The smallest eigenvalue distribution of incomplete Laguerre Unitary Ensemble.

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Thomas Bothner (University of Michigan), Alexander R. Its, Andrei Prokhorov (IUPUI)

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Laguerre Unitary Ensemble

Consider the point process on the positive real axis with the probability density function

$$f(\lambda_1,\ldots,\lambda_n)=\frac{1}{Z_n}\prod_{1\le k<j\le n}(\lambda_j-\lambda_k)^2\prod_{j=1}^{n}\lambda_j^{\alpha}e^{-\lambda_j},\quad \alpha>0,\quad 0<\lambda_1<\lambda_2<\ldots<\lambda_n. \qquad (1)$$

Such model can be refered as Laguerre Unitary Ensemble. Its k-correlation functions are expressed in terms of Christoffel-Darboux kernel of Laguerre orthogonal polynomials

$$R_k(\lambda_1,\ldots,\lambda_k)=\frac{n!}{(n-k)!}\int_0^{\infty}\cdots\int_0^{\infty}f(\lambda_1,\ldots,\lambda_n)\prod_{j=k+1}^{n}d\lambda_j=\det\left[K_n(\lambda_j,\lambda_l)\right]_{j,l=1}^{k}.$$

Complex Wishart matrices

Consider the $m\times n$ matrix X with complex entries. Suppose that the real and imaginary part of these entries are Gaussian random variables with mean zero and variance $\frac{1}{2}$. The matrix X^*X is called complex Wishart matrix. Its eigenvalue distribution is given by (1) with $\alpha=n-m$.

Fredholm determinant

The limiting eigenvalue density is described by Marchenko-Pastur law

$$\lim_{n\to\infty}R_1(n\lambda)=\frac{1}{2\pi}\sqrt{\frac{4-\lambda}{\lambda}}\chi_{(0,4)}(\lambda).$$

The limiting distribution of the smallest eigenvalue is given by Fredholm determinant (see [5])

$$\lim_{n\to\infty}\mathbb{P}\left(\frac{\lambda_{\min}}{4n}\ge t\right)=\det(I-K_{\mathrm{Bess}}),$$

where K_{Bess} is acting in $L^2(0,t)$ with the kernel

$$K_{\mathrm{Bess}}(\lambda,\mu)=\frac{\phi(\lambda)\psi(\mu)-\phi(\mu)\psi(\lambda)}{\lambda-\mu}, \qquad (2)$$

where

$$\phi(\lambda)=J_\alpha(\sqrt{\lambda}),\quad \psi(\lambda)=\frac{1}{2}\sqrt{\lambda}J_\alpha'(\sqrt{\lambda}). \qquad (3)$$

Thinning procedure

Thinning procedure consists of removing every eigenvalue independently with probability $1-\gamma\in[0,1)$. The distribution of corresponding minimal eigenvalue $\lambda_{\min}(\gamma)$ is given by (see [3])

$$\lim_{n\to\infty}\mathbb{P}\left(\frac{\lambda_{\min}(\gamma)}{4n}\ge t\right)=\det(I-\gamma K_{\mathrm{Bess}}). \qquad (4)$$

Main result

Theorem. *[2]*

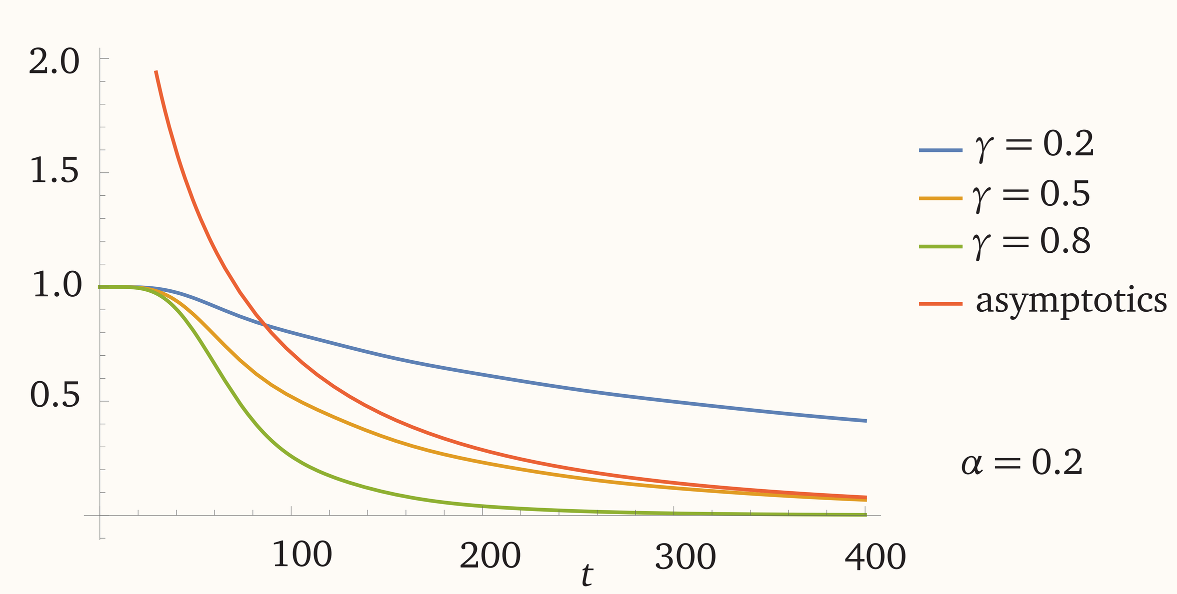
$$\lim_{n\to\infty}\mathbb{P}\left(\frac{\lambda_{\min}(\gamma)}{4n}\ge t\right)=1-\frac{\gamma}{\Gamma^2(\alpha+2)}\left(\frac{t}{4}\right)^{\alpha+1}(1+o(1)),\quad t\to 0,$$

$$\lim_{n\to\infty}\mathbb{P}\left(\frac{\lambda_{\min}(\gamma)}{4n}\ge t\right)=e^{-\frac{v}{\pi}\sqrt{t}}(16t)^{\frac{v^2}{8\pi^2}}e^{\frac{\alpha v}{2}}\left|G\left(1+\frac{iv}{2\pi}\right)\right|^2(1+o(1)),\quad t\to+\infty,$$

where $v=-\ln(1-\gamma)$ and $G(\lambda)$ is the Barnes G-function.

The case of complete spectrum was resolved in [4]

$$\lim_{n\to\infty}\mathbb{P}\left(\frac{\lambda_{\min}(\gamma)}{4n}\ge t\right)$$



Riemann Hilbert problem

The Bessel kernel (2) is of integrable type. The corresponding Riemann-Hilbert problem has form

1. $Y(\lambda)$ is analytic outside of the interval $(0,t)$, oriented from left to right,
2. $Y_+(\lambda)=Y_-(\lambda)\left[I-2\pi i\gamma\begin{pmatrix}\phi(\lambda)\psi(\lambda) & -\phi^2(\lambda)\\ \psi^2(\lambda) & -\phi(\lambda)\psi(\lambda)\end{pmatrix}\right]$,
3. $Y(\lambda)=I+\frac{Y_1}{\lambda}+O\left(\frac{1}{\lambda^2}\right),\quad \lambda\to\infty.$

Here $\phi(\lambda)$ and $\psi(\lambda)$ are given by (3).
The Hamiltonian (7) is given by

$$H=\frac{Y_{1,12}}{2}.$$

Undressing

Introduce

$$\Psi_\alpha(\lambda)=\sqrt{\pi}e^{-i\frac{\pi}{4}}\begin{pmatrix}I_\alpha\left((e^{-i\pi}\lambda)^{\frac{1}{2}}\right) & -\frac{i}{\pi}K_\alpha\left((e^{-i\pi}\lambda)^{\frac{1}{2}}\right)\\ (e^{-i\pi}\lambda)^{\frac{1}{2}}I_\alpha'\left((e^{-i\pi}\lambda)^{\frac{1}{2}}\right) & -\frac{i}{\pi}(e^{-i\pi}\lambda)^{\frac{1}{2}}K_\alpha'\left((e^{-i\pi}\lambda)^{\frac{1}{2}}\right)\end{pmatrix},$$

where $I_\alpha(\lambda)$ and $K_\alpha(\lambda)$ are modified Bessel functions. Consider

$$X(\lambda)=\begin{pmatrix}1 & 0\\ \frac{4\alpha^2+3}{8} & 1\end{pmatrix}Y(\lambda)\Psi_\alpha(\lambda). \qquad (5)$$

We have (see [1])

1. $X(\lambda)$ is analytic outside of the positive real axis, oriented from left to right,
2. $X_+(\lambda)=X_-(\lambda)\begin{pmatrix}e^{-i\pi\alpha} & 1-\gamma\\ 0 & e^{i\pi\alpha}\end{pmatrix},\quad 0\le\lambda\le t,$

   $X_+(\lambda)=X_-(\lambda)\begin{pmatrix}e^{-i\pi\alpha} & 1\\ 0 & e^{i\pi\alpha}\end{pmatrix},\quad \lambda\ge t,$
3. $X(\lambda)=\left(I+O\left(\frac{1}{\lambda}\right)\right)(e^{-i\pi}\lambda)^{-\frac{1}{4}\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}e^{-i\frac{\pi}{4}\sigma_3}\exp\left((e^{-i\pi}\lambda)^{\frac{1}{2}}\sigma_3\right),\quad \lambda\to\infty,$

   $X(\lambda)=\hat{X}_0(\lambda)(e^{-i\pi}\lambda)^{\frac{\alpha}{2}\sigma_3}\begin{pmatrix}1 & -\frac{i}{2}\frac{1-\gamma}{\sin(\pi a)}\\ 0 & 1\end{pmatrix},\quad \lambda\to 0,\quad \alpha\notin\mathbb{Z},$

   $X(\lambda)=\hat{X}_0(\lambda)(e^{-i\pi}\lambda)^{\frac{\alpha}{2}\sigma_3}\left[I-\frac{e^{i\pi\alpha}}{2\pi i}(1-\gamma)\ln(e^{-i\pi}\lambda)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right],\quad \lambda\to 0,\quad \alpha\in\mathbb{Z},$

   $X(\lambda)=\hat{X}_t(\lambda)\left(I+\frac{\gamma}{2\pi i}\begin{pmatrix}-1 & -e^{-i\pi\alpha}\\ e^{i\pi\alpha} & 1\end{pmatrix}\ln(\lambda-t)\right)\begin{cases}\begin{pmatrix}e^{-i\pi\alpha} & 1\\ -1 & 0\end{pmatrix}, \Im\lambda>0\\ \begin{pmatrix}1 & 0\\ -e^{i\pi\alpha} & 1\end{pmatrix}, \Im\lambda<0\end{cases},\ \lambda\to t.$

Asymptotics

Asymptotics $\gamma\to 0$ and $t\to 0$ are evaluated by iterating RHP for $Y(\lambda)$. For asymptotics $t\to+\infty$ we open lenses in the problem for (5). The global parametrix is constructed explicitly. The parametrix at one is expressed in terms of confluent hypergeometric function. The parametrix at zero is expressed in terms of Bessel function.

Action integral representation

Using (9) we get

$$\ln\left(\det(I-\gamma K_{\mathrm{Bess}})\right)=\int_0^t pq'-Hds+2Ht+L.$$

Action integral is convenient for taking derivatives. More precisely, we can rewrite

$$\ln\left(\det(I-\gamma K_{\mathrm{Bess}})\right)=\int_0^{\gamma}\left(p\frac{\partial q'}{\partial\tilde{\gamma}}\right)d\tilde{\gamma}+2Ht+L.$$

The last expression allows us to evaluate constant factor at the asymptotics $t\to+\infty$ of the determinant.

Hamiltonian structure

The Fredholm determinant (4) can be represented as

$$\det(I-\gamma K_{\mathrm{Bess}})=\exp\left(\int_0^t H(s)\,ds\right), \qquad (6)$$

where the Hamiltonian H is given by

$$H(p,q,t,\alpha)=\frac{q^2-1}{4t}p^2-\frac{q^2}{4}-\frac{\alpha^2}{4t(q^2-1)}. \qquad (7)$$

The corresponding Hamiltonian system is equivalent to equation (see [7])

$$t\,(q^2-1)(tq')'=q\,(tq')^2+\frac{1}{4}(t-\alpha^2)q+\frac{1}{4}tq^3(q^2-2). \qquad (8)$$

The solution corresponding to (6) is fixed by behavior at zero

$$q(t)=\frac{\sqrt{\gamma}\,t^{\frac{\alpha}{2}}}{2^{\alpha}\Gamma(1+\alpha)}(1+o(1)).$$

Painlevé equation

Consider

$$y(s)=\frac{-\alpha q(4s^2)+8s^2}{2sq(4s^2)\cdot(q^2(4s^2)-1)},$$

where $q(t)$ satisfies equation (8). Then $y(s)$ satisfies Painlevé-III equation

$$y''=\frac{y'^2}{y}-\frac{y'}{s}+\frac{4}{s}(-\alpha y^2+1+\alpha)+4y^3-\frac{4}{y}.$$

Jimbo-Miwa-Ueno differential

Jump matrices for the function (5) are independent of λ. Therefore we can consider isomonodromic deformations of $X(\lambda,t)$.

$$\frac{\partial X}{\partial\lambda}=\left(-\frac{1}{2}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}+\frac{A_0}{\lambda}+\frac{A_t}{\lambda-t}\right)X(\lambda)=A(\lambda)X(\lambda),\quad \frac{\partial X}{\partial t}=-\frac{A_t}{\lambda-t}.$$

One can express coefficients of matrices A_0, A_t in terms of $q(t)$. The compatibility condition gives (8). Following [6] we extend the JMU form

$$\omega=\operatorname*{res}_{\lambda=0}\mathrm{Tr}(A(\lambda)\partial(\hat{X}_0)\hat{X}_0^{-1})+\operatorname*{res}_{\lambda=t}\mathrm{Tr}(A(\lambda)\partial(\hat{X}_t)\hat{X}_t^{-1})$$
$$+\operatorname*{res}_{\lambda=\infty}\mathrm{Tr}(A(z)\partial\left[X\exp\left(-(e^{-i\pi}\lambda)^{\frac{1}{2}}\sigma_3\right)\right]\exp\left((e^{-i\pi}\lambda)^{\frac{1}{2}}\sigma_3\right)X^{-1}).$$

After taking the residues we get

$$\omega=pdq-Hdt+d(2Ht+\alpha L)-Ld\alpha,\qquad L(t)=-\int_0^t\frac{\alpha^2q^2}{2s(q^2-1)}ds. \qquad (9)$$

$L(t)$ is expressible in terms of $X(\lambda)$.

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