Arithmetic statistics for knots and knot invariants

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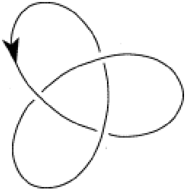
June 25, 2019

Knots



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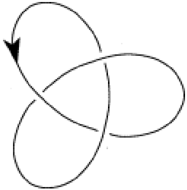

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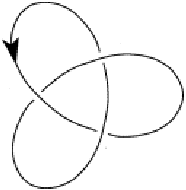

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We can also define knots in higher dimensions:

Definition

An n-knot K is an embedded submanifold of S^{n+2} homeomorphic to S^n.

Knot Equivalence

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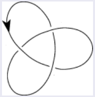
Note

We consider knots to be *oriented*, which means that we keep track of the orientation on K as well as on the ambient S^{n+2}.



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- The class group of a random real quadratic field is a 2-group times a finite group drawn from a given distribution(in particular the average size of the odd part is bounded).
- The class group of a random imaginary quadratic field is a 2-group times a cyclic group $\sim 98\%$ of the time.

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Connection to arithmetic statistics

Families of knots can be parametrized by arithmetic data.

The family of simple n-knots

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An n-knot K is *simple* if $\pi_i(S^{n+2} - K) = \pi_i(S^1)$ for $i \leq (n-1)/2$.

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Alexander polynomial

The Alexander module Alex_K is a complicated object, but it turns out that a lot of the information from it is contained in one polynomial, namely the Alexander polynomial $\Delta_K(t) \in \mathbb{Z}[t]$ of a simple $4a+1$-knot K.

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- $\Delta_K(1) = 1$
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If $\Delta_K(t)$ is a quadratic polynomial, it must have the form

$$mt^2 + (1-2m)t + m$$

for some $m \in \mathbb{Z}$.

Alexander Module vs Alexander Polynomial

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Goal

Obtain a quantitative form of this finiteness statement.

Seifert hypersurfaces and Seifert pairings

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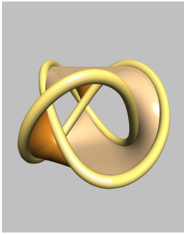

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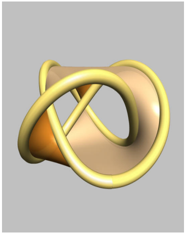

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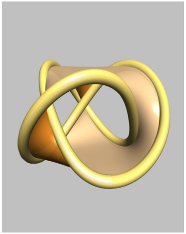

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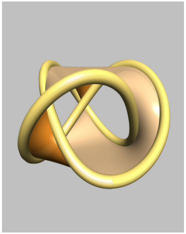

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Seifert hypersurfaces exist for all knots, but are not unique.
We say that V^{n+1} is a *simple Seifert hypersurface* if V is $\lfloor \frac{n}{2} \rfloor$-connected.
Simple Seifert hypersurfaces exist for all simple knots.

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If you pick a basis for $H_{2a+1}(V, \mathbb{Z})$ in which the intersection pairing has matrix equal to the standard skew-symmetric matrix J, the matrix P of the Seifert pairing will satisfy $P - P^t = J$.

Seifert pairing and Alexander module

Theorem (Kearton, Levine)

If V^{4a+2} is a Seifert surface for K^{4a+1} with nondegenerate Seifert matrix P, then the $\mathbb{Z}[t, t^{-1}]$-module Alex_K is presented by the matrix

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$$\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 6 \end{pmatrix}$$

Seifert pairings and orbits

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Observation

Isomorphism classes of Seifert pairings are equivalent to orbits of the group $\mathrm{Sp}_{2g}(\mathbb{Z})$ on the set of $2g \times 2g$- matrices P with $P - P^t = J$. (Here $X \in \mathrm{Sp}_{2g}(\mathbb{Z})$ acts by $P \mapsto XPX^t$.)

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Conclusion

Isomomorphism classes of Seifert pairings are orbits for the representation $\mathrm{Sym}^2(2g)$ of $\mathrm{Sp}_{2g}(\mathbb{Z})$!

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There is also a generalization to higher degree Alexander polynomials.

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So we get the same results for isomorphism classes of Seifert pairings, or of simple Seifert hypersurfaces.

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Average-case counting simple knots of genus 1, equivalently genus 1 Alexander modules with Blanchfield pairing, is harder because we need to know the average size of the narrow class group of $R_\Delta = \mathbb{Z}[\frac{1}{m}][\frac{1+\sqrt{1-4m}}{2}]$.

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Case: $|m|$ is composite

For any $p \mid |m|$, the ideal $(p, \gamma_m)^2$ lies in the kernel of the map $\mathrm{NCl}(\mathbb{Z}[\gamma_m]) \to \mathrm{NCl}(R_m)$

Our Heuristic

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- *There are* $\sim X^{3/2}/\log X$ *isomorphism classes of simple* $2q-1$ *knots whose Alexander polynomial has the form* $pt^2+(1-2p)t+p$ *for some prime* p *in the range* $[1, X]$.

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- *There are* $\sim X \log X$ *isomorphism classes of simple* $2q-1$ *knots whose Alexander polynomial has the form* $mt^2 + (1-2m)t + m$ *for some* $m \in [-X, X]$ *such* m *is not a positive prime.*

Bounds on contribution from the prime case

Theorem

The total number of isotopy classes of simple $4a+1$-knots having Alexander polynomial equal to Δ_p for $0 \leq p \leq X$ is $\gg (X^{3/2-\epsilon})$ for all $\epsilon > 0$.

Theorem

The total number of isotopy classes of simple $4a+1$-knots having Alexander polynomial equal to Δ_p for p running over all primes in the range $[1, X]$ is (unconditionally) bounded above by $O(X^{3/2}/\log X)$.

Bounds on total

The lower bound for the contribution of primes also gives us a lower bound for the total:

Theorem

The total number of isotopy classes of simple $4a + 1$*-knots having Alexander polynomial equal to* Δ_m *for* $|m| <\leq X$ *is* $\gg (X^{3/2-\epsilon})$ *for all* $\epsilon > 0$*.*

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Work in progress

The total number of Sp_{2g}-orbits of symmetric matrices Q such that $\mathrm{ht}(e_Q) < X$ is asymptotic to $X^{g(g+\frac{1}{2})}$.
As in quadratic case, most orbits come from the cases when the stabilizer of Q finite, when happens when $\mathbb{Q}[x]/e_Q(x)$ is a CM field. Can also count the other orbits weighted by regulator.

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For the knot question, expect again largest contribution when c_{2g} is prime.

Thank you!