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# ARITHMETIC TRACES OF NON-HOLOMORPHIC MODULAR INVARIANTS

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We extend results of Bringmann and Ono that relate certain generalized traces of Maass-Poincaré series to Fourier coefficients of modular forms of half-integral weight. By specializing to cases in which these traces are usual traces of algebraic numbers, we generalize results of Zagier describing arithmetic traces associated to modular forms. We define correspondences $\mathcal{Z}_{\lambda,N,m} : M^{\sharp}_{2-2\lambda}(\Gamma_0(N)) \to M^{!}_{\lambda+\frac{1}{2}}(\Gamma_0(4N))$ and $\mathcal{Z}'_{\lambda,N,n} : M^{\sharp}_{2-2\lambda}(\Gamma_0(N)) \to M^{!}_{\frac{3}{2}-\lambda}(\Gamma_0(4N))$. We show that if $f$ is a modular form of non-positive weight $2-2\lambda$ and odd level $N$, holomorphic away from the cusp at infinity, then the traces of values at Heegner points of a certain iterated non-holomorphic derivative of $f$ are equal to Fourier coefficients of the half-integral weight modular forms $\mathcal{Z}_{\lambda,N,m}(f)$ and $\mathcal{Z}'_{\lambda,N,n}(f)$.

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## 1. Introduction and Statement of Results

Inspired by work of Borcherds [2] on product expansions for meromorphic modular forms, Zagier [11] discovered that the coefficients of certain weakly holomorphic forms of half-integral weight can be interpreted as traces of the values of associated

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modular invariants at Heegner points. For example, let $J := j - 744$ be the Hauptmodul for $\mathrm{SL}_2(\mathbb{Z})$. Borcherds proved that the algebraic trace of a value of $J$ at a Heegner point of discriminant $d$ is equal to the $d$th Fourier coefficient of the unique weight $\frac{3}{2}$ weakly holomorphic modular form in Kohnen's plus-space for $\Gamma_0(4)$ having Fourier development of the form $-q^{-1} + O(1)$, where $q := e(z) := e^{2\pi iz}$. Zagier obtained a variety of similar results by arithmetic arguments. His methods rely on the existence of certain 'nice bases' for the relevant Kohnen spaces of half-integral weight forms.

Bringmann and Ono [3] showed that many of Zagier's results can be viewed as special cases of more generic trace identities relating pairs of weak Maass forms. By interpreting these forms as Poincaré series, Bringmann and Ono were able to work with explicit Fourier developments. They then derived identities giving explicit formulas for traces by relating the Kloosterman sums found in these expansions to Salié sums that have interpretations in terms of binary quadratic forms. We extend their results to broader categories of Maass forms and consider cases in which these traces have arithmetic interpretations as traces of algebraic numbers.

We use the following notion of trace as a sum of values of a modular invariant at Heegner points of a fixed discriminant (this generalizes the twisted traces defined in [3] and [11]). These traces cannot necessarily be interpreted as the usual trace of a single algebraic number, so we will be particularly interested in cases in which they can.

Let $\mathcal{Q}_D$ denote the set of positive definite integral binary quadratic forms $Q = [a_Q, b_Q, c_Q] := a_Q X^2 + b_Q XY + c_Q Y^2$ of discriminant $-D = b_Q^2 - 4a_Q c_Q$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{Q}_D$ in the usual manner, and the subset of quadratic forms $Q$ with $N \mid a_Q$ is invariant under $\Gamma_0(N)$. For a fundamental discriminant $D_1$, let $\chi_{D_1}$ denote the associated genus character for positive definite binary quadratic forms whose discriminants are multiples of $D_1$ (see definition in Section 2). If $\lambda$ is an integer, $D_2$ is a nonzero integer with $(-1)^\lambda D_2 \equiv 0, 1 \pmod 4$ and $(-1)^\lambda D_1 D_2 < 0$, and $f$ is a function on the upper half plane $\mathbb{H}$ invariant under the action of $\Gamma_0(N)$, we let $\mathrm{Tr}_{N,D_1}(f; D_2)$ denote the twisted trace

$$\mathrm{Tr}_{N,D_1}(f; D_2) := \sum_{\substack{Q \in \mathcal{Q}_{|D_1 D_2|}/\Gamma_0(N) \\ a_Q \equiv 0 \pmod N}} \frac{\chi_{D_1}(Q) f(\tau_Q)}{\omega_Q}, \tag{1.1}$$

where $\omega_Q$ is the cardinality of the stabilizer of $Q$ in $\mathrm{PSL}_2(\mathbb{Z})$ and $\tau_Q$ is the unique root of $Q(X, 1) = a_Q X^2 + b_Q X + c_Q$ in $\mathbb{H}$.

**Remark 1.1.** For $Q$ as above, we call the point $\tau_Q$ a *Heegner point of level $N$*. Note that there are no Heegner points of level $N$ with discriminant $-D$ if $-D$ is not a square mod $N$, so $\mathrm{Tr}_{N,D_1}(f; D_2) = 0$ when $\left(\frac{-|D_1 D_2|}{N}\right) = -1$.

We will use the notation

$$\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right\}$$

for the translations in $\mathrm{SL}_2(\mathbb{Z})$.

For positive integers $\lambda, \nu, N$ with $N$ odd, we define a large family $\mathfrak{F}_{\lambda,N,\nu}(z)$ of weak Maass forms (of weight 0 and eigenvalue $\lambda(1-\lambda)$) whose traces will turn out to be Fourier coefficients of certain half-integral weight modular forms. They are given by the Maass-Poincaré series

$$\mathfrak{F}_{\lambda,N,\nu}(z) := \pi\nu^{\lambda-1} \sum_{A\in\Gamma_\infty\backslash\Gamma_0(N)} \mathrm{Im}(\nu Az)^{\frac{1}{2}} I_{\lambda-\frac{1}{2}}(2\pi\,\mathrm{Im}(\nu Az))e(-\mathrm{Re}(\nu Az)), \quad (1.2)$$

where $I_s(x)$ denotes the usual modified Bessel function of the first kind.

We also follow Bringmann and Ono (see [3]) in defining families of weak Maass forms of half-integral weight $k := \lambda + \frac{1}{2}$, which we have generalized to arbitrary level $4N$.

First, for $s \in \mathbb{C}$ and $y \in \mathbb{R} - \{0\}$, define

$$\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2}\,\mathrm{sgn}(y), s-\frac{1}{2}}(|y|),$$

where $M_{\nu,\mu}(z)$ is the usual $M$-Whittaker function. For $m \geq 1$ with $(-1)^{\lambda+1}m \equiv 0, 1 \pmod 4$, define

$$\varphi_{-m,s}(z) := \mathcal{M}_s(-4\pi m\,\mathrm{Im}(z))e(-m\,\mathrm{Re}(z)).$$

Then define the Poincaré series

$$\mathcal{F}_{\lambda,N}(-m, s; z) := \sum_{A\in\Gamma_\infty\backslash\Gamma_0(4N)} (\varphi_{-m,s}|_k A)(z) \quad (1.3)$$

(see Section 2 for definition of $|_k$), which converges for $\mathrm{Re}(s) > 1$.

We now define another family of weak Maass forms of half-integral weight by applying Kohnen's projection operator $\mathrm{pr}_\lambda$ (as defined in [7]) to (1.3) for certain special values of the parameter $s$. More explicitly, for any integer $\lambda$ let

$$F_{\lambda,N}(-m, z) := \begin{cases} \frac{3}{2}\mathcal{F}_{\lambda,N}(-m, \frac{k}{2}; z)|\,\mathrm{pr}_\lambda & \text{if } \lambda \geq 1 \\ \frac{3}{2(1-k)\Gamma(1-k)}\mathcal{F}_{\lambda,N}(-m, 1-\frac{k}{2}; z)|\,\mathrm{pr}_\lambda & \text{if } \lambda \leq 0. \end{cases} \quad (1.4)$$

If $\lambda \geq 1$, then $\varphi_{-m,\frac{k}{2}}(z) = e(-mz)$, so $F_{\lambda,N}(-m, z)$ is actually a weakly holomorphic modular form (of weight $\lambda + \frac{1}{2}$ and level $4N$) satisfying the Kohnen plus-condition (note: if $\lambda = 1$, then these definitions require analytic continuation, and thus the form may still be non-holomorphic). Recall that a modular form is called *weakly holomorphic* if it is holomorphic away from the cusps, and it satisfies the Kohnen plus-condition if its Fourier development is supported on exponents that are 0 or $(-1)^\lambda$ mod 4. The Kohnen space $M^!_{\lambda+\frac{1}{2}}(\Gamma_0(4N))$ is the space of weakly holomorphic modular forms on $\Gamma_0(4N)$ satisfying the Kohnen plus-condition. Thus $F_{\lambda,N}(-m; z)$ has Fourier expansion

$$F_{\lambda,N}(-m; z) = q^{-m} + \sum_{\substack{n\geq 0 \\ (-1)^\lambda n\equiv 0,1 \pmod 4}} b_{\lambda,N}(-m; n)q^n. \quad (1.5)$$

For $\lambda \leq 1$, $F_{\lambda,N}(-m,z)$ is a weak Maass form, and it has Fourier expansion

$$\begin{aligned} F_{\lambda,N}(-m,z) = q^{-m} &+ \sum_{\substack{n\geq 0 \\ (-1)^\lambda n \equiv 0,1 \pmod 4}} b_{\lambda,N}(-m;n)q^n \\ &+ b^-_{\lambda,N}(-m;0)y^{\frac{1}{2}-\lambda} + \sum_{\substack{n>0 \\ (-1)^{\lambda+1} n \equiv 0,1 \pmod 4}} b^-_{\lambda,N}(-m;n)\frac{\Gamma(1-k,4\pi ny)}{\Gamma(1-k)}q^{-n}, \end{aligned} \tag{1.6}$$

where $y := \mathrm{Im}(z)$ and

$$\Gamma(a,x) := \int_x^\infty e^{-t}t^a\frac{dt}{t}$$

is the incomplete Gamma function.

Our first main result is:

**Theorem 1.2.** *Suppose that $\lambda, m, n, \nu, N \geq 1$ are integers with $N$ odd. If $(-1)^{\lambda+1}m$ is a fundamental discriminant, then for each $n$ with $(-1)^\lambda n \equiv 0, 1 \pmod 4$ we have*

$$\begin{aligned} &\mathrm{Tr}_{N,(-1)^{\lambda+1}m}(\mathfrak{F}_{\lambda,N,\nu};n) = \\ &\qquad \frac{1}{2}(-1)^{\left[\frac{\lambda+1}{2}\right]}\nu^{2\lambda-1}m^{\frac{\lambda}{2}}n^{\frac{1-\lambda}{2}} \cdot \sum_{d\mid\nu} d^{-\lambda}\left(\frac{(-1)^{\lambda+1}m}{d}\right)b_{\lambda,\frac{N}{(N,d)}}\left(\frac{-\nu^2 m}{d^2};n\right). \end{aligned}$$

*Alternatively, if $(-1)^\lambda n$ is a fundamental discriminant, then for each $m$ with $(-1)^{\lambda+1}m \equiv 0, 1 \pmod 4$ we have*

$$\begin{aligned} &\mathrm{Tr}_{N,(-1)^{\lambda}n}(\mathfrak{F}_{\lambda,N,\nu};m) = \\ &\qquad \frac{1}{2}(-1)^{\left[\frac{\lambda-1}{2}\right]}m^{\frac{\lambda}{2}}n^{\frac{1-\lambda}{2}} \cdot \sum_{d\mid\nu} d^{\lambda-1}\left(\frac{(-1)^{\lambda}n}{d}\right)b_{1-\lambda,\frac{N}{(N,d)}}\left(\frac{-\nu^2 n}{d^2};m\right). \end{aligned}$$

**Remark 1.3.** If $\nu = N = 1$, this is Theorem 1.2 of [3]. If in addition we require $\lambda = 1$, this is Theorem 6 of [11]. Alternatively, for $\lambda = m = \nu = 1$, this is closely related to Theorem 8 of [11].

It is not immediately clear that the above trace formula generalizes Theorem 11 of [11], which describes traces of certain non-holomorphic derivatives of modular forms. In general, the Poincaré series $\mathfrak{F}_{\lambda,N,\nu}$ need not be such a non-holomorphic derivative. However, it turns out that we can write any non-holomorphic derivative as a linear combination of the $\mathfrak{F}_{\lambda,N,\nu}$ and apply Theorem 1.2 to obtain formulas for its traces.

We recall the definition of the weight $h$ non-holomorphic derivative

$$\partial_h := \frac{1}{2\pi i}\frac{\partial}{\partial z} - \frac{h}{4\pi\,\mathrm{Im}\,z}. \tag{1.7}$$

If $f$ is a modular form of weight $2-2\lambda$ where $\lambda \geq 1$, we define a corresponding modular invariant

$$\mathcal{D}(f) := \partial_{-2}\partial_{-4}\cdots\partial_{2-2\lambda}f. \tag{1.8}$$

Let $M^{\sharp}_{2-2\lambda}(\Gamma_0(N))$ denote the space of modular forms of weight $2-2\lambda$ and level $N$ that are holomorphic away from the cusp at infinity.

**Remark 1.4.** If $f \in M^{\sharp}_{2-2\lambda}(\Gamma_0(N))$, then $\mathcal{D}(f)$ is real-analytic on the upper half plane. It is easily verified that the non-holomorphic derivative $\partial_h$ maps weak Maass forms of weight $h$ and eigenvalue $\xi$ to weak Maass forms of weight $h+2$ and eigenvalue $\xi + h$. Hence $\mathcal{D}(f)$ is actually a weak Maass form of weight 0 and eigenvalue $\lambda(1-\lambda)$. The Maass-Poincaré series $\mathfrak{F}_{\lambda,N,\nu}$ is, by construction, also a weak Maass form of this weight and eigenvalue. Thus it is natural to ask when $\mathcal{D}(f)$ can be written as a linear combination of the $\mathfrak{F}_{\lambda,N,\nu}$.

It turns out that $\mathcal{D}(f)$ has a simple expression in terms of the principal part of $f$. In general, for $n \in \mathbb{Z}$ let $A_n(f)$ denote the $n$th Fourier coefficient of a modular form $f$; in the following theorem, we will only be interested in the case $n < 0$, since such coefficients describe the principal part of $f$.

**Theorem 1.5.** *Suppose that $\lambda \geq 2$, $N \geq 1$ is odd. For any $f \in M^{\sharp}_{2-2\lambda}(\Gamma_0(N))$, we have*

$$\sum_{\nu>0} A_{-\nu}(f)\mathfrak{F}_{\lambda,N,\nu} = \frac{(-1)^{\lambda+1}}{2}\mathcal{D}(f).$$

*If $\lambda = 1$, the two sides differ only by a constant.*

This result allows us to describe the trace of $\mathcal{D}(f)$ for any $f \in M^{\sharp}_{2-2\lambda}(\Gamma_0(N))$. We phrase this description in terms of lifting operators $\mathcal{Z}_{\lambda,N,m}$ and $\mathcal{Z}'_{\lambda,N,n}$. For each $\lambda, N, m \geq 1$ such that $N$ is odd and $(-1)^{\lambda+1}m$ is a fundamental discriminant, let

$$\mathcal{Z}_{\lambda,N,m}(f) := \sum_{\nu>0} A_{-\nu}(f)\nu^{2\lambda-1}\sum_{d\mid\nu} d^{-\lambda}\left(\frac{(-1)^{\lambda+1}m}{d}\right)F_{\lambda,\frac{N}{(N,d)}}\left(\frac{-\nu^2 m}{d^2};z\right). \tag{1.9}$$

Unless $\lambda = m = 1$, the individual terms in the above summation are weakly holomorphic modular forms and thus $\mathcal{Z}_{\lambda,N,m}(f)$ is also weakly holomorphic, of weight $\lambda + \frac{1}{2}$, so we have defined a linear operator

$$\mathcal{Z}_{\lambda,N,m} : M^{\sharp}_{2-2\lambda}(\Gamma_0(N)) \to M^{!}_{\lambda+\frac{1}{2}}(\Gamma_0(4N)).$$

**Remark 1.6.** When $\lambda = 1$, note that the above definition of $\mathcal{Z}_{\lambda,N,m}(f)$ is not affected by adding a constant to $f$. If $m > 1$, this makes sense because the twisted traces of a constant function $\mathrm{Tr}_{N,m}(1;n)$ all vanish. However, the untwisted trace $\mathrm{Tr}_{N,1}(1;n)$ will be non-zero in general. This suggests that an additional term should be added to the above summation when defining $\mathcal{Z}_{1,N,1}$. This can be done, and $\mathcal{Z}_{1,N,1}(f)$ will be a weak Maass form of weight $\frac{3}{2}$.

Similarly, for each $\lambda, N, n \geq 1$ such that $N$ is odd and $(-1)^\lambda n$ is a fundamental discriminant, we define

$$\mathcal{Z}'_{\lambda,N,n} : M^\sharp_{2-2\lambda}(\Gamma_0(N)) \to M^!_{\frac{3}{2}-\lambda}(\Gamma_0(4N))$$

by

$$\mathcal{Z}'_{\lambda,N,n}(f) := \sum_{\nu>0} A_{-\nu}(f) \sum_{d|\nu} d^{\lambda-1} \left(\frac{(-1)^\lambda n}{d}\right) F_{1-\lambda,\frac{N}{(N,d)}}\left(\frac{-\nu^2 n}{d^2}; z\right). \tag{1.10}$$

The individual terms in the above summations are not necessarily holomorphic, but we will prove in Section 3.4 that this definition actually does give a weakly holomorphic modular form.

**Corollary 1.7.** *Suppose that $\lambda, N, m, n \geq 1$ are integers such that $N$ is odd and $(-1)^{\lambda+1}m, (-1)^\lambda n \equiv 0, 1 \pmod 4$. For any $f \in M^\sharp_{2-2\lambda}(\Gamma_0(N))$, we have*

$$\mathrm{Tr}_{N,(-1)^{\lambda+1}m}(\mathcal{D}(f); n) = (-1)^{\left[\frac{\lambda}{2}+1\right]} m^{\frac{\lambda}{2}} n^{\frac{1-\lambda}{2}} \cdot A_n(\mathcal{Z}_{\lambda,N,m}(f))$$

*if $(-1)^{\lambda+1}m$ is a fundamental discriminant and*

$$\mathrm{Tr}_{N,(-1)^\lambda n}(\mathcal{D}(f); m) = (-1)^{\left[\frac{\lambda}{2}\right]} m^{\frac{\lambda}{2}} n^{\frac{1-\lambda}{2}} \cdot A_m(\mathcal{Z}'_{\lambda,N,n}(f))$$

*if $(-1)^\lambda n$ is a fundamental discriminant.*

**Remark 1.8.** This corollary implies that the linear transformations $\mathcal{Z}_{\lambda,N,m}$ and $\mathcal{Z}'_{\lambda,N,n}$ commute with the actions of the Hecke algebra on the relevant spaces of modular forms of integral and half-integral weights. More explicitly, for any prime $p \nmid N$,

$$\mathcal{Z}_{\lambda,N,m}(f)|_k T_{4N}(p^2) = p^{2\lambda-1} \mathcal{Z}_{\lambda,N,m}(f|_{2-2\lambda} T_N(p))$$

and

$$\mathcal{Z}'_{\lambda,N,n}(f)|_{2-k} T_{4N}(p^2) = \mathcal{Z}'_{\lambda,N,n}(f|_{2-2\lambda} T_N(p))$$

where $T_{4N}(p^2)$ and $T_N(p)$ are the half-integral and integral weight Hecke operators, respectively.

Since $\mathrm{Tr}_{N,(-1)^{\lambda+1}m}(\mathcal{D}(f); n) = \mathrm{Tr}_{N,(-1)^\lambda n}(\mathcal{D}(f); m)$ if both $(-1)^{\lambda+1}m$ and $(-1)^\lambda n$ are fundamental discriminants, Corollary 1.7 immediately yields the following duality property.

**Corollary 1.9.** *Suppose that $\lambda, N \geq 1$ are integers with $N$ odd, and that $f \in M^\sharp_{2-2\lambda}(\Gamma_0(N))$. Then for any $m, n \geq 1$ such that $(-1)^{\lambda+1}m$ and $(-1)^\lambda n$ are fundamental discriminants, we have*

$$A_n(\mathcal{Z}_{\lambda,N,m}(f)) = -A_m(\mathcal{Z}'_{\lambda,N,n}(f)).$$

Our correspondences[a] $\mathcal{Z}_{\lambda,N,m}$ and $\mathcal{Z}'_{\lambda,N,n}$ answer the open problem posed by Zagier at the end of [11], where he conjectures the existence of lifting operators possessing the above properties.

In Section 2 we state the definitions and auxiliary results needed for Theorem 1.2, which is proved in Section 3.1. We then prove a general result in Section 3.2 concerning the algebraicity of the values of $\mathcal{D}(f)$ at Heegner points for those $f$ with rational Fourier coefficients. We connect these non-holomorphic derivatives to our Maass-Poincaré series in Section 3.3, where we prove Theorem 1.5. In Section 3.4, we prove that $\mathcal{Z}'_{\lambda,N,n}(f)$ is always a weakly holomorphic modular form. In Section 4, we apply Corollary 1.7 to two concrete examples.

## 2. Preliminaries and Fourier Expansions

We now recall basic definitions relating to the construction of half-integral weight Poincaré series such as $\mathcal{F}_{\lambda,N}(-m;z)$. For each $A = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \Gamma_0(4)$, let

$$j(A,z) := \left(\frac{\gamma}{\delta}\right)\epsilon_\delta^{-1}(\gamma z+\delta)^{\frac{1}{2}}.$$

be the usual factor of automorphy for half-integral weight modular forms (see [9]), where

$$\epsilon_\delta := \begin{cases} 1 & \text{if } \delta \equiv 1 \pmod 4\\ i & \text{if } \delta \equiv 3 \pmod 4. \end{cases}$$

Now suppose that $\lambda$ is an integer and $k = \lambda + \frac{1}{2}$. For a function $f : \mathbb{H} \to \mathbb{C}$ and $A \in \Gamma_0(4)$, we let

$$(f|_k A)(z) := j(A,z)^{-2\lambda-1} f(Az).$$

Recall that $F_{\lambda,N}(-m;z)$ is defined by applying the Kohnen projection operator $\mathrm{pr}_\lambda$ to the Poincaré series

$$\mathcal{F}_{\lambda,N}(-m,s;z) := \sum_{A\in\Gamma_\infty\backslash\Gamma_0(4N)} (\varphi_{-m,s}|_k A)(z).$$

The Fourier development of $F_{\lambda,N}(-m;z)$ involves certain Kloosterman sums, which are defined as follows. If $\lambda$ is an integer and $c$ is a positive integer multiple of 4, the weight $\lambda + \frac{1}{2}$ Kloosterman sum is given by

$$K_\lambda(m,n,c) := \sum_{v \pmod c^*} \left(\frac{c}{v}\right)\epsilon_v^{2\lambda+1} e\left(\frac{m\bar{v}+nv}{c}\right) \tag{2.1}$$

where $v$ runs over the primitive residue classes mod $c$ and $\bar{v}$ denotes the inverse of $v$ mod $c$.

[a]Duke and Jenkins [5] recently defined these correspondences $\mathcal{Z}, \mathcal{Z}'$ in the case $N = 1$ using similar methods.

The coefficients $b_{\lambda,N}(-m;n)$ in the Fourier developments (1.5) and (1.6) of $F_{\lambda,N}(-m;z)$ are given by the following theorem, which is a straightforward generalization of Theorem 2.1 of [3]. (Note that we have also included the coefficients of the nonholomorphic part of this Fourier development for $\lambda \le 0$.) We need the following definitions:

$$\delta_{\text{odd}}(m) := \begin{cases} 1 & \text{if } m \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \qquad \delta_{\square}(m) := \begin{cases} 1 & \text{if } m \text{ is a square} \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 2.1.** *Suppose that $\lambda$ is an integer, $m, n$ are positive integers such that $(-1)^{\lambda+1}m, (-1)^{\lambda}n \equiv 0, 1 \pmod 4$, and $N > 0$ is odd.*

*(1) If $\lambda \ge 2$, then $b^-_{\lambda,N}(-m;n) = 0$ for all $n \ge 0$, $b_{\lambda,N}(-m;0) = 0$, and for $n > 0$ we have*

$$b_{\lambda,N}(-m;n) = (-1)^{[(\lambda+1)/2]}\pi\sqrt{2}(n/m)^{\frac{\lambda}{2}-\frac{1}{4}}(1-(-1)^{\lambda}i) \times \sum_{\substack{c>0 \\ c\equiv 0 \pmod{4N}}} (1+\delta_{\text{odd}}(c/4))\frac{K_{\lambda}(-m,n,c)}{c} \cdot I_{\lambda-\frac{1}{2}}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

*(2) If $\lambda = 1$, then $b_{\lambda,N}(-m;0) = 0$ and for $n > 0$ we have*

$$b_{\lambda,N}(-m;n) = -\pi\sqrt{2}(n/m)^{\frac{1}{4}}(1+i) \times \sum_{\substack{c>0 \\ c\equiv 0 \pmod{4N}}} (1+\delta_{\text{odd}}(c/4))\frac{K_{1}(-m,n,c)}{c} \cdot I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

*(3) If $\lambda \le 0$, then $b^-_{\lambda,N}(-m;0) = 0$,*

$$b_{\lambda,N}(-m;0) = (-1)^{[(\lambda+1)/2]}\pi^{\frac{3}{2}-\lambda}2^{1-\lambda}m^{\frac{1}{2}-\lambda}(1-(-1)^{\lambda}i) \times \frac{1}{(\frac{1}{2}-\lambda)\Gamma(\frac{1}{2}-\lambda)} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{4N}}} (1+\delta_{\text{odd}}(c/4))\frac{K_{\lambda}(-m,0,c)}{c^{\frac{3}{2}-\lambda}},$$

*and for $n > 0$ we have*

$$b_{\lambda,N}(-m;n) = (-1)^{[(\lambda+1)/2]}\pi\sqrt{2}(n/m)^{\frac{\lambda}{2}-\frac{1}{4}}(1-(-1)^{\lambda}i) \times \sum_{\substack{c>0 \\ c\equiv 0 \pmod{4N}}} (1+\delta_{\text{odd}}(c/4))\frac{K_{\lambda}(-m,n,c)}{c} \cdot I_{\frac{1}{2}-\lambda}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

*and*

$$b^-_{\lambda,N}(-m;n) = -\delta_{n,m} + (-1)^{[(\lambda+1)/2]}\pi\sqrt{2}(n/m)^{\frac{\lambda}{2}-\frac{1}{4}}(1-(-1)^{\lambda}i) \times \sum_{\substack{c>0 \\ c\equiv 0 \pmod{4N}}} (1+\delta_{\text{odd}}(c/4))\frac{K_{\lambda}(-m,-n,c)}{c} \cdot J_{\frac{1}{2}-\lambda}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

**Remark 2.2.** Note that Theorem 2.1 says nothing about the non-holomorphic Fourier coefficients $b^-_{\lambda,N}(-m;n)$ when $\lambda = 1$; we will not need these coefficients for our results.

**Proof.** (Sketch) The proof follows *mutatis mutandis* as in the proof of Theorem 2.1 of [3]. We first obtain the Fourier coefficients of the Poincaré series $\mathcal{F}_{\lambda,N}(-m;z)$. This is a standard application of Poisson summation (similar to Section 3.2 of [6]), after which we can evaluate the resulting integral as a Bessel function and interpret part of the sum as a half-integral weight Kloosterman sum. This yields a sum of products of Kloosterman sums and Bessel functions. Applying Kohnen's projection operator (see Proposition 3 of [7]) then gives the desired formula for the Fourier coefficients of $F_{\lambda,N}(-m;z)$. $\square$

We now define Salié sums and state identities relating them both to Kloosterman sums and to sums over quadratic forms. We will use the consequent result relating Kloosterman sums to sums over quadratic forms in our proof of Theorem 1.2.

If $D_1 \equiv 0, 1 \pmod 4$ is a nonzero integer, $D_2$ is an integer with $(-1)^\lambda D_2 \equiv 0, 1 \pmod 4$, and $\lambda$ and $\nu$ are integers, we define the *Salié sum* $S_\lambda(D_1, D_2, 4a, \nu)$ as

$$S_\lambda(D_1, D_2, 4a, \nu) := \sum_{\substack{x \pmod{4a} \\ x^2 \equiv (-1)^\lambda D_1 D_2 \pmod{4a}}} \chi_{D_1}\left(a, x, \frac{x^2 - (-1)^\lambda D_1 D_2}{4a}\right) e\left(\frac{x\nu}{2a}\right), \tag{2.2}$$

where the genus character $\chi_{D_1}(a, b, c)$ of a binary quadratic form $Q = [a, b, c]$ with discriminant divisible by $D_1$ is given by

$$\chi_{D_1}(a, b, c) := \begin{cases} 0 & \text{if } (a, b, c, D_1) > 1, \\ \left(\frac{D_1}{r}\right) & \text{if } (a, b, c, D_1) = 1 \text{ and } Q \text{ represents } r \text{ with } (r, D_1) = 1. \end{cases}$$

The following proposition of Kohnen (see Proposition 5 of [7]) allows us to express these Salié sums as linear combinations of Kloosterman sums.

**Proposition 2.3.** *Suppose that $a$ is a positive integer. If $\lambda, \nu$ are integers, $D_1$ is a fundamental discriminant, and $D_2$ is a nonzero integer such that $(-1)^\lambda D_2 \equiv 0, 1 \pmod 4$, then*

$$S_\lambda(D_1, D_2, 4a, \nu) =$$
$$(1 - (-1)^\lambda i) \sum_{d | (a,\nu)} \left(\frac{D_1}{d}\right) (4a/d)^{-1/2} (1 + \delta_{odd}(a/d)) K_\lambda \left((-1)^\lambda \frac{\nu^2}{d^2} D_1, D_2, 4a/d\right).$$

The following result, which is a simple generalization of Proposition 3.3 of [3], interprets Salié sums as sums over Heegner points.

**Proposition 2.4.** *Suppose that $\lambda, D_1, D_2, \nu$, and $a$ satisfy the conditions of Proposition 2.3. If $(-1)^\lambda D_1 D_2 < 0$, then for any odd positive integer $N$ dividing $a$,*

$$S_\lambda(D_1, D_2, 4a, \nu) = 2 \sum_{Q \in \mathcal{Q}_{|D_1D_2|}/\Gamma_0(N)} \frac{\chi_{D_1}(Q)}{\omega_Q} \sum_{\substack{A \in \Gamma_\infty \backslash \Gamma_0(N) \\ \operatorname{Im}(A\tau_Q) = \frac{\sqrt{|D_1D_2|}}{2a}}} e(-\nu \operatorname{Re}(A\tau_Q)).$$

**Proof.** We combine the two summations on the right hand side, using the fact that $\omega_Q$ is the cardinality of the stabilizer of $Q$ in $\Gamma_0(N)/\{\pm 1\}$ when $N \mid a_Q$, to get

$$2 \sum_{\substack{Q \in \mathcal{Q}_{|D_1D_2|}/\Gamma_\infty \\ \operatorname{Im}(\tau_Q) = \frac{\sqrt{|D_1D_2|}}{2a}}} \chi_{D_1}(Q) e(-\nu \operatorname{Re}(\tau_Q)). \tag{2.3}$$

If $Q = [a_Q, b_Q, c_Q]$ is a positive definite quadratic form of discriminant $-|D_1D_2|$, then the corresponding Heegner point is

$$\tau_Q = \frac{-b_Q + i\sqrt{|D_1D_2|}}{2a_Q}.$$

The above expression (2.3) can thus be rewritten as

$$2 \sum_{\substack{Q \in \mathcal{Q}_{|D_1D_2|}/\Gamma_\infty \\ Q = [a, b_Q, c_Q]}} \chi_{D_1}(Q) e\left(\frac{b_Q \nu}{2a}\right).$$

The action of $\Gamma_\infty$ identifies values of $b_Q$ that are congruent mod $2a$, so we can rewrite this as a sum over the possible values of $x = b_Q \pmod{2a}$. These values are precisely the solutions to the congruence $x^2 \equiv -|D_1D_2| = (-1)^\lambda D_1D_2 \pmod{4a}$. This gives the desired Salié sum. $\square$

## 3. Proofs of Our Results

### 3.1. *Proof of Theorem 1.2*

Our proof is a straightforward generalization of the proof of Theorem 1.2 of [3].

**Proof.** (Theorem 1.2) For simplicity, we only consider the case $\lambda \geq 2$; the argument for $\lambda = 1$ is identical.

Using Theorem 2.1 and setting $c = 4a/d$, we obtain

$$\begin{aligned} &\frac{1}{2}(-1)^{[\frac{\lambda+1}{2}]} \nu^{2\lambda-1} m^{\frac{\lambda}{2}} n^{\frac{1-\lambda}{2}} \cdot \sum_{d|\nu} d^{-\lambda} \left(\frac{(-1)^{\lambda+1} m}{d}\right) b_{\lambda, \frac{N}{(N,d)}} \left(\frac{-\nu^2 m}{d^2}; n\right) \\ &= \frac{\pi}{\sqrt{2}} (mn)^{\frac{1}{4}} \nu^{\lambda - \frac{1}{2}} (1 - (-1)^\lambda i) \sum_{d|\nu} \left(\frac{(-1)^{\lambda+1} m}{d}\right) d^{\frac{1}{2}} \\ &\quad \times \sum_{\substack{a>0 \\ a \equiv 0 \pmod{N} \\ a \equiv 0 \pmod{d}}} (1 + \delta_{\text{odd}}(a/d)) \frac{K_\lambda\left(-\frac{\nu^2 m}{d^2}, n, \frac{4a}{d}\right)}{4a} I_{\lambda - \frac{1}{2}} \left(\frac{\pi \nu \sqrt{mn}}{a}\right). \end{aligned}$$

We interchange summation and apply Proposition 2.3 to eliminate the sum over $d$, yielding

$$(mn)^{\frac{1}{4}}\nu^{\lambda-\frac{1}{2}}\frac{\pi}{2\sqrt{2}}\sum_{\substack{a>0\\ a\equiv 0 \pmod N}} S((-1)^{\lambda+1}m,n,4a,\nu)\frac{I_{\lambda-\frac{1}{2}}\left(\frac{\pi\nu\sqrt{mn}}{a}\right)}{\sqrt{a}}$$

We now use Proposition 2.4 to rewrite the Salié sum as a sum over equivalence classes of quadratic forms, as follows:

$$(mn)^{\frac{1}{4}}\nu^{\lambda-\frac{1}{2}}\frac{\pi}{\sqrt{2}}\sum_{\substack{a>0\\ a\equiv 0 \pmod N}}\sum_{Q\in\mathcal{Q}_{mn}/\Gamma_0(N)}\frac{\chi_{(-1)^{\lambda+1}m}(Q)}{\omega_Q}$$
$$\times\sum_{\substack{A\in\Gamma_\infty\backslash\Gamma_0(N)\\ \mathrm{Im}(A\tau_Q)=\frac{\sqrt{mn}}{2a}}} e(-\nu\,\mathrm{Re}(A\tau_Q))\frac{I_{\lambda-\frac{1}{2}}\left(\frac{\pi\nu\sqrt{mn}}{a}\right)}{\sqrt{a}}$$

Note that the inner sum vanishes unless $a_Q\equiv 0 \pmod N$. We can replace the triple sum with a sum over equivalence classes of quadratic forms in $\mathcal{Q}_{mn}/\Gamma_\infty$ subject to the restriction that $N\mid a_Q$. This single sum can be expanded as follows:

$$\nu^{\lambda-\frac{1}{2}}\pi\sum_{\substack{Q\in\mathcal{Q}_{mn}/\Gamma_0(N)\\ N\mid a_Q}}\frac{\chi_{(-1)^{\lambda+1}m}(Q)}{\omega_Q}$$
$$\times\sum_{A\in\Gamma_\infty\backslash\Gamma_0(N)} e(-\nu\,\mathrm{Re}(A\tau_Q))I_{\lambda-\frac{1}{2}}(2\pi\nu\,\mathrm{Im}(A\tau_Q))\,\mathrm{Im}(A\tau_Q)^{\frac{1}{2}}.$$

We recognize the inner sum as the Poincaré series for $\mathfrak{F}_{\lambda,N,\nu}$, and substitute to obtain

$$\sum_{\substack{Q\in\mathcal{Q}_{mn}/\Gamma_0(N)\\ N\mid a_Q}}\frac{\chi_{(-1)^{\lambda+1}m}(Q)}{\omega_Q}\mathfrak{F}_{\lambda,N,\nu}(\tau_Q)=\mathrm{Tr}_{N,(-1)^{\lambda+1}m}(\mathfrak{F}_{\lambda,N,\nu};n).$$

We approach the second part of the theorem similarly. First, by using Theorem 2.1 and letting $c=4a/d$, we obtain that

$$\frac{1}{2}(-1)^{[\frac{\lambda-1}{2}]}m^{\frac{\lambda}{2}}n^{\frac{1-\lambda}{2}}\cdot\sum_{d\mid\nu}d^{\lambda-1}\left(\frac{(-1)^\lambda n}{d}\right)b_{1-\lambda,\frac{N}{(N,d)}}\left(\frac{-\nu^2 n}{d^2};m\right)$$
$$=\frac{\pi}{\sqrt{2}}(mn)^{\frac{1}{4}}\nu^{\lambda-\frac{1}{2}}(1+(-1)^\lambda i)\sum_{d\mid\nu}\left(\frac{(-1)^\lambda n}{d}\right)d^{\frac{1}{2}}$$
$$\times\sum_{\substack{a>0\\ a\equiv 0 \pmod N\\ a\equiv 0 \pmod d}}(1+\delta_{\mathrm{odd}}(a/d))\frac{K_{1-\lambda}\left(-\frac{\nu^2 n}{d^2},m,\frac{4a}{d}\right)}{4a}I_{\lambda-\frac{1}{2}}\left(\frac{\pi\nu\sqrt{mn}}{a}\right).$$

Interchanging summation and applying Proposition 2.3 yields

$$(mn)^{\frac{1}{4}}\nu^{\lambda-\frac{1}{2}}\frac{\pi}{2\sqrt{2}}\sum_{\substack{a>0\\ a\equiv 0 \pmod N}} S((-1)^\lambda n, m, 4a, \nu)\frac{I_{\lambda-\frac{1}{2}}\left(\frac{\pi\nu\sqrt{mn}}{a}\right)}{\sqrt{a}},$$

and the remainder of the proof is exactly as in the first part of the theorem. □

### 3.2. *Algebraicity of non-holomorphic derivatives at Heegner points*

We recall the weight $h$ non-holomorphic derivative

$$\partial_h = \frac{1}{2\pi i}\frac{\partial}{\partial z} - \frac{h}{4\pi \operatorname{Im} z}.$$

We can obtain a modular invariant from any modular form of non-positive weight $2-2\lambda$ by iterating this non-holomorphic derivative; thus let $\mathcal{D}(f) := \partial_{-2}\partial_{-4}\cdots\partial_{2-2\lambda}f$.

This proposition, well-known to experts, establishes the rationality of the traces of $\mathcal{D}(f)$.

**Proposition 3.1.** *If $\lambda$ is a positive integer and $f$ is a meromorphic modular form of weight $2-2\lambda$ on $\Gamma_0(N)$ with rational Fourier coefficients, then the modular invariant $\mathcal{D}(f)$ has algebraic values at Heegner points of level $N$. Moreover, for a fixed discriminant $-D<0$, the multiset of values at $\Gamma_0(N)$-equivalence classes of Heegner points is a union of Galois orbits in the ring class field of the quadratic order of discriminant $-D$.*

**Proof.** Observe that $\partial_k = \tilde{\partial}_k + \frac{k}{12}E_2^*$, where $\tilde{\partial}_k := \frac{1}{2\pi i}\frac{\partial}{\partial z} - \frac{k}{12}E_2$ is the holomorphic differential defined in [10] and

$$E_2^*(z) := 1 - \frac{3}{\pi \operatorname{Im} z} - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n$$

is the non-holomorphic Eisenstein series of weight 2. Thus

$$\mathcal{D}(f) = g_0 + g_1E_2^* + \cdots + g_{\lambda-1}(E_2^*)^{\lambda-1}, \tag{3.1}$$

where each $g_i$ is a meromorphic modular form of weight $-2i$ on $\Gamma_0(N)$ with rational Fourier coefficients. Each $g_i$ can then be written in the form $g_i(z) = \left(\frac{E_4(z)}{E_6(z)}\right)^i S_i(j(z), j(Nz))$ for some rational function $S_i \in \mathbb{Q}(X,Y)$. We now describe the values of $E_2^*$.

Fix a discriminant $-D<0$ and let

$$D_0 = \begin{cases} \frac{D}{4} & \text{if } D \equiv 0 \pmod 4 \\ D & \text{otherwise.}\end{cases}$$

Consider the function $r_{D_0}(z) := E_2^*(z) - D_0E_2^*(D_0z)$. This is a modular form of weight 2 on $\Gamma_0(D_0)$, so it can be written as $\frac{2E_6(z)}{E_4(z)}R_{D_0}(j(z), j(D_0z))$, where

$R_{D_0}(X,Y) \in \mathbb{Q}(X,Y)$. Suppose $Q = [a,b,c]$ is a primitive quadratic form of discriminant $-D$ with root $\tau = \tau_Q$. Let $M := \left(\begin{smallmatrix} -b/2 & c \\ a & b/2 \end{smallmatrix}\right)$ if $b$ is even, and $\left(\begin{smallmatrix} -b & 2c \\ 2a & b \end{smallmatrix}\right)$ if $b$ is odd, so $\det M = D_0$ and $M\tau = \tau$. Because $M$ is primitive, we can find $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$ such that $M = \gamma_1^{-1} \left(\begin{smallmatrix} 1 & 0 \\ 0 & D_0 \end{smallmatrix}\right) \gamma_2$. Applying modular transformation properties with factor of automorphy $j_\gamma(z) := (\det\gamma)^{-\frac{1}{2}}(gz+h)$ for $\gamma = \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)$, we find that

$$\begin{aligned} E_2^*(\tau) &= j_{\gamma_1}(\tau)^{-2} E_2^*(\gamma_1\tau) \\ &= j_{\gamma_1}(\tau)^{-2}(r_{D_0}(\gamma_1\tau) + D_0 E_2^*(\gamma_2\tau)) \\ &= j_{\gamma_1}(\tau)^{-2}(r_{D_0}(\gamma_1\tau) + D_0 j_{\gamma_2}(\tau)^2 E_2^*(\tau)), \end{aligned}$$

and so

$$E_2^*(\tau) = \frac{j_{\gamma_1}(\tau)^{-2} r_{D_0}(\gamma_1(\tau))}{1 - j_{\gamma_1}(\tau)^{-2} j_{\gamma_2}(\tau)^2 D_0}. \tag{3.2}$$

We now note that

$$\begin{aligned} D_0 j_{\gamma_2}(\tau)^2 j_{\gamma_1}(\tau)^{-2} &= D_0 j_{\gamma_2}(\tau)^2 j_{\gamma_1^{-1}}\left(\begin{pmatrix} 1 & \\ & D_0 \end{pmatrix} \gamma_2\tau\right)^2 \\ &= D_0 j_{\gamma_2}(\tau)^2 j_{\left(\begin{smallmatrix} 1 & \\ & D_0 \end{smallmatrix}\right)}(\gamma_2\tau)^{-2} j_{\gamma_1^{-1}\left(\begin{smallmatrix} 1 & \\ & D_0 \end{smallmatrix}\right)}(\gamma_2\tau)^2 \\ &= j_{\gamma_1^{-1}\left(\begin{smallmatrix} 1 & \\ & D_0 \end{smallmatrix}\right)\gamma_2}(\tau)^2 \\ &= \frac{1}{D}(2a\tau + b)^2 = \frac{1}{D}(4a(a\tau^2 + b\tau + c) - 4ac + b^2) = -1. \end{aligned}$$

We substitute this into the denominator of (3.2) and obtain

$$\begin{aligned} E_2^*(\tau) &= \frac{1}{2} j_{\gamma_1}(\tau)^{-2} r_{D_0}(\gamma_1(\tau)) \\ &= \frac{1}{2} j_{\gamma_1}(\tau)^{-2} \frac{2E_6(\gamma_1\tau)}{E_4(\gamma_1\tau)} R_{D_0}(j(\gamma_1\tau), j(D_0\gamma_1\tau)) \\ &= \frac{E_6(\tau)}{E_4(\tau)} R_{D_0}(j(\tau), j(\tau)). \end{aligned}$$

Substituting into (3.1), we find that

$$\mathcal{D}(f)(\tau) = \sum_{i=0}^{\lambda-1} S_i(j(\tau), j(N\tau)) R_{D_0}(j(\tau), j(\tau))^i.$$

Note that a Heegner point $\tau$ of discriminant $-D$ is of level $N$ if and only if $N\tau$ is also a Heegner point of discriminant $-D$. From the theory of complex multiplication (see Theorem 11.1 of [4]), we know that the values of $j$ at the Heegner points of discriminant $-D$ belong to the ring class field of the quadratic order of discriminant $-D$, and thus $\mathcal{D}(f)(\tau)$ does as well. Let $\Phi_N(X,Y) \in \mathbb{Q}[X,Y]$ be the level $N$ modular equation, so $\Phi_N(x,y) = 0$ if and only if there exists $\alpha$ such that $x = j(\alpha)$, $y = j(N\alpha)$. If $\sigma$ is an element of the absolute Galois group of $\mathbb{Q}$, then $\Phi_N(\sigma(j(\tau)),\sigma(j(N\tau))) = \sigma(\Phi_N(j(\tau),j(N\tau))) = 0$, so there exists $\tau'$ such that $j(\tau') = \sigma(j(\tau))$ and $j(N\tau') = \sigma(j(N\tau))$. Because $j(z) = j(z')$ implies that $z$ and $z'$ are identified by the action of $\mathrm{SL}_2(\mathbb{Z})$ and the values of $j$ at Heegner points of discriminant $-D$ form a complete set of Galois conjugates (see [4]), $\tau'$ and $N\tau'$ are also Heegner points of discriminant $-D$. Thus $\tau'$ is actually a Heegner point of level $N$ and discriminant $-D$. Since $\sigma(\mathcal{D}(f)(\tau)) = \mathcal{D}(f)(\tau')$, the set of values of $\mathcal{D}(f)$ at Heegner points of level $N$ and discriminant $-D$ is a union of Galois orbits, as desired. $\square$

### 3.3. *Explicit Fourier developments and proof of Theorem 1.5*

We will prove Theorem 1.5 by comparing the Fourier developments of both sides. We first recall the definition of the integral weight Kloosterman sums that occur in these developments, which are given by

$$\mathcal{S}(n,m;c) := \sum_{v \pmod{c}^*} e\left(\frac{m\bar{v}+nv}{c}\right). \tag{3.3}$$

Analogously to Equation 4.2 of [3], it turns out that the Fourier development of $\mathfrak{F}_{\lambda,N,\nu}(z)$ is given by

$$\mathfrak{F}_{\lambda,N,\nu}(z) = \pi\nu^{\lambda-1}\,\mathrm{Im}(\nu z)^{\frac{1}{2}} I_{\lambda-\frac{1}{2}}(2\pi\,\mathrm{Im}(\nu z))e(-\mathrm{Re}(\nu z)) + 2\sum_{n\in\mathbb{Z}} b_{\lambda,N,\nu}(n,y)q^n,$$

where we define $b_{\lambda,N,\nu}(n,y)$ to equal

$$\begin{cases} \frac{\nu^{2\lambda-1}y^{1-\lambda}\pi^{\lambda+1}}{(2\lambda-1)(\lambda-1)!} \displaystyle\sum_{\substack{c>0 \\ c\equiv 0 \pmod{N}}} \frac{\mathcal{S}(0,-\nu;c)}{c^{2\lambda}} & n=0, \\ \pi\nu^{\lambda-\frac{1}{2}}y^{\frac{1}{2}}e^{2\pi ny}K_{\lambda-\frac{1}{2}}(2\pi ny) \displaystyle\sum_{\substack{c>0 \\ c\equiv 0 \pmod{N}}} \frac{\mathcal{S}(n,-\nu;c)}{c} I_{2\lambda-1}\left(\frac{4\pi\sqrt{n\nu}}{c}\right) & n>0, \\ \pi\nu^{\lambda-\frac{1}{2}}y^{\frac{1}{2}}e^{2\pi ny}K_{\lambda-\frac{1}{2}}(2\pi|n|y) \displaystyle\sum_{\substack{c>0 \\ c\equiv 0 \pmod{N}}} \frac{\mathcal{S}(n,-\nu;c)}{c} J_{2\lambda-1}\left(\frac{4\pi\sqrt{|n|\nu}}{c}\right) & n<0. \end{cases} \tag{3.4}$$

Here $J_s(x)$ and $K_s(x)$ are the $J$- and $K$- Bessel functions, respectively.

The following lemma gives that certain linear combinations of the coefficients $b_{\lambda,N,\nu}(n,y)$ vanish for negative $n$.

**Lemma 3.2.** *If $\lambda \geq 1$ and $f \in M^{\sharp}_{2-2\lambda}(\Gamma_0(N))$, then for any positive integer $n$ we have*

$$\sum_{\nu>0} A_{-\nu}(f)\nu^{\lambda-\frac{1}{2}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod{N}}} \frac{\mathcal{S}(n,\nu;c)}{c} J_{2\lambda-1}\left(\frac{4\pi\sqrt{n\nu}}{c}\right) = \frac{(-1)^{\lambda+1}}{2\pi} n^{\lambda-\frac{1}{2}} A_{-n}(f).$$

**Proof.** Define $\hat{f}$ by $\hat{f}(z) = \overline{f(-\bar{z})}$. It is clear that $f \in M^{\sharp}_{2-2\lambda}(\Gamma_0(N))$ implies that $\hat{f} \in M^{\sharp}_{2-2\lambda}(\Gamma_0(N))$. For any $g \in S_{2\lambda}(\Gamma_0(N))$, $\hat{f}g$ is a modular form of weight 2 which is holomorphic away from the cusp at infinity and cuspidal at all other cusps. We interpret this form as a meromorphic differential form on the compactification of the quotient surface $\Gamma_0(N)\backslash\mathbb{H}$, which is then holomorphic everywhere except at infinity. Thus the residue of this form at infinity is 0, so the constant term vanishes. Hence $\sum_{\nu>0}\overline{A_{-\nu}(f)}A_\nu(g) = A_0(\hat{f}g) = 0$. Let $P_{\nu,N}$ be the $\nu$th holomorphic Poincaré series of weight $2\lambda$ on $\Gamma_0(N)$, as defined in [6]. Using the well-known formula for $A_\nu(g)$ in terms of the Petersson inner product $\langle g, P_{\nu,N}\rangle$,

$$\sum_{\nu>0}\overline{A_{-\nu}(f)}\frac{(4\pi\nu)^{2\lambda-1}}{\Gamma(2\lambda-1)}\langle g, P_{\nu,N}\rangle = 0.$$

Because the Petersson inner product is non-degenerate, this implies

$$\sum_{\nu>0} A_{-\nu}(f)\nu^{2\lambda-1}P_{\nu,N} = 0.$$

Using the Fourier development for $P_{\nu,N}$ given in [6], we take the $n$th Fourier coefficient, and obtain that

$$\sum_{\nu>0} A_{-\nu}(f)\nu^{2\lambda-1}\left(\frac{n}{\nu}\right)^{\lambda-\frac{1}{2}}\left(\delta_{\nu n} + (-1)^\lambda 2\pi \sum_{\substack{c>0\\ c\equiv 0 \pmod N}} \frac{\mathcal{S}(n,\nu;c)}{c} J_{2\lambda-1}\left(\frac{4\pi\sqrt{n\nu}}{c}\right)\right)$$

vanishes for any $n$. Simplifying and rearranging gives the desired result. □

We now define a related family $g_{\lambda,N,\nu}$ of $q$-series for $\lambda \geq 1$ by

$$\begin{aligned} g_{\lambda,N,\nu}(z) := q^{-\nu} &+ \frac{4(-1)^{\lambda+1}\pi^{2\lambda+\frac{1}{2}}\nu^{2\lambda-1}}{\Gamma(\lambda-\frac{1}{2})(2\lambda-1)(\lambda-1)!}\sum_{\substack{c>0\\ c\equiv 0 \pmod N}}\frac{\mathcal{S}(0,-\nu;c)}{c^{2\lambda}} \\ &+ 2\pi(-1)^{\lambda+1}\nu^{\lambda-\frac{1}{2}}\sum_{n\geq 1} n^{-\lambda+\frac{1}{2}}\sum_{\substack{c>0\\ c\equiv 0 \pmod N}}\frac{\mathcal{S}(n,-\nu;c)}{c} I_{2\lambda-1}\left(\frac{4\pi\sqrt{n\nu}}{c}\right)q^n. \end{aligned} \tag{3.5}$$

For $\lambda \geq 2$, any modular form $f \in M^\sharp_{2-2\lambda}(\Gamma_0(N))$ can be written in terms of the $g_{\lambda,N,\nu}$ by

$$f = \sum_{\nu>0} A_{-\nu}(f) g_{\lambda,N,\nu}.$$

(See for example page 314 of [8]). For $\lambda = 1$, the above formula holds up to a constant.

This allows us to obtain explicit formulas for the Fourier development of $f$ in terms of its principal part.

**Proof.** (Theorem 1.5) By a standard calculation involving Bessel functions (see Chapter 4 of [1]),

$$\partial_{-2}\partial_{-4}\cdots\partial_{2-2\lambda}(q^n) = 2n^{\lambda-\frac{1}{2}}y^{\frac{1}{2}}e^{2\pi ny}K_{\lambda-\frac{1}{2}}(2\pi ny)q^n$$

for all $n \in \mathbb{Z}$. We now apply this formula to differentiate the Fourier development of $\frac{(-1)^{\lambda+1}}{2} f = \frac{(-1)^{\lambda+1}}{2} \sum_{\nu>0} A_{-\nu}(f) g_{\lambda,N,\nu}$ given by (3.5) term-by-term. Using Lemma 3.2, we observe that this Fourier development is the same as that of $\sum_{\nu>0} A_{-\nu}(f)\mathfrak{F}_{\lambda,N,\nu}$ given by (3.4). □

### 3.4. *Proof that $\mathcal{Z}'_{\lambda,N,n}(f)$ is holomorphic*

We adopt the definition and notation for the Kohnen plus-space Poincaré series $P_{\lambda,N,m} \in S^+_{\lambda+\frac{1}{2}}(\Gamma_0(4N))$ defined by Kohnen (see page 250 of [7]). This form is characterized by its Petersson inner product

$$\langle g, P_{\lambda,N,m}\rangle = i_{4N}^{-1} \frac{\Gamma(\lambda - 1/2)}{(4\pi m)^{\lambda-\frac{1}{2}}} A_m(g) \tag{3.6}$$

for $g \in S^+_{\lambda+\frac{1}{2}}(\Gamma_0(4N))$.

**Lemma 3.3.** *Suppose $\lambda, N, m, n$ are positive integers such that $N$ is odd and $(-1)^{\lambda+1}m, (-1)^\lambda n \equiv 0, 1 \pmod 4$. Then*

$$b_{1-\lambda,N}(-m; n) = -\frac{3}{2}\left(\frac{m}{n}\right)^{\lambda-\frac{1}{2}} A_n(P_{\lambda,N,m}).$$

**Proof.** This follows immediately from comparing the formula for $b_{1-\lambda,N}(-m; n)$ given in Theorem 2.1 with Kohnen's computation of the Fourier coefficients of $P_{\lambda,N,m}$ (see Proposition 4 of [7]). □

**Proposition 3.4.** *Suppose $\lambda, N, n, e$ are positive integers such that $N$ is odd, $(-1)^\lambda n$ is a fundamental discriminant, and $e \mid N$. Also suppose that $f \in M^\sharp_{2-2\lambda}$. Then*

$$\sum_{\nu>0} A_{-e\nu}(f) \sum_{\substack{d|\nu \\ (d,\frac{N}{e})=1}} d^{\lambda-1}\left(\frac{(-1)^\lambda n}{d}\right) F_{1-\lambda,\frac{N}{e}}\left(\frac{-\nu^2 n}{d^2}; z\right)$$

*is a weakly holomorphic modular form of weight $\frac{3}{2} - \lambda$ on $\Gamma_0(\frac{4N}{e})$.*

**Proof.** It suffices to show that the non-holomorphic terms of the Fourier development of this weak Maass form vanish. By the previous lemma, this is equivalent to showing that

$$\sum_{\nu>0} A_{-e\nu}(f) \sum_{\substack{d|\nu \\ (d,\frac{N}{e})=1}} d^{\lambda-1}\left(\frac{(-1)^\lambda n}{d}\right)\left(\frac{\nu^2 n}{d^2 m}\right)^{\lambda-\frac{1}{2}} A_m(P_{\lambda,\frac{N}{e},\frac{\nu^2 n}{d^2}}) = 0 \tag{3.7}$$

for any $m > 0$ such that $(-1)^{\lambda+1}m \equiv 0, 1 \pmod 4$.

Now, suppose $g \in S^+_{\lambda+\frac{1}{2}}(\Gamma_0(\frac{4N}{e}))$. Let $h = g|\mathcal{S}_{\lambda,\frac{N}{e},(-1)^\lambda n}$ be the Shimura lift (as defined in [7], for example) of $g$ corresponding to the fundamental discriminant

$(-1)^\lambda n$, so $h \in S_{2\lambda}(\Gamma_0(\frac{N}{e}))$ and the Fourier coefficients of $g$ and $h$ are related by

$$A_t(h) = \sum_{\substack{d|t \\ (d,\frac{N}{e})=1}} d^{\lambda-1}\left(\frac{(-1)^\lambda n}{d}\right) A_{\frac{t^2 n}{d^2}}(g). \tag{3.8}$$

As in the proof of Lemma 3.2, we let $\hat{f}(z) := \overline{f(-\bar{z})}$. Then $\hat{f}(z)h(ez)$ is a meromorphic modular form of weight 2 on $\Gamma_0(N)$ with a single pole at the cusp at infinity. Thus by the residue theorem for compact Riemann surfaces, we have that $A_0(\hat{f}(z)h(ez)) = 0$. Expanding and using equation (3.8), we have that

$$0 = \sum_{\nu>0} \overline{A_{-e\nu}(f)} A_\nu(h) = \sum_{\nu>0} \overline{A_{-e\nu}(f)} \sum_{\substack{d|\nu \\ (d,\frac{N}{e})=1}} d^{\lambda-1}\left(\frac{(-1)^\lambda n}{d}\right) A_{\frac{\nu^2 n}{d^2}}(g).$$

We use (3.6) to rewrite this as a Petersson inner product, and conclude that

$$\langle g, \sum_{\nu>0} A_{-e\nu}(f) \sum_{\substack{d|\nu \\ (d,\frac{N}{e})=1}} d^{\lambda-1}\left(\frac{(-1)^\lambda n}{d}\right)\left(\frac{\nu^2 n}{d^2}\right)^{\lambda-1} P_{\lambda,\frac{N}{e},\frac{\nu^2 n}{d^2}}\rangle = 0$$

for any $g \in S^+_{\lambda+\frac{1}{2}}(\Gamma_0(\frac{4N}{e}))$, which implies the desired equation (3.7) because the Petersson inner product is nondegenerate. $\square$

This proposition implies that formula (1.10) for $\mathcal{Z}'_{\lambda,N,n}(f)$ yields a weakly holomorphic modular form on $\Gamma_0(4N)$ because

$$\mathcal{Z}'_{\lambda,N,n}(f) = \sum_{e|N} e^{\lambda-1}\left(\frac{(-1)^\lambda n}{e}\right)$$
$$\times \sum_{\nu>0} A_{-e\nu}(f) \sum_{\substack{d|\nu \\ (d,\frac{N}{e})=1}} d^{\lambda-1}\left(\frac{(-1)^\lambda n}{d}\right) F_{1-\lambda,\frac{N}{e}}\left(\frac{-\nu^2 n}{d^2}; z\right).$$

## 4. Examples

Here we give two examples applying the main results of this paper.

**Example 4.1.** Let $\lambda = 9$, $N = 1$, $m = 1$. Then the form

$$f = \frac{E_4^2}{\Delta^2} = q^{-2} + 528q^{-1} + 86184 + \ldots$$

has the appropriate weight $-16$. Calculation yields

$$\mathrm{Tr}_{1,1}(\mathcal{D}(f); 3) = 507008 \quad \mathrm{Tr}_{1,1}(\mathcal{D}(f); 4) = 4172040.$$

Then Corollary 1.7 implies that

$$\mathcal{Z}_{9,1,1}(f) = 2^{17}q^{-4} + (2^8+528)q^{-1} - \sum_{\substack{n\geq 3 \\ n\equiv 0,3 \pmod 4}} n^4 \operatorname{Tr}_{1,1}(\mathcal{D}(f);n)q^n$$
$$= 131072q^{-4} + 784q^{-1} - 41067648q^3 - 1068042240q^4 - \dots$$

is a modular form of weight $\frac{19}{2}$ on $\Gamma_0(4)$.

**Example 4.2.** Let $\lambda = 1$, $N = 25$, $n = 3$, and let $f$ be the Hauptmodul

$$\frac{\eta(z)}{\eta(25z)} + 1 = q^{-1} - q + q^4 + q^6 - q^{11} - \dots.$$

Calculation gives that the first few traces of $f$ (twisted by discriminant $-3$) which are not the empty sum are

$$\operatorname{Tr}_{25,-3}(f;8) = -\sqrt{8} \qquad \operatorname{Tr}_{25,-3}(f;12) = \sqrt{12}$$
$$\operatorname{Tr}_{25,-3}(f;13) = -\sqrt{13} \qquad \operatorname{Tr}_{25,-3}(f;17) = \sqrt{17}.$$

Then Corollary 1.7 implies that

$$\mathcal{Z}'_{1,25,3}(f) = q^{-3} + \sum_{\substack{m\geq 1 \\ n\equiv 0,1 \pmod 4}} \frac{1}{\sqrt{m}} \operatorname{Tr}_{25,-3}(f;m)q^m = q^{-3} - q^8 + q^{12} - q^{13} + q^{17} + \dots$$

is a modular form of weight $\frac{1}{2}$ on $\Gamma_0(100)$.

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