplectic cut with the following example.

Example 22.12. Consider the cylinder $X = \mathbb{R} \times S^1$ with symplectic form $\omega = dp \wedge dq$, where p and q are the coordinates on $\mathbb{R}$ and S^1 respectively. The Hamiltonian

$$\begin{aligned} H: X &\rightarrow \mathbb{R} \\ (p, q) &\mapsto p \end{aligned}$$

generates a (free) circle action which rotates the S^1 factor. The symplectic reduction at any level c is a single point, so if we only quotient $H^{-1}(c)$ by this circle action, the space we get is the cylinder “pinched” at level c . This pinched cylinder is the union of two symplectic disks $X_{H \leq c}$ and $X_{H \geq c}$ whose intersection is $H^{-1}(c)/S^1$ (see Figure 43).

Definition 22.13 (Naive definition of the symplectic cut). If a Hamiltonian H generates an S^1 action, then the symplectic cut at (a regular value) c is

$$H^{-1}((c, \infty)) \cup (H^{-1}(c)/S^1).$$

The naive definition does not make it clear that the result is a manifold (or orbifold), and does not show the result has a symplectic form. The following rather clever definition addresses these issues.

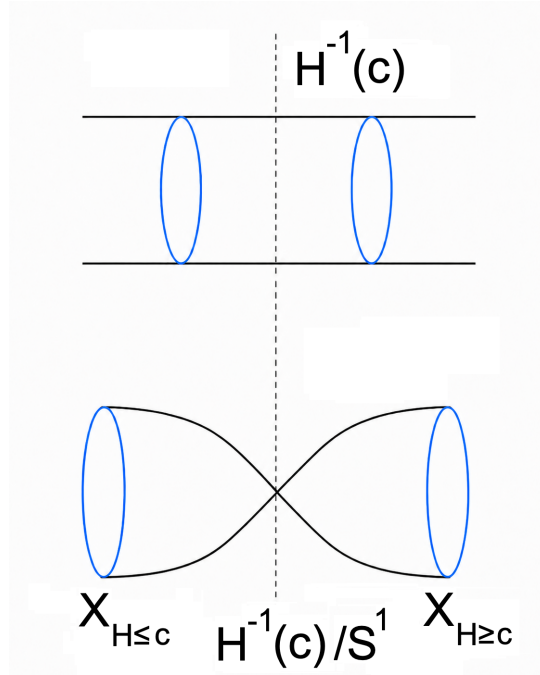


FIGURE 43. The symplectic cut of a cylinder

Definition 22.14 (Actual definition of the symplectic cut). Let (X, ω) be a symplectic manifold and suppose that H is a Hamiltonian generating a circle action. Let c be a regular value for which the circle action on $H^{-1}(c)$ is free.

Consider $X \times \mathbb{C}$ with symplectic form $\omega + dp \wedge dq$ (where $\xi := p + iq \in \mathbb{C}$). Define the Hamiltonian

$$\hat{H}_c(x, \xi) = H(x) - c - \frac{1}{2}|\xi|^2.$$

The Hamiltonian $\hat{H}_c(x, \xi)$ generates a circle action on $X \times \mathbb{C}$ by sending (x, ξ) to $(\phi_t^H(x), e^{-it}\xi)$. The *symplectic cut* of X at level c is defined to be $\hat{H}_c^{-1}(0)/S^1$.

Note

- $\hat{H}_c^{-1}(0) = \{(x, \xi) : H(x) - \frac{1}{2}|\xi|^2 = c\}$.
- $\text{Proj}_X(\hat{H}_c^{-1}(0)) = \{x : H(x) \geq c\}$.
- If $\xi \neq 0$, the circle action kills ξ .
- If $\xi = 0$, the circle action really quotients $H^{-1}(c)$.

The actual definition of the symplectic cut works very nicely and realizes the naive definition of the symplectic cut.

23. SYMPLECTIC CUTS AND FOCUS-FOCUS SINGULARITIES (04/07, ZH, YD)

The sources for this lecture are [Eva23, Sections 4.4, 4.5, 5.1, and 6.1], however note that unlike Evans our matrices act from the left.

We begin with the following meta-example: Suppose X is a toric manifold and $\mu: X \rightarrow \mathbb{R}^n$ is the moment map with moment image $\Delta = \mu(X)$. Pick an integral linear map $s: \mathbb{R}^n \rightarrow \mathbb{R}$ and take $H = s \circ \mu$. If we symplectically cut at level c , where c is a regular value of H , the result is still toric and the moment image is

$$\{x \in \Delta : s(x) \geq c\}.$$

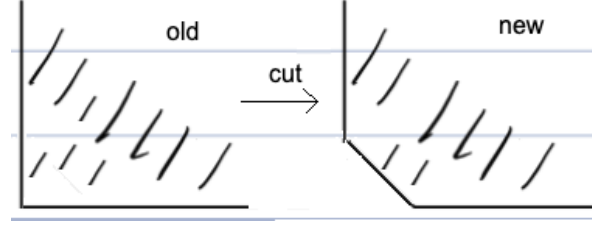


FIGURE 44

Exercise: We need s to be integral linear so that $H = s \circ \mu$ generates an S^1 -action. This implies the following existence theorem:

Theorem 23.1. *Any convex rational polytope Δ occurs as the moment image of a toric Hamiltonian system (on a possibly singular space).*

Example 23.2. Suppose X is $\mathbb{C}^n$ and $H(z) = \frac{1}{2} \sum_{i=1}^n |z_i|^2$. The symplectic reduction at level $c > 0$ is $\mathbb{C}P^{n-1}$. If we instead do the cut at c , we get a toric manifold with image

$$\{(b_1, \dots, b_n) \in \mathbb{R}_{\geq 0}^n : \sum_{i=1}^n b_i \geq c\},$$

where $b_i = \frac{1}{2}|z_i|^2$. This is the symplectic blow-up of X in the ball

$$B_c = \{z \in \mathbb{C}^n : |z|^2 \leq 2c\}.$$

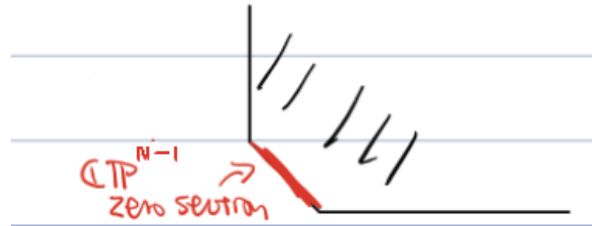


FIGURE 45

We can use this symplectic blowup to desingularize. We will describe how to desingularize the space from Example 21.13 by resolving the singularity of this space at the origin (for more details, see [Eva23, Example 4.30]).

Example 23.3. Consider the cyclic quotient singularity $\frac{1}{n}(1, a)$ from Example 21.13. Here, $n \geq 2$, $0 < a < n$, and $\gcd(a, n) = 1$. Its moment image $\pi(n, a)$ is the convex cone in the plane generated by $(0, 1)$ and (n, a) (see Figure 46).

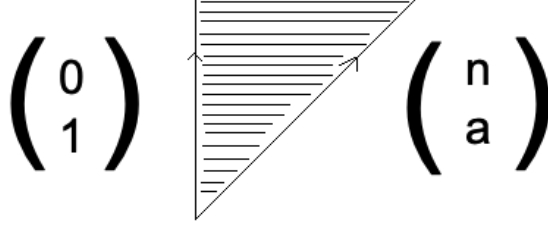


FIGURE 46. The moment image $\pi(n, a)$

Since $n \geq 2$ and $\gcd(a, n) = 1$, the origin is a non-Delzant vertex of the moment image and we can desingularize this vertex as follows. We symplectically cut along a horizontal slice at level $c > 0$ to obtain a polygon $\tilde{\pi}(n, a)$ that has two vertices (see Figure 47): a Delzant corner (on the left side) and another corner (on the right side) that might not be Delzant.

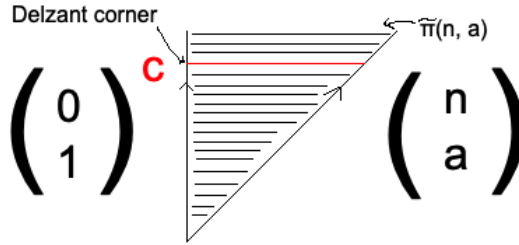


FIGURE 47. The polygon $\tilde{\pi}(n, a)$ obtained from symplectically cutting

We can apply a matrix M in $\mathrm{GL}_2(\mathbb{Z})$ to the primitive integral vectors in the two edge directions $((-1, 0)$ and $(n, a))$ coming from this vertex to put these two primitive integral vectors in the form $(0, 1)$ and (x, y) for a certain (x, y) (we will clarify what exactly (x, y) should be soon). More specifically, we want to construct $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ so that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} -1 & n \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & x \\ 1 & y \end{pmatrix}.$$

Since the left-hand side is equal to

$$\begin{pmatrix} -A & An + Ba \\ -C & Cn + Da \end{pmatrix},$$

we should have that $A = 0$ and $C = -1$. Therefore, the matrix

$$\begin{pmatrix} -A & An + Ba \\ -C & Cn + Da \end{pmatrix},$$

simplifies to

$$\begin{pmatrix} 0 & Ba \\ 1 & -n + Da \end{pmatrix}.$$

Set B to be 1 and D to be $\lfloor \frac{n}{a} \rfloor$. We have that $-n + Da$ is the unique integer in $[0, a)$ congruent to $-n \pmod{a}$ and we denote this integer by “ $-n \pmod{a}$ ”. As a result, we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ \text{“} -n \pmod{a} \text{”} \end{pmatrix}.$$

To summarize, by applying a symplectic cut to $\pi(n, a)$ horizontally at a level above the singularity, we obtain a new polygon $\tilde{\pi}(n, a)$ consisting of a Delzant corner and another corner that can be modeled by $\frac{1}{a}(1, \text{“} -n \pmod{a} \text{”})$. This second corner might not be Delzant, but we can repeat this procedure applying a symplectic cut to $\tilde{\pi}(n, a)$ horizontally. From it, we get a decreasing sequence of positive integers, and so this procedure must eventually terminate.

In particular, this procedure replaces one non-Delzant vertex with finitely many Delzant vertices and we get a Delzant polygon. The sequence of positive integers we get is related to the continued fraction expansion of $\frac{a}{n}$, although the continued fraction expansion of $\frac{a}{n}$ that we get here is slightly non-standard. In addition, we get a chain of symplectic spheres, corresponding to the edges of this Delzant polygon, and whose self-intersection numbers can be understood through this continued fraction expansion.

23.1. Brief scan of Chapter 5 of [Eva23]. Suppose we have the same setup as before. One can ask:

Question 23.4. What Lagrangians can I “see” from the setup?

The most obvious answer is the fibers, but it is possible to carefully mix fiber and base directions while still being Lagrangian. But the nicest way to do this is constrained as follows.

Theorem 23.5 ([Eva23, Theorem 5.1]). *Consider the integrable Hamiltonian system*

$$\begin{aligned} H: \mathbb{R}^n \times T^n &\rightarrow \mathbb{R}^n \\ (p, q) &\mapsto p \end{aligned}$$

and the symplectic form on $\mathbb{R}^n \times T^n$ is $\sum_i dp_i \wedge dq_i$. Let $L \subset \mathbb{R}^n \times T^n$ be a Lagrangian submanifold. Suppose $H|_L: L \rightarrow \mathbb{R}^n$ factors as $H|_L = f \circ g$, where $g: L \rightarrow K$ is a bundle over a k -dimensional manifold, with $k < n$ and with compact fibres, and $f: K \rightarrow \mathbb{R}^n$ is an embedding (so $K \cong f(K)$). Then $K \cong f(K) \subset \mathbb{R}^n$ is contained in a k -dimensional affine linear subspace of $\mathbb{R}^n$ which is rational with respect to the lattice $(2\pi\mathbb{Z})^n$.

Definition 23.6. Lagrangians which project this way are called *visible Lagrangians*.

Example 23.7. Consider $X = \mathbb{R}^2 \times T^2$ with coordinates (p_1, p_2, q_1, q_2) . The Lagrangian embedding

$$\begin{aligned} i: \mathbb{R} \times S^1 &\rightarrow X \\ (s, t) &\mapsto (s, 0, 0, t) \end{aligned}$$

is visible for the projection $(p, q) \mapsto p$ by composing the bundle map

$$\begin{aligned} g: \mathbb{R} \times S^1 &\rightarrow \mathbb{R} \\ (s, \theta) &\mapsto s \end{aligned}$$

with the embedding

$$\begin{aligned} f: \mathbb{R} &\rightarrow \mathbb{R}^2 \\ s &\mapsto (s, 0). \end{aligned}$$

The Lagrangian torus

$$\begin{aligned} j: S^1 \times S^1 &\rightarrow X \\ (s, t) &\mapsto (\sin(s), 0, s, t) \end{aligned}$$

is also visible for this projection map: the Lagrangian torus projects to $[-1, 1] \times \{0\}$ and we have a bundle over $(-1, 1) \times \{0\}$ (the preimage over a point in $(-1, 1) \times \{0\}$ is a pair of circles).¹⁶

23.2. Focus-Focus singularities. We now briefly mention focus-focus singularities (we will say more about this in the next lecture).

Example 23.8 (standard focus-focus system). Consider the following pair of Hamiltonians on $(\mathbb{R}^4, dp_1 \wedge dq_1 + dp_2 \wedge dq_2)$:

$$\begin{aligned} F_1 &= -p_1 q_1 - p_2 q_2 \\ F_2 &= p_2 q_1 - p_1 q_2. \end{aligned}$$

If we use the (non-compatible) complex coordinates

$$p = p_1 + ip_2, \quad q = q_1 + iq_2, \quad F := F_1 + iF_2$$

then (check) $F = -\bar{p}q$.

Lemma 23.9 (Exercise). F_1 and F_2 generate the following flow

$$\phi_t^{F_1}(p, q) = (e^t p, e^{-t} q), \quad \phi_t^{F_2}(p, q) = e^{it}(p, q).$$

In addition, the Hamiltonians F_1, F_2 Poisson commute.

We can understand the ϕ^{F_1} -flowlines and the ϕ^{F_2} -flowlines via an auxiliary map (that is not the moment map) $(p, q) \mapsto (|p|, |q|)$ (see Figure 48). The orbits of ϕ^{F_1} map to hyperbolas. The ϕ^{F_2} orbits project to points.

24. FOCUS-FOCUS SINGULARITIES AND MONODROMY (04/09, AK, JZ)

Reference: [Eva23, Section 6.3], [Sym02, Section 2.3].

¹⁶Technically, the projection of the Lagrangian torus to $[-1, 1] \times \{0\}$ is not a bundle. The issue of the projection map to $[-1, 1] \times \{0\}$ not being a bundle stems from the fact that in the map $S^1 \rightarrow [-1, 1], s \mapsto \sin(s)$, the preimage of $x \in [-1, 1]$ is two points except when $x = \pm 1$.

Focus-focus singularities. Recall the standard focus-focus system $(\mathbb{R}^4, dp_1 \wedge dq_1 + dp_2 \wedge dq_2)$ with complex coordinates $p = p_1 + ip_2, q = q_1 + iq_2$. We have Hamiltonians $F_1 = \text{Re}(-\bar{p}q)$ and $F_2 = \text{Im}(-\bar{p}q)$ which generate the following flows:

$$\phi_t^{F_1}(p, q) = (e^t p, e^{-t} q), \quad \phi_t^{F_2}(p, q) = e^{it}(p, q).$$

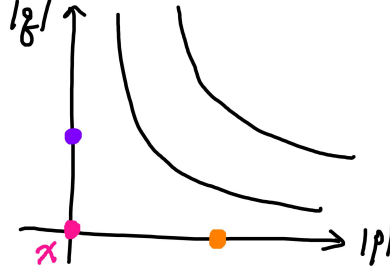


FIGURE 48. Plot of $|p|, |q|$ coordinate plane. The hyperbolae are flow lines of ϕ^{F_1} .

Note that on the $|p|, |q|$ coordinate plane, the flow generated by F_1 moves points along the hyperbolae $|p| \cdot |q| = c$, while the flow generated by F_2 fixes every point. The orbits along the axes above look like a cylinder with its center circle pinched to a point:

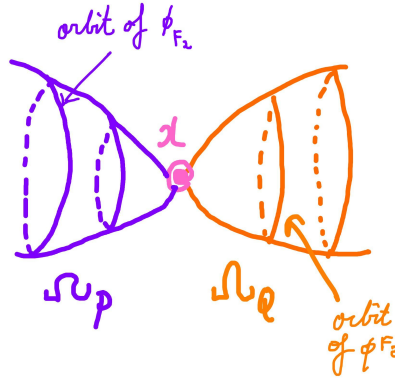


FIGURE 49. Orbits of points on the coordinate axes.

Note that the orbits of points on the coordinate axes form a pinched cylinder. The central point x (pink) is completely fixed by both flows. The flow generated by ϕ^{F_2} rotates, whereas the flow of ϕ^{F_1} moves points along the pinched cylinder.

Definition 24.1. A **focus-focus chart** for an integrable Hamiltonian system $H : X \rightarrow \mathbb{R}^2$ is a pair of embeddings:

$$\begin{array}{ccc} \mathbb{R}^4 \supset U & \xhookrightarrow{E} & (X, \omega) \\ \downarrow F & & \downarrow H \\ \mathbb{R}^2 \supset V & \xhookrightarrow{e} & \mathbb{R}^2 \end{array}$$

where

- $U \subset \mathbb{R}^4$ is a neighborhood of the origin with $E^*\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$;
- $V = F(U)$, where $F = (F_1, F_2)$ is as in the standard focus-focus system;
- $H \circ E = e \circ F$.

We say $H : X \rightarrow \mathbb{R}^2$ has a **focus-focus critical point** or a **focus-focus singularity** at $x \in X$, where $E(0) = x$.

Lemma 24.2. *Let $H : X \rightarrow \mathbb{R}^2$ be an integrable Hamiltonian system with a focus-focus critical point at x and no other critical points. Then $H^{-1}(0)$ is homeomorphic to a pinched torus.*

Proof. By assumption, the fiber $H^{-1}(0)$ is compact and connected. Recall that $H^{-1}(0)$ is a disjoint union of orbits. Moreover, since $H^{-1}(0) \setminus \{x\}$ consists of regular points, each orbit other than $\{x\}$ is either $\mathbb{R}^2$, $\mathbb{R} \times S^1$, or T^2 . But the last case cannot occur since it would be clopen in $H^{-1}(0)$ and hence would give a connected component of $H^{-1}(0)$ not containing x . In particular, the orbits other than $\{x\}$ must contain closed circle orbits and be non-compact. The only possible such orbit is $\mathbb{R} \times S^1$.

Let Ω_P be the orbit containing the line $\{(p, 0) : p \neq 0\}$, and Ω_Q be the orbit containing $\{(0, q) : q \neq 0\}$, as depicted in Figure 49. The key claim is that $\Omega_P = \Omega_Q$. Suppose not, then $\Omega_P \cup \Omega_Q \cup x$ is not compact. This implies that $H^{-1}(0)$ is not compact; use the fact that $H^{-1}(0) \setminus \{x\}$ is a 2-manifold. Hence $\Omega_P = \Omega_Q$, so $\Omega_P \cup \{x\}$ is a compact pinched torus. Since $H^{-1}(0)$ is connected and $H^{-1}(0) \setminus \{x\}$ is a manifold, this pinched torus must be all of $H^{-1}(0)$. $\square$

Monodromy around F-F singularities. Note that a punctured neighborhood of the base of a focus-focus chart admits an integral affine structure, as (X, ω) admits a torus fibration once we remove the singular fiber. One naturally wonders how this integral affine structure changes as we go around the origin, i.e. what kind of *monodromy* we have in this system. The answer is that the monodromy acts as a *Dehn twist* around the curve pinched at the focus-focus singularity. We give a brief description and justification for this claim. For a more rigorous explanation, see [Eva23, Section 6.3].

Recall that the local model of the singularity is

$$\mathbb{R}^4 \cong \mathbb{C}^2 \rightarrow \mathbb{R}^2, (p, q) \mapsto (\operatorname{Re}(F), \operatorname{Im}(F)),$$

where $F = -\bar{p}q$. Now making an $\mathbb{R}$ -linear change of coordinates

$$u = -\bar{p} = -p_1 + ip_2, \quad v = q = q_1 + iq_2,$$

we obtain $F = uv$. Consider the level sets

$$F^{-1}(t) = \{(u, v) \in \mathbb{C}^2 : uv = t\}.$$

In particular, $F^{-1}(0)$ are two copies of $\mathbb{C}$ meeting at the origin forming a nodal singularity. We describe the setting in the following picture:

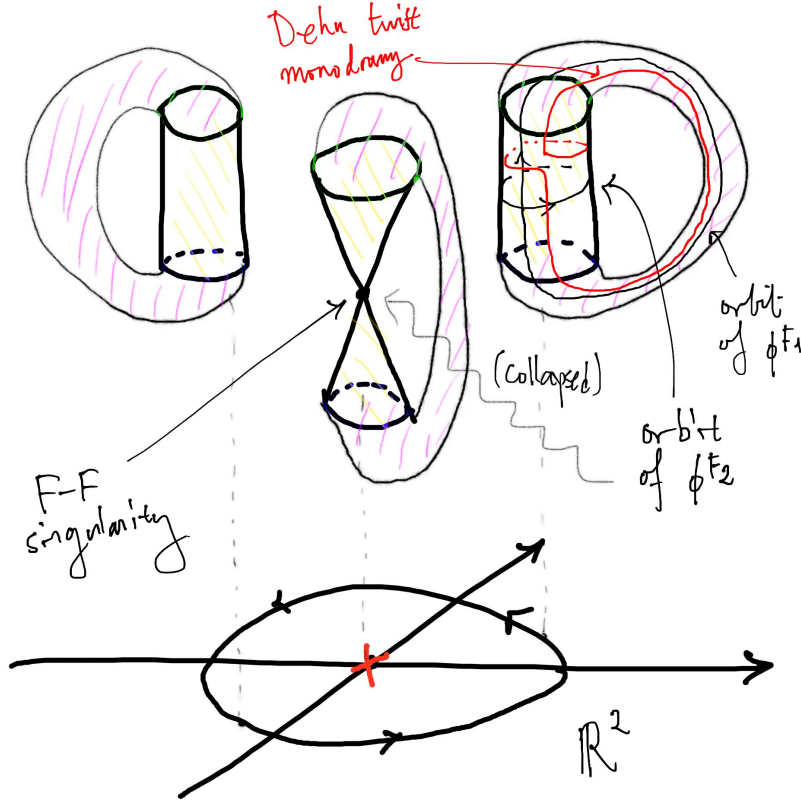


FIGURE 50. Sketch of focus-focus singularity. The origin of the base, marked by the red cross, has the pinched torus as its fiber.

The local model describes the cylinders, including the pinched one, with thick outlines filled in with light yellow. Note that there is a consistent choice of two waist circles on the fibers including the singular fiber. This splits the fibers into two parts, one colored in light yellow and the other in magenta.

The magenta cylinders form a fiber bundle over the base $B \cong D^2$, which can be trivialized and hence does not exhibit any nontrivial monodromy around the origin. On the other hand, the yellow cylinders form a fiber bundle only over the punctured disc $B - (0, 0)$. Hence we obtain a monodromy map for cylinder C :

$$(\dagger) \quad \mathbb{Z} \cong \pi_1(B - (0, 0)) \rightarrow \text{Mod}(C, \partial C) \cong \mathbb{Z}.$$

Let us analyze this map in local coordinates. Taking $t = \rho e^{i\theta} \neq 0$ with $\rho > 0$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, we can parametrize the fiber $F^{-1}(t)$ in the following polar form:

$$u = \sqrt{\rho} e^{s+i\phi}, \quad v = \sqrt{\rho} e^{-s+i(\theta-\phi)},$$

where $s \in \mathbb{R}$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$. Restricting to the neighborhood $U = \{(u, v) : |u|, |v| \leq \varepsilon\}$ corresponds to restricting to $|s| \leq L$, where $L = \log \frac{\varepsilon}{\sqrt{\rho}}$. Hence in both the u and v coordinates, the fiber is an annulus with the same inner/outer radii and corresponds to

the yellow cylinders above. The annuli in the u, v coordinates are identified as

$$(*) \quad v = \frac{t}{u} = \frac{\rho}{u} e^{i\theta}.$$

Let us fix $\rho \ll \varepsilon^2$ and analyze the annular fiber $F_\theta = F^{-1}(\rho e^{i\theta})$ as we vary $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. To find the mapping class we fix the coordinates on parts of F_θ that meet the boundary ∂U . Looking at F_θ in the v -coordinate plane, the outer boundary of the annuli has $|v| = \varepsilon$, so is fixed as we vary θ . On the contrary, the inner boundary has $|u| = \varepsilon$ so we fix its u -coordinate. By (*), this means that on the v -coordinate plane the inner boundary is rotating by angle θ .

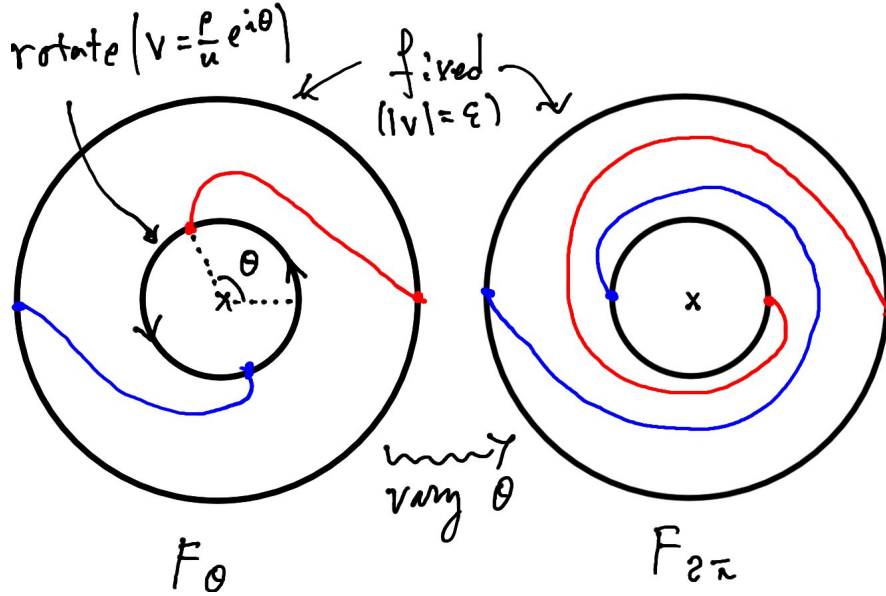


FIGURE 51. The annular fiber F_θ in the v -coordinate plane. The outer boundary is fixed, while the inner boundary is rotated by θ . We smoothly interpolate this action to the interior and trace out the radial line segments in red/blue as we vary θ from 0 to 2π .

Smoothly interpolating this action to the interiors of the annuli and varying $0 \leq \theta \leq 2\pi$, we see that the mapping class on the cylinder as we go around the puncture in the base is precisely a Dehn twist. Thus we have justified that the monodromy map $(\dagger)$ is nontrivial.

Addendum. An additional argument is required to justify the previous calculation. Yes, the magenta cylinders form a trivial bundle over the disc, but why can we pick such a trivialization to agree with the implicit chosen trivialization of the boundary of the yellow part defined by the outer boundary of the u annulus in fixed u -coordinates and the outer boundary of the v annulus in fixed v -coordinates?

The reason is that these target trivializations (that we want to agree with) are defined over the whole disc. We can pick any preliminary trivialization of the magenta bundle and adjust it as required by boundary reparametrizations (such as rotations near the

boundary): make the required adjustment on the boundary and taper it off as we move slightly into the cylinder. This gives the desired trivialization of the magenta part.

There are two key points here: an isotopy prescribed on the boundary can be extended over a collar of the boundary; and the fact that the target trivializations are defined over the whole disc.

Three notions of monodromy. There are three notions of ‘monodromy’ in play here:

- the mapping class group monodromy in $\pi_0(\text{Homeo}(T^2))$;
- the induced push-forward action on $H_1(T^2) \cong \mathbb{Z}^2$;
- the linear part of the holonomy of the integral affine structure.

In this case, all three of these turn out to be equivalent:

- (i) There is a natural map

$$\text{Mod}(T^2) = \pi_0(\text{Homeo}(T^2)) \rightarrow \text{Aut}(H_1(T^2)) \cong GL(2, \mathbb{Z}),$$

which is an isomorphism specifically for the torus and not for higher genera surfaces.

- (ii) The induced action on $H_1(T^2)$ contains the same information as the linear part of the affine holonomy. This is a more general phenomenon compared to (i). In fact, the affine coordinates can be written as

$$\frac{1}{2\pi} \left(\int_{\alpha} \lambda, \int_{\beta} \lambda \right)$$

for a basis $\{\alpha, \beta\}$ of $H_1(T^2, \mathbb{Z})$ and the Liouville 1-form λ on the cotangent bundle of T^2 . Recall that we have $d\lambda = \omega$ after identifying a neighborhood of the zero section of T^*T^2 with a local trivialization of X .

Why is this true? A quick answer is that X is locally symplectomorphic to $B \times T^2$ with $\omega = db_1 \wedge d\theta_1 + db_2 \wedge d\theta_2$ and $\lambda = b_1 d\theta_1 + b_2 d\theta_2$. From this we immediately see that

$$(b_1, b_2) = \frac{1}{2\pi} \left(\int_0^{2\pi} b_1 d\theta_1, \int_0^{2\pi} b_2 d\theta_2 \right).$$

But more conceptually, this equivalence is baked into the definition of the integral affine structure via 1-forms:

$$\Lambda_b^H = H_1(\pi^{-1}(b)) = H_1(\mathbb{R}^n/\Lambda) = \Lambda.$$

Addendum. The linear part of holonomy and the action on H_1 are transpose of each other. Indeed, fixing a loop γ in the base, suppose the action on homology is $M \in GL(2, \mathbb{Z})$. Analytically continuing the coordinates

$$B = \frac{1}{2\pi} \left(\int_{\alpha} \lambda, \int_{\beta} \lambda \right)$$

around the chosen loop gives the coordinates

$$\frac{1}{2\pi} \left(\int_{M\alpha} \lambda', \int_{M\beta} \lambda' \right).$$

Here λ' is the analytic continuation of λ , and we have $d\lambda = \omega = d\lambda'$. Since λ and λ' differ by a closed form, the analytically continued coordinates have the same linear part as

$$B' = \frac{1}{2\pi} \left(\int_{M_\alpha} \lambda, \int_{M_\beta} \lambda \right).$$

So, in other words, the starting coordinates are pairing with λ , and the end coordinates are multiply by M and then pair with λ . As is usual, the pullback defined by pre-composition with a linear map gives the formula

$$B' = M^T B,$$

if we write B' and B now as column vectors.

We now recall the equation linking holonomy to the developing map

$$\text{hol}(\gamma x) = \rho(\gamma) \text{hol}(x).$$

Up to constants, B' is $\text{hol}(\gamma x)$ and B is $\text{hol}(x)$. So we get $M^T B = \rho(\gamma) B$, and $\rho(\gamma) = M^T$.

To confuse matters, there is yet another operation in play: You can ask about parallel translating tangent vectors along γ . That operation is computed by $(M^T)^{-1}$. (It is the inverse of $\rho(\gamma)$ because the parallel translate points in a constant direction under the developing map: so the parallel translate of v must be the vector equal to v after applying M^T .)

25. THE AUROUX SYSTEM (04/14, JZ, AK)

Reference: [Eva23, Chapter 7.1].

Now, we turn to a key example of a focus-focus system, the Auroux system. This will serve as a crucial example which will allow us to “mess up” toric behavior to *almost* toric behavior. This gives much more flexibility for the objects that we can study. Most of today’s section will be about the Auroux system.

Example 25.1 (The Auroux system). Let c be positive. Consider the Hamiltonians $H = (H_1, H_2): \mathbb{C}^2 \rightarrow \mathbb{R}^2$ given by

$$H_1(z_1, z_2) = |z_1 z_2 - c|^2, \quad H_2(z_1, z_2) = \frac{1}{2}(|z_1|^2 - |z_2|^2).$$

One should think of H_1 as being “weird” while H_2 is similar to Hamiltonians we have seen before. In particular, one can easily compute $\phi_t^{H_2}(z_1, z_2) = (e^{it} z_1, e^{-it} z_2)$ (note that the minus sign in the definition of H_2 leads to a minus sign in the second coordinate of the flow). Since $(e^{it} z_1)(e^{-it} z_2) = z_1 z_2$, it follows that the flow $\phi_t^{H_2}$ preserves the level sets of H_1 . Then, Lemma 13.10 implies $\{H_1, H_2\} = 0$.

While the flow of H_1 is harder to compute, we can think about it geometrically using the auxiliary map $\pi: \mathbb{C}^2 \rightarrow \mathbb{C}$ defined by $\pi(z_1, z_2) = z_1 z_2$. This map gives a conic fibration since for each $t \in \mathbb{C}$, $\pi^{-1}(t) = \{z_1 z_2 = t\}$ is a conic. Note that there is one singular fiber, $\pi^{-1}(0) = \{z_1 z_2 = 0\}$ which is a union of two complex lines. Consider the following pictures of the generic fibers:

There is also a picture of the singular fiber over $t = 0$ at the top of the next page in Figure 53.

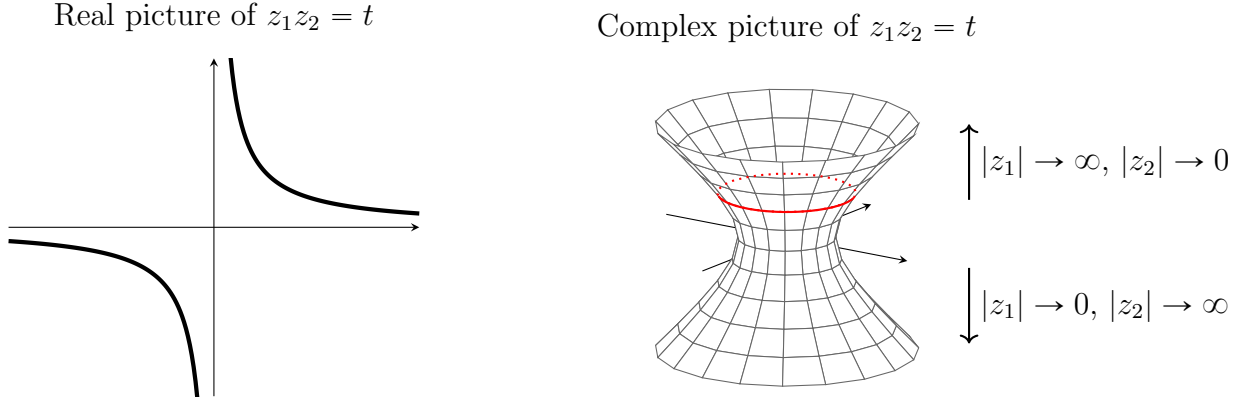


FIGURE 52. The real and complex points of the generic fiber for the map $\pi(z_1, z_2) = z_1 z_2$. A particular level set of H_2 is pictured in red.

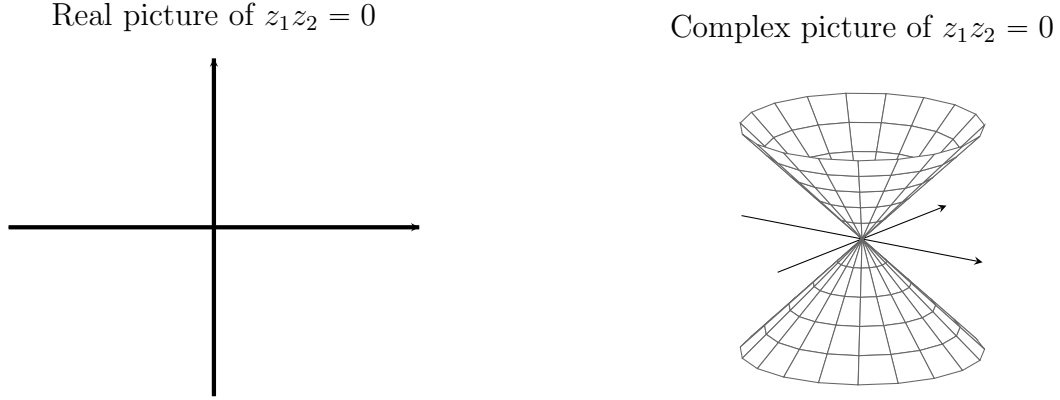


FIGURE 53. The real and complex points of the singular fiber $\pi^{-1}(0)$.

To summarize, for fixed $t \neq 0$, the fiber $\pi^{-1}(t)$ is a conic. Then for fixed b_2 and $t \neq 0$, the set

$$(\pi, H_2)^{-1}(t, b_2) = \{z = (z_1, z_2) : \pi(z) = t, H_2(z) = b_2\}$$

contains all points on the conic with a fixed height $w = H_2 = b_2$, i.e., the fiber $(\pi, H_2)^{-1}(t, b_2)$ is just a circle. Another way of seeing this is to just solve for z_1 in $\pi(z) = z_1 z_2 = t$ and $H_2(z) = \frac{1}{2}(|z_1|^2 - |z_2|^2) = b_2$. This determines $|z_1|$, but the argument of can be anything. Then, z_1 determines z_2 . This is another way to show the fiber $(\pi, H_2)^{-1}(t, b_2)$ is a circle. As expected, the circle action induced by H_2 (which gives a global torus action) is just to rotate along these circles.

Then, is useful to note that

$$H_1(z_1, z_2) = \text{dist}(c, z_1 z_2)^2 = \text{dist}(c, \pi(z))^2.$$

Therefore, the fiber $H_1^{-1}(r^2)$ is exactly the set of points in $\pi^{-1}(t)$ ranging over all $t \in \mathbb{C}$ that are r away from c . In other words, the fiber is the union of conics over the circle of radius r with center c . In particular, assuming $r \neq c$, the fiber $H_1^{-1}(r^2)$ is $S^1 \times \text{conic}$

(which makes sense since we expect fibers of H_1 to have 3 real dimensions). This can be pictured as in Figure 54:

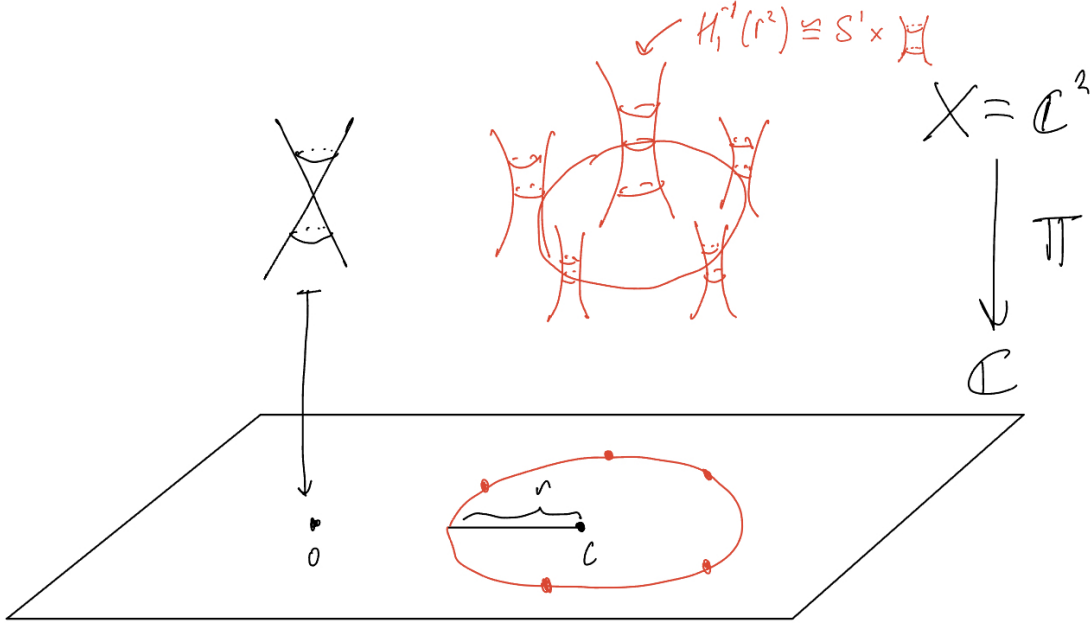


FIGURE 54. A generic fiber $H_1^{-1}(r^2)$ is shown as well as the fiber of π over $t = 0$.

Finally, the generic fiber $(H_1, H_2)^{-1}(r^2, b_2)$ equals a union of $(\pi, H_2)^{-1}(t, b_2)$'s over the circle $|t - c| = r$ — the fiber is just $S^1 \times S^1$!

There are also some special fibers (pictured below in Figure 55): For example, consider $H^{-1}(c^2, 0)$, which is a torus with one curve pinched. Another example is $H^{-1}(0, b_2)$ which involves points over the circle a radius 0 centered at c . Such a circle is a single point, i.e., the fiber is just a single circle contained in the conic above c . Since $H_1(z) \geq 0$ for all $z \in \mathbb{C}^2$, you can think of this as occurring at the edge of the Delzant polytope. One can check that $(z_1, z_2) = (0, 0)$ is the focus-focus singular point of the system. In particular, $(0, 0)$ is the unique point on the singular fiber $\pi^{-1}(0)$ with height $w = H_2(z) = 0$. Technically, we still need to check that a neighborhood of $(0, 0)$ in $\mathbb{C}^2$ can be given a focus-focus chart. As explained on the top of page 103 in Evans, it suffices to check order 2 information, in which case this becomes a computation.

Remark 25.2. Although the system is locally toric on the regular fibers, the focus-focus singularity prevents it from being the moment map of a global Hamiltonian torus action! While the Hamiltonians H_1, H_2 define an integrable system, they fail to be action coordinates because they don't define a global torus action. Actually, as already noted H_2 does induce a global action and the problem occurs with H_1 . We will show how the focus-focus singularity creates monodromy that obstructs the action being

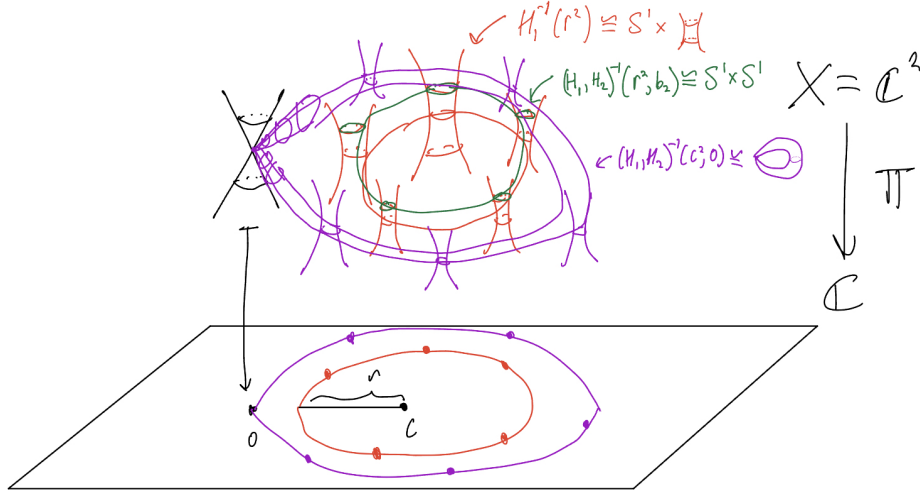


FIGURE 55. Adding to Figure 54, we have labeled a generic fiber of H , $H^{-1}(r^2, b_2) \cong S^1 \times S^1$ and the singular focus-focus fiber $H^{-1}(c^2, 0)$.

globally toric. Another way to see this is to just compute the equations for the action to find that the period of the H_1 action blows up as $(r^2, b_2) \rightarrow (c^2, 0)$.

Next, consider the image of H . Since $H_1 \geq 0$ and H_2 can be anything, the image is the right half plane. The singular values come in two types: There is the isolated focus-focus singular value at $(c^2, 0)$ and there is the toric boundary line of elliptic critical values whose fibers are circles instead of tori, occurring along the y -axis.

Let $B = \{(b_1, b_2) : b_1 \geq 0\} \setminus \{(c^2, 0)\}$ be the set of nonsingular fibers (including the toric boundary). The existence of monodromy prevents there from being global action coordinates. However, making a branch cut, consider the simply connected $B_0 = \{(b_1, b_2) : b_1 \geq 0\} \setminus \{(b_1, 0) : b_1 \geq c^2\}$. As will be explained, we can define global action coordinates $\alpha : B_0 \rightarrow \mathbb{R}^2$.

Remark 25.3. It is particularly informative to consider the case of $c = 0$, where we can just use the entire right half plane $B = \{(b_1, b_2) : b_1 > 0\}$ (which is simply connected) instead of passing to B_0 . Now we have $H = (H_1, H_2)$ where $H_1(z) = |z_1 z_2|^2$ and $H_2(z) = \frac{1}{2}(|z_1|^2 - |z_2|^2)$. Compare this to $\mu = (\mu_1, \mu_2)$ where $\mu_1(z) = \frac{1}{2}|z_1|^2$ and $\mu_2(z) = \frac{1}{2}|z_2|^2$. In particular, $H_1 = 4\mu_1\mu_2$, $H_2 = \mu_1 - \mu_2$ and the values of μ determine the values of H are vice versa (eg, given H_1, H_2 recover μ_1 as the nonnegative root of $H_1 = 4\mu_1(\mu_1 - H_2)$). The upshot is that H and μ have the same fibers. Since we understand μ really well (its Delzant polytope is just the right quadrant), we can use this get global action angle coordinates for the system determined by H .

Returning to the case of $c > 0$, the base still has an integral affine structure, but it is only defined locally. Because of the focus-focus singularity, this affine structure has nontrivial monodromy, so there are no global action coordinates. However, after passing to the universal cover, we do get global action coordinates. It is useful to think of B_0 as being a fundamental domain for the $\pi_1(B) = \mathbb{Z}$ action on $\tilde{B}$. A good model

for $\tilde{B}$ is pictured below in Figure 56. As you travel around the focus-focus singularity in the base, you move from one copy of B_0 to another copy in $\tilde{B}$. Therefore, if we give one B_0 an integral affine structure, we can analytically extend to another copy via the affine monodromy. As explained in the previous lecture, traveling around the focus-focus singularity induces monodromy i.e., the local integral affine structure is sheered compared to the structure on the previous sheet.

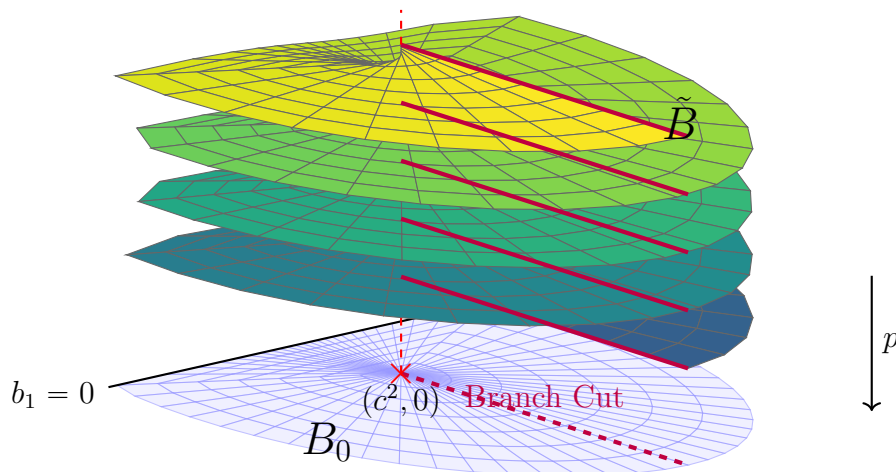


FIGURE 56. The universal cover $\tilde{B}$ over the base space B . The fundamental domain B_0 is formed by making a branch cut along $b_1 \geq c^2$. The universal cover $\tilde{B}$ consists of infinitely many glued copies of B_0 , glued along the branch cut.

As mentioned, we can define global action coordinates on $\tilde{B}$. Moreover, we have the following:

Proposition 25.4. *Viewing B_0 as a fundamental domain for the $\pi_1(B)$ action on $\tilde{B}$, we can define global action coordinates on $\tilde{B}$ such that the image of B_0 under the action coordinates is*

$$\{(b_1, b_2) : 0 \leq b_1 \leq \phi(b_2)\} \setminus \{(b_1, 0) : b_1 \geq m\}$$

for some $m > 0$ and $\phi: \mathbb{R} \rightarrow (0, \infty]$.

Proof. The existence of action angle coordinates (Theorem 17.4) is only ensured locally, and because of monodromy, there is no globally defined change of coordinates $\alpha: B \rightarrow \mathbb{R}^2$ such that $\alpha \circ H$ generates a global torus action. However, since the fundamental domain B_0 is simply connected, we can patch together local action-coordinate charts to obtain global action coordinates $\alpha: B_0 \rightarrow \mathbb{R}^2$. Since H_2 already generates a torus action, the second component of α can stay unchanged, i.e., $\alpha(b_1, b_2) = (G(b_1, b_2), b_2)$ for some function G . If we add back the branch cut, then we can still consider $\alpha: B \rightarrow \mathbb{R}^2$ with $\alpha(b_1, b_2) = (G(b_1, b_2), b_2)$, except we need to allow G to be multi-valued. In particular, when you go around the focus-focus singularity, the value $(G(b_1, b_2), b_2)$ is replaced with $(G(b_1, b_2) + b_2, b_2)$. In other words, G is the multi valued function which takes on exactly

the values $G(b_1, b_2) + b_2\mathbb{Z}$, where the amount of b_2 's added on depends on how many loops around the focus-focus you have traveled from some original reference. In this way, α can be thought of as a single-valued global action coordinate map $\tilde{B} \rightarrow \mathbb{R}^2$.

In order to prove the claim, note that it suffices to show one of the B_0 's making up $\tilde{B}$ has its y -axis mapping to the y -axis of $\mathbb{R}^2$ (or even just a vertical line). For then we have a function $\alpha(b_1, b_2) = (G(b_1, b_2), b_2)$ which only moves points horizontally and which preserves the y -axis. We can also assume G is always ≥ 0 . Then, the claim follows immediately by taking $\phi(b_2) = \sup_{b_1 \geq 0} G(b_1, b_2)$ and $m = G(c^2, 0)$.

It suffices to show we can find some fundamental domain such that α sends the y -axis to a vertical line. Since $l := \{(0, b_2)\} \subset B_0$ is a toric boundary, Proposition 19.5 implies its image under α is a line with rational slope. Then, we must somehow argue that the slope is $1/n$ for some $n \in \mathbb{Z}$ (see Evans for more details about how to show this).

Then, by post-composing with $\begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}$, we can assume l gets mapped to a vertical line, which can be made the y -axis after a horizontal shift.

Note that we can't use an arbitrary integral affine map to change coordinates so that l is vertical because then we will no longer have α moving points horizontally. We actually need to ensure we can make l vertical via a shear.

□

In the next lecture, we will define *almost toric fibrations*, and will see how the Auroux system is the prototypical example of such a phenomenon.

26. ALMOST TORIC FIBRATIONS AND NODAL TRADES (04/16, MN, YD)

This lecture covers material from [Eva23, Chapter 8]. Another excellent reference is [Sym02].

In this lecture, we will explore how to modify an existing Lagrangian fibration on a symplectic 4-manifold to construct new Lagrangian fibrations. Two concrete applications of constructing new Lagrangian fibrations are that they allow us to: (i) better understand and “visualize” the topology of the symplectic 4-manifold, and (ii) find new Lagrangians inside symplectic 4-manifolds. For example, in [Sym02, Section 11], Symington constructs an almost toric fibration that allows us to visualize the homology classes and intersection form of a K3 surface¹⁷:

$$H_2(\text{K3}, \mathbb{Z}) \simeq E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}.$$

Recall that a toric 4-manifold is a symplectic manifold X^4 equipped with a Hamiltonian $\mathbb{T}^2$ -action associated with a moment map $\mu: X \rightarrow \mathbb{R}^2$ (cf. Definition 19.3). In particular, the fibers are either $\mathbb{T}^2$, $\mathbb{T}^1$, or a single point. We now relax this definition by allowing the map to have focus-focus singularities, thus permitting a pinched 2-torus to also be a fiber.

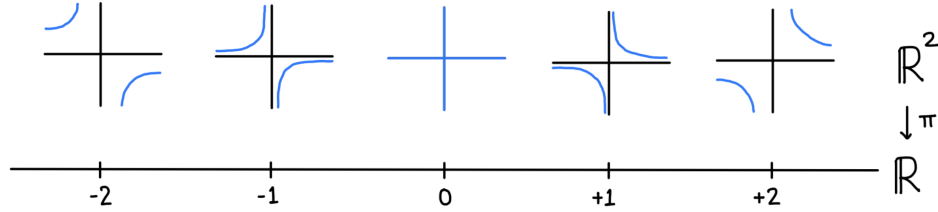
¹⁷This is especially exciting because, in standard K3 literature, the isomorphism $H_2(\text{K3}, \mathbb{Z}) \simeq E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$ is not explicit and is typically established via the classification of even indefinite lattices of a given rank and signature.

Definition 26.1. An **almost toric fibration** is a Lagrangian torus fibration $f: X \rightarrow B$ on a 4-dimensional symplectic manifold such that the discriminant locus, $B \setminus B^{\text{reg}}$, in the base B comprises a collection of 0- and 1-dimensional strata (cf. Definition 18.5). Furthermore, the smooth structure on B extends over these strata, and f is smooth with respect to this extended smooth structure, possessing either toric or focus-focus critical points there.¹⁸

Remark 26.2. We emphasize that the symplectic manifold X , which is the total space of the almost toric fibration, has no singularities and is completely smooth, even though individual fibers may be singular, i.e. a pinched torus. The situation is completely analogous to an ordinary double point (or A_1) singularity in algebraic geometry/symplectic fibrations where the total space is smooth, but the fiber over the critical value has a nodal singularity.

Explicitly, working over $\mathbb{R}$, consider

$$\begin{aligned} \pi: \mathbb{R}^2 &\rightarrow \mathbb{R} \\ (x_1, x_2) &\mapsto x_1 x_2 \end{aligned}$$



Over $\mathbb{C}$, we have

$$\begin{aligned} \pi: \mathbb{C}^2 &\rightarrow \mathbb{C} \\ (z_1, z_2) &\mapsto z_1 z_2 \end{aligned}$$

Continuing the general discussion on almost toric fibrations, notice that away from the fibers containing focus-focus critical points,

$$f|_{X \setminus f^{-1}(f(\text{FF}))}: X \setminus f^{-1}(f(\text{FF})) \rightarrow B \setminus f(\text{FF}),$$

the total space $X \setminus f^{-1}(f(\text{FF}))$ is a (non-compact) locally toric manifold. In particular, $B \setminus f(\text{FF})$ admits an integral affine structure.

We proceed to study almost toric fibrations by studying the base B with certain singularities.

¹⁸To expand, a point $b \in B \setminus B^{\text{reg}}$ is called toric critical value if there is a neighborhood, N_b , of b so that $f^{-1}(N_b)$ is a toric manifold on which f is the toric moment map; in particular, the fiber over b is a smooth torus, $\mathbb{T}^k$. Next, $b \in B \setminus B^{\text{reg}}$ is called focus-focus critical value if it contains a focus-focus critical point in its fiber (see Definition 24.1).

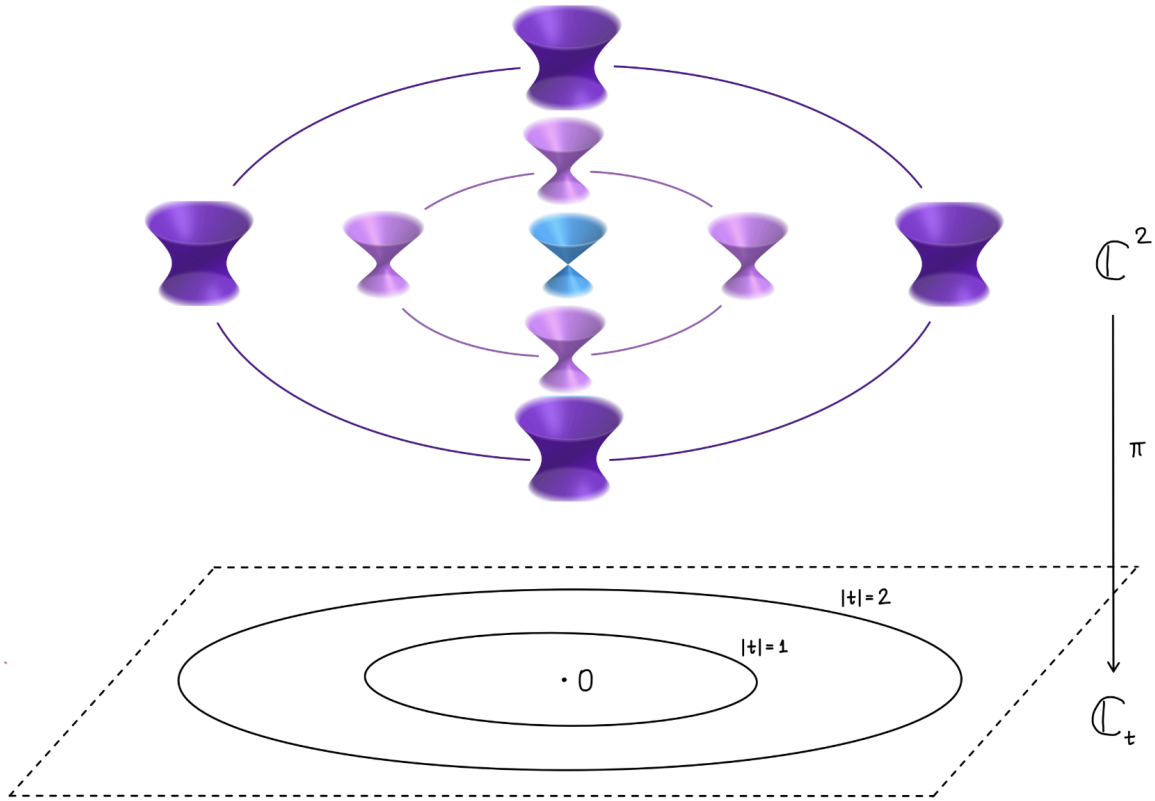


FIGURE 57. Analogous to the real case above, the singular fiber over 0 is a union of two copies of $\mathbb{C}$ intersecting transversely at a single point. This singular fiber along with the other smooth fibers “fit” together inside the smooth total space $\mathbb{C}^2$.

Example 26.3. The Auroux system, introduced in the previous lecture, is the primary one-parameter family of almost toric fibrations. It serves as the local model (in a neighborhood of the focus-focus fibers) for the constructions that follow. Recall that it is a global, explicit Hamiltonian system depending on a real parameter $c > 0$, defined as:

$$\begin{aligned} H: \mathbb{C}^2 &\rightarrow \mathbb{R}^2 \\ H_1(z_1, z_2) &= |z_1 z_2 - c|^2 \\ H_2(z_1, z_2) &= \frac{1}{2}(|z_1|^2 - |z_2|^2) \end{aligned}$$

The critical point at the origin $(0, 0) \in \mathbb{C}^2$ is a focus-focus singularity, which corresponds to the nodal singularity of the pinched 2-torus fiber over the point $(c^2, 0)$ in the base. To use this as a local model, we examine the integral affine structure on the regular part of the base $B \setminus \{(c^2, 0)\}$ (see Figure 58).

The key observations are:

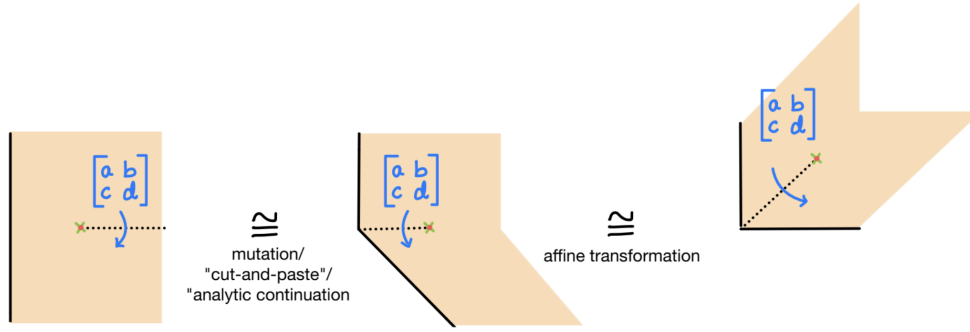


FIGURE 58. Three equivalent representations of the integral affine structure on the base $B \setminus \{(c^2, 0)\}$ of the Auroux system.

- (1) There are several ways to represent the integral affine structure on the base $B \setminus \{(c^2, 0)\}$, all of which are integral affine equivalent.
- (2) The one-dimensional stratum is always a straight line. Certain representations depict a “illusionary corner” created by the choice of branch cut and the resulting monodromy around the nodal point.
- (3) In all three representations in Figure 58, the dotted line (the branch cut) acts as an eigenline for the monodromy matrix.
- (4) The monodromy around the nodal point is the unique integral affine map that fixes the dotted line pointwise and “straightens” the illusionary corner into a straight line.

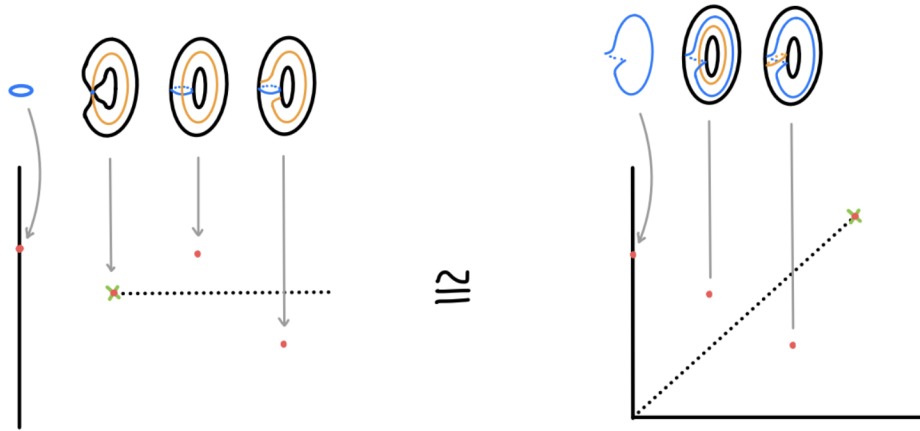
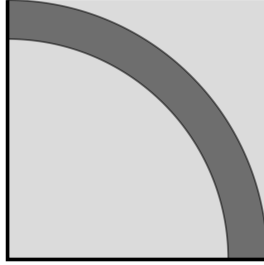


FIGURE 59. Fibers of the Auroux system over different points in the base, showing the evolution of homology cycles. The cycles on distinct fibers are related via the Gauss–Manin connection (Ehresmann’s theorem).

26.1. Nodal Trade. The basic operation used to produce examples of almost toric fibrations is the **nodal trade**. It is a local operation in a neighborhood of the origin

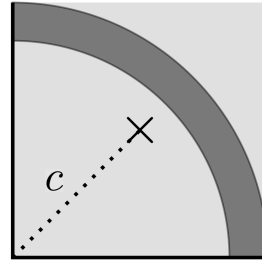
in $\mathbb{C}^2$ that allows us to “trade” a Delzant corner for a focus-focus singularity. This operation is performed by replacing a neighborhood of the Delzant corner with a neighborhood of the Auroux system as illustrated below:

Before Trade (Toric)



- No focus-focus singularity.
- One Delzant corner.
- The corner is “real.”
- Toric boundary: two symplectic discs intersecting transversely at $(0, 0)$.

After Trade (Almost toric)



- One focus-focus singularity.
- No Delzant corner.
- The corner is “illusionary.”
- Toric boundary: a single symplectic cylinder.

Explanations:

- **Operation.** The shaded circular regions in the two pictures are integrally affine equivalent and the preimages of these two regions are fiber-preserving symplectomorphic. This allows us to excise the Delzant corner and replace it with the Auroux system by gluing along the common symplectomorphic preimages.
- **Toric boundary.** This is the preimage of the boundary of the moment image. A nodal trade “smooths” two symplectic discs intersecting transversely into a single symplectic cylinder.

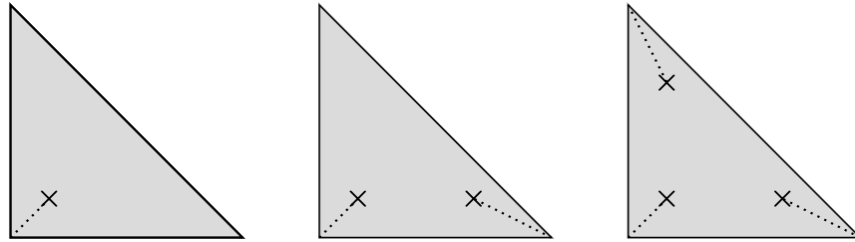


FIGURE 60. Three different almost toric fibrations on $\mathbb{CP}^2$ obtained by performing one, two, and three nodal trades.

Example 26.4. Almost toric fibrations on $\mathbb{CP}^2$ can be obtained by performing nodal trades at one or more vertices of the standard Delzant triangle of $\mathbb{CP}^2$, as depicted in Figure 60. Furthermore, observe that:

- (1) The direction of the dotted line at any vertex is uniquely determined by the rule that the dotted line, together with the Delzant corner, is integrally affine equivalent to the standard diagonal dotted line paired with the standard “quadrant” Delzant corner.
- (2) The monodromy around any dotted line is precisely the unique integral affine map that fixes the dotted line pointwise and straightens the delusional corner into a straight line.
- (3) The toric boundaries in the three examples are:
 - (a) two symplectic spheres intersecting transversely at two points, where one sphere has twice the symplectic area of the other.
 - (b) a pinched torus, obtained from a symplectic sphere with a single self-intersection, having a symplectic area of three.
 - (c) a torus having a symplectic area of three.

Remark 26.5. There is a closely related operation called a *generalized nodal trade* that can be used to smooth the singularities of toric orbifolds. To illustrate this, consider a singular cubic surface:

$$\{z_0 z_1 z_2 = z_3^3\} \subseteq \mathbb{CP}^3$$

with three A_2 singularities at $[1 : 0 : 0 : 0]$, $[0 : 1 : 0 : 0]$, and $[0 : 0 : 1 : 0]$. This cubic admits a torus action given by:

$$[z_0 : z_1 : z_2 : z_3] \mapsto [e^{3i\theta} z_0 : e^{3i\varphi} z_1 : z_2 : e^{i\theta+i\varphi} z_3]$$

Performing generalized nodal trades on the moment polytope associated with this action yields a smooth cubic surface.

26.2. Nodal Slide. Recall that the Auroux system is a family of systems parametrized by $c > 0$. Each value of c gives a different almost-toric fibration with different lengths for the dotted line¹⁹. Such a family of fibrations is called a **nodal slide** (see Figure 61).

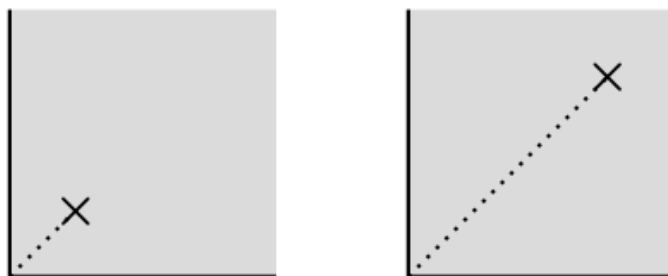


FIGURE 61. A nodal slide is the operation of changing the length of the branch cut.

¹⁹Question: What is the relationship between the affine length of the dotted line and parameter c ?

Theorem 26.6. *Suppose that X and X' are almost toric manifolds whose almost toric base diagrams B and B' are related by a nodal slide or a nodal trade. Suppose that the base diagrams have the homotopy type of a punctured 2-dimensional surface. Then X and X' are symplectomorphic.*

26.3. Classification of closed 4-manifolds admitting toric and almost toric fibrations. In the theory of toric varieties, the following classification theorem is established through the study of *fans* associated with the varieties.

Theorem 26.7. *Let X be a smooth, complete (i.e., closed) toric variety over $\mathbb{C}$ of complex dimension 2. Then X is obtained via a finite sequence of toric blow-ups (at torus-fixed points) from one of the following minimal models:*

- The complex projective plane $\mathbb{CP}^2$.
- A Hirzebruch surface $\mathcal{H}_r = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r))$ for an integer $r \geq 0$, with $r \neq 1$.

Analogously, in [Sym02, Appendix A], Symington discusses the classification of smooth closed 4-manifolds that admit a symplectic toric fibration by studying their moment images.

Theorem 26.8. *Let X be a smooth closed 4-manifold that admits a symplectic toric fibration. Then X is diffeomorphic to one of the following or their blow-ups (i.e., the connected sum with finitely many $\overline{\mathbb{CP}}^2$):*

- The complex projective plane $\mathbb{CP}^2$.
- The product of two spheres $S^2 \times S^2$, which is diffeomorphic to $\mathcal{H}_r$ for all even r .²⁰

Moving to almost toric fibrations, Leung and Symington [LS10, Theorem 2.6] show that the collection of closed manifolds admitting almost toric fibrations is a modest expansion of those admitting toric fibrations.

Theorem 26.9. *Let X be a smooth closed 4-manifold that admits an almost toric fibration. Then X is diffeomorphic to one of the following:*

- $S^2 \times S^2$ or $S^2 \times T^2$,
- blow ups of $n \geq 0$ points on $\mathbb{CP}^2$ or the non-trivial oriented sphere bundle over the torus,
- the K3 surface or the Enriques surface,
- a specific kind of torus bundle over the torus or over the Klein bottle.

27. MUTATIONS AND THE MARKOV EQUATION (04/21, YW, MN)

27.1. Shearing along the eigenline. Let x be a node in $\mathbf{R}^2$, i.e., the image of a focus-focus singularity. The one eigenline of monodromy bisects the base into two parts. We can shear the branch cut either CW or CCW:

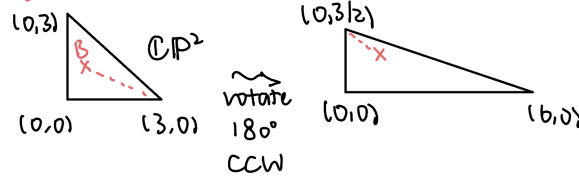
$$\begin{array}{ccc} \text{CW} \curvearrowright & x & \text{D}_1 \\ & \vdots & \\ \text{CCW} \curvearrowleft & & \text{D}_2 \end{array}$$

²⁰The odd Hirzebruch surfaces, $\mathcal{H}_r$ with odd r , are all diffeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$.

In each case, the affine structure on the base can be described as

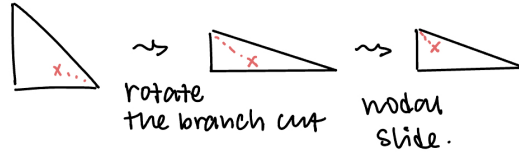
$$\begin{array}{ccc} D_1 & & \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} D_1 \\ \text{---} x & \text{or} & \text{---} x \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} D_2 & & D_2 \end{array}$$

This rotation gives us different pictures of the *same* object: neither the total space (X, ω) nor the almost toric fibration have changed. To better understand this, consider the following example, keeping in mind that the two edges adjacent to a vertex with a node is actually a straight line:



Depending on the situation, the word *mutation* can have two different meanings:

- (1) it is the rotational move above;
- (2) it is the rotational move above followed by a nodal slide, which *does* change the fibration structure.

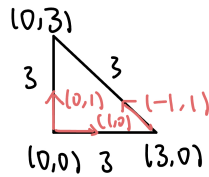


Definition 27.1. Given a rational triangle whose edges are of affine lengths ℓ_1, ℓ_2, ℓ_3 , we define new coordinates called *the Markov coordinates*:

$$p_1 = 3(\ell_2 \ell_3)^{-1/2}, \quad p_2 = 3(\ell_1 \ell_3)^{-1/2}, \quad p_3 = 3(\ell_1 \ell_2)^{-1/2}.$$

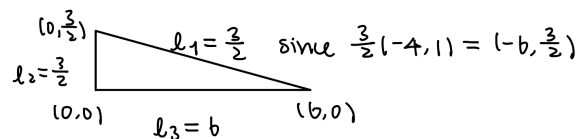
Because $\ell_1 = 3p_1/(p_2 p_3)$ and similarly for ℓ_2, ℓ_3 , this change is invertible.

Example 27.2. Consider the triangle



where the affine lengths are $\ell_1 = \ell_2 = \ell_3 = 3$. Then $(p_1, p_2, p_3) = (1, 1, 1)$.

Example 27.3. Consider the triangle



where the affine lengths are $\ell_1 = \ell_2 = 3/2, \ell_3 = 6$. Then $(p_1, p_2, p_3) = (1, 1, 2)$, which are integral.

Definition 27.4. The Markov equation is

$$p_1^2 + p_2^2 + p_3^2 = 3p_1p_2p_3.$$

In ℓ_i -coordinates, this becomes

$$\frac{9}{\ell_2\ell_3} + \frac{9}{\ell_1\ell_3} + \frac{9}{\ell_1\ell_2} = \frac{9^2}{\ell_1\ell_2\ell_3}.$$

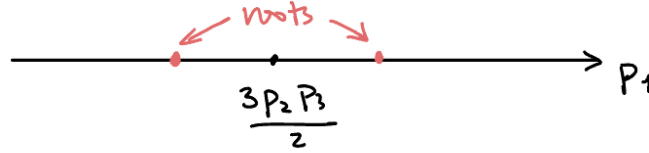
Multiplying both sides by $\ell_1\ell_2\ell_3$, this implies $\ell_1 + \ell_2 + \ell_3 = 9$, that is, the affine perimeter of the corresponding triangle is 9.

27.2. Elementary discussion of the Markov equation. First, it is quadratic in each variable. For example, in p_1 it can be written as

$$p_1^2 + (-3p_2p_3)p_1 + (p_2^2 + p_3^2) = 0.$$

The roots are

$$p_1 = \frac{3p_2p_3 \pm \sqrt{9p_2^2p_3^2 - 4p_2^2 - 4p_3^2}}{2}.$$



$$\text{involution } p_1 \mapsto -p_1 + 3p_2p_3$$

Definition 27.5. The Vieta involution is

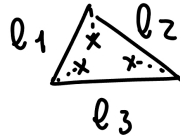
$$\sigma_1(p_1, p_2, p_3) = (-p_1 + 3p_2p_3, p_2, p_3).$$

It preserves the Markov equation. Similarly we can define σ_2, σ_3 .

Remark 27.6. The involutions $\sigma_1, \sigma_2, \sigma_3$ do not commute. In fact, $\langle \sigma_1, \sigma_2, \sigma_3 \rangle = (\mathbf{Z}/2) * (\mathbf{Z}/2) * (\mathbf{Z}/2)$.

From now on, we restrict to triangles obtained from the standard $\mathbb{CP}^2$ triangle by repeated mutations, with the normalization above.

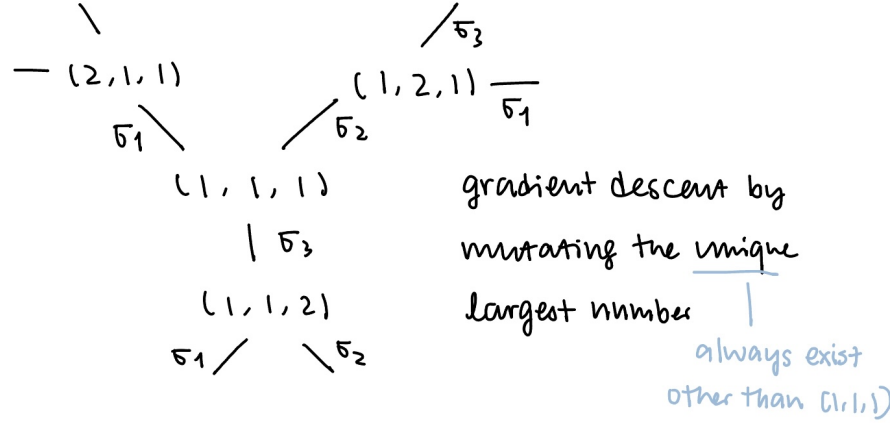
Lemma 27.7. *If we mutate the diagram below at vertex i*



then the Markov coordinates change from (p_1, p_2, p_3) to $\sigma_i(p_1, p_2, p_3)$.

Corollary 27.8. *All triangles arising from the repeated mutation from the triangle with affine lengths $\ell_1 = \ell_2 = \ell_3 = 3$ have Markov coordinates that form an integral solution to the Markov equation.*

Conversely, it is a fact that any positive integral solution of the Markov equation can be obtained from the the initial triangle with all three affine edge lengths equal to 3 by repeated Vieta involution, and all such solutions together form an infinite trivalent tree:



Corollary 27.9. *Positive integral Markov triples describe the triangular almost toric diagrams arising from these mutations*

There are many consequences of this result, and we will only list the statements:

- (1) There are infinitely many *exotic* monotone Lagrangian tori in $\mathbf{CP}^2$ (the tori over a certain center point of each mutated triangle).
- (2) There is a famous problem as follows: consider the ellipsoid

$$E(1, a) = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^2 + \frac{|z_2|^2}{a} \leq 1\}$$

and set

$$c(a) := \inf\{\mu : \exists \text{ symplectic embedding } E(1, a) \hookrightarrow B(0; \sqrt{\mu}) \subseteq \mathbf{C}^2\}.$$

The graph of $c(a)$ turns out to contain an infinite staircase as depicted in Figure 62. The embeddings can be concretely constructed using the shape of moment polytope of $E(1, a)$, which can be embedded into balls using Delzant uniqueness.

27.3. Almost toric blowups. In addition to the symplectic blow-up discussed in Example 23.2 using symplectic cut, there is another way to achieve a blow-up for toric/almost-toric manifolds called *almost toric blowup*. This involves modifying the moment image as follows:

This operation introduces a new nodal fiber. The following figure compares (a) the earlier symplectic blowup with (b) the almost-toric blowup applied on $\mathbf{CP}^2$ with the standard toric fibration.

Performing nine almost-toric blowups along with three nodal trades on $\mathbf{CP}^2$ gives an

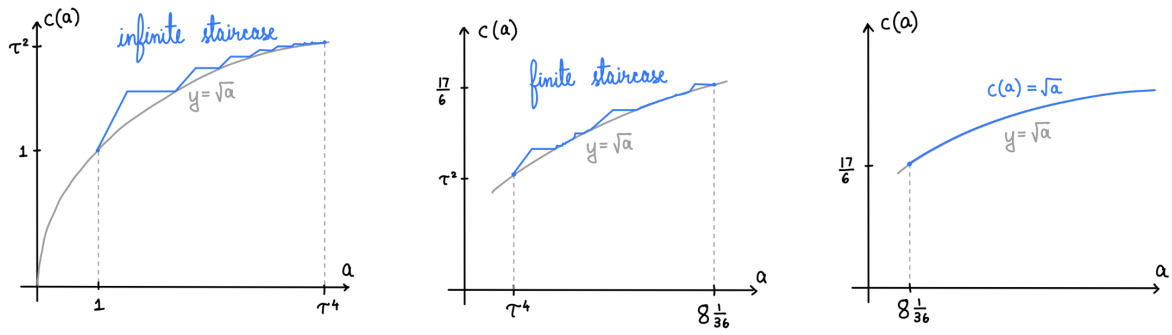
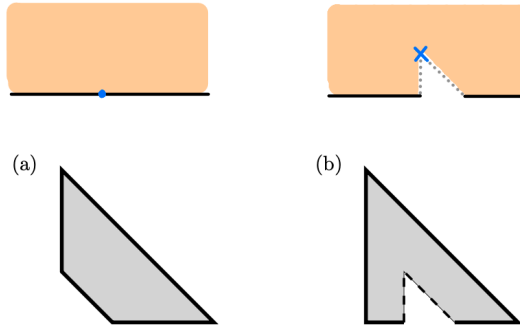
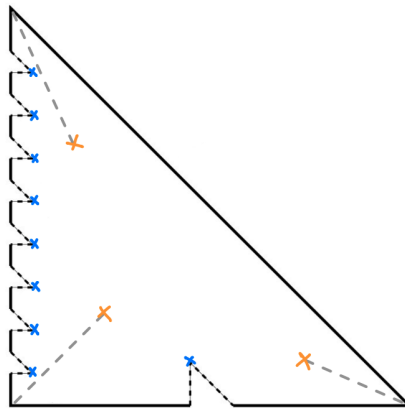


FIGURE 62. The graph of $c(a)$ vs a in three domains (See [MS12, Theorem 1.1.2.]). Note $c(a) \geq \sqrt{a}$ as symplectic embeddings are volume preserving.



almost toric fibration on $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$:



This manifold is the rational elliptic surface $E(1)$. Notice the toric boundary of this almost-toric manifold is a single smooth torus. Gluing two copies of $E(1)$ appropriately along the toric boundaries leads to a K3 surface! For more details, see [Sym02].

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