On some Hamiltonian properties of isomonodromic tau functions.

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Hamiltonian system

For a Hamiltonian $H(q,p,t)$ we can consider the Hamiltonian system

$$\begin{cases} \dfrac{dq}{dt} = \dfrac{\partial H}{\partial p}, \\ \dfrac{dp}{dt} = -\dfrac{\partial H}{\partial q}, \end{cases} \qquad (1)$$

We introduce the classical action and the tau function

$$S(t_1,t_2) = \int_{t_1}^{t_2} p\frac{dq}{dt} - H dt, \quad \ln\tau(t_1,t_2) = \int_{t_1}^{t_2} H dt.$$

Painlevé-I equation

Painlevé-I equation is given by

$$\frac{d^2q}{dt^2} = 6q^2 + t.$$

It is equivalent to the Hamiltonian system with Hamiltonian

$$H = \frac{p^2}{2} - 2q^3 - tq.$$

We have the following relation

$$\ln\tau(t_1,t_2) = S(t_1,t_2) + \left(\frac{4}{5}Ht - \frac{2}{5}pq\right)\Big|_{t_1}^{t_2}$$

Painlevé-II equation

Painlevé-II equation is given by

$$\frac{d^2q}{dt^2} = 2q^3 + tq + \alpha.$$

It is equivalent to the Hamiltonian system with Hamiltonian

$$H = \frac{p^2}{4} - tq^2 - q^4 - 2\alpha q.$$

We have the following relation

$$\ln\tau(t_1,t_2) = S(t_1,t_2) - \alpha\frac{\partial S(t_1,t_2)}{\partial\alpha} + \left(\frac{2}{3}Ht - \frac{1}{3}pq + p\alpha\frac{\partial q}{\partial\alpha}\right)\Big|_{t_1}^{t_2}.$$

Painlevé-III equation

Painlevé-III equation is given by

$$\frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{1}{t}(\alpha q^2 + \beta) + \gamma q^3 + \frac{\delta}{q},$$

where

$$\alpha = 8\theta_0, \quad \beta = 4 - 8\theta_\infty, \quad \gamma = 4, \quad \delta = -4.$$

It is equivalent to the Hamiltonian system with Hamiltonian

$$H = \frac{2p^2q^2}{t} + 2p(1-q^2) + \frac{pq}{t}(4\theta_\infty - 1) - 2q(\theta_0 + \theta_\infty).$$

We have the following relation

$$\ln\tau(t_1,t_2) = S(t_1,t_2) + \left(\frac{1-4\theta_\infty}{4}\right)\frac{\partial S(t_1,t_2)}{\partial\theta_\infty} - \left(\frac{1+4\theta_0}{4}\right)\frac{\partial S(t_1,t_2)}{\partial\theta_0}$$
$$+ \left[Ht - \left(\frac{1-4\theta_\infty}{4}\right)p\frac{\partial q}{\partial\theta_\infty} + \left(\frac{1+4\theta_0}{4}\right)p\frac{\partial q}{\partial\theta_0}\right]\Big|_{t_1}^{t_2}.$$

References

[1] A. Its, A. Prokhorov, *On Some Hamiltonian Properties of the Isomonodromic Tau Functions* , Reviews in Mathematical Physics 30:7, (2018).

[2] A. Its, O. Lisovyy, A. Prokhorov, *Monodromy dependence and connection formulæ for isomonodromic tau functions*, , Duke Math. J. 167:7 (2018), 1347-1432.

Alternative formula for the action

Assume that we can parametrize the solutions for the system (1) by parameters (m_1, m_2) which are usually called monodromy data. Then we can notice that

$$\frac{\partial S(t_1,t_2)}{\partial m_j} = \int_{t_1}^{t_2}\left[\frac{\partial p}{\partial m_j}\frac{dq}{dt} + p\frac{d}{dt}\left(\frac{\partial q}{\partial m_j}\right) - \frac{\partial H}{\partial m_j}\right]dt$$

$$= \int_{t_1}^{t_2}\left[\frac{\partial p}{\partial m_j}\frac{dq}{dt} + p\frac{d}{dt}\left(\frac{\partial q}{\partial m_j}\right) - \frac{dp}{dt}\frac{\partial q}{\partial m_j} + \frac{dq}{dt}\frac{\partial p}{\partial m_j}\right]dt = \left(p\frac{\partial q}{\partial m_j}\right)\Big|_{t_1}^{t_2}$$

Therefore we have

$$S(t_1,t_2) = \int_{(m_1^0,m_2^0)}^{(m_1,m_2)} \left(p\frac{\partial q}{\partial m_1}\right)\Big|_{t_1}^{t_2} dm_1 + \left(p\frac{\partial q}{\partial m_2}\right)\Big|_{t_1}^{t_2} dm_2.$$

Painlevé-IV equation

Painlevé-IV equation is given by

$$\frac{d^2q}{dt^2} = \frac{1}{2q}\left(\frac{dq}{dt}\right)^2 + \frac{3}{2}q^3 + 4tq^2 + 2(t^2-\alpha)q + \frac{\beta}{q},$$

where

$$\alpha = 2\theta_\infty - 1, \quad \beta = -8\theta_0^2.$$

It is equivalent to the Hamiltonian system with Hamiltonian

$$H = 2p^2q + p(q^2 + 2qt + 4\theta_0) + q(\theta_0 + \theta_\infty).$$

We have the following relation

$$\ln\tau(t_1,t_2) = S(t_1,t_2) - \theta_0\frac{\partial S(t_1,t_2)}{\partial\theta_0} - \theta_\infty\frac{\partial S(t_1,t_2)}{\partial\theta_\infty}$$
$$+ \left(\frac{1}{2}Ht - \frac{1}{2}pq + \theta_0 p\frac{\partial q}{\partial\theta_0} + \theta_\infty p\frac{\partial q}{\partial\theta_\infty}\right)\Big|_{t_1}^{t_2}.$$

Painlevé-V equation

Painlevé-V equation is given by

$$\frac{d^2q}{dt^2} = \left(\frac{1}{2q} + \frac{1}{q-1}\right)\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{(q-1)^2}{t^2}\left(\alpha q + \frac{\beta}{q}\right) + \gamma\frac{q}{t} + \delta\frac{q(q+1)}{(q-1)},$$

where

$$\alpha = \frac{(\theta_0 - \theta_1 + \theta_\infty)^2}{2}, \quad \beta = -\frac{(\theta_0 - \theta_1 - \theta_\infty)^2}{2}, \quad \gamma = (1 - 2\theta_0 - 2\theta_1), \quad \delta = -\frac{1}{2}.$$

It is equivalent to the Hamiltonian system with Hamiltonian

$$H = \frac{p^2(q-1)^2q}{t} + p\left(\frac{q^2}{t}(\theta_0 + 3\theta_1 + \theta_\infty) + \frac{q}{t}(t - 2\theta_\infty - 4\theta_1) + \frac{1}{t}(\theta_\infty + \theta_1 - \theta_0)\right)$$
$$+ \frac{2q\theta_1}{t}(\theta_\infty + \theta_1 + \theta_0).$$

We have the following relation

$$\ln\tau(t_1,t_2) = S(t_1,t_2) - \theta_0\frac{\partial S(t_1,t_2)}{\partial\theta_0} - \theta_1\frac{\partial S(t_1,t_2)}{\partial\theta_1} - \theta_\infty\frac{\partial S(t_1,t_2)}{\partial\theta_\infty}$$
$$+ \left[Ht + \theta_0 p\frac{\partial q}{\partial\theta_0} + \theta_1 p\frac{\partial q}{\partial\theta_1} + \theta_\infty p\frac{\partial q}{\partial\theta_\infty}\right]\Big|_{t_1}^{t_2}.$$

Conjecture for Hamiltonian structure

We denote by δ the differential in the configuration space which does not include isomonodromic times. Consider the form in this space

$$\alpha = \sum_{a_\nu}\mathrm{res}_{z=a_\nu}\mathrm{Tr}\left(\frac{\partial A(z)}{\partial t}\delta G_\nu(z)G_\nu(z)^{-1}\right) - \sum_{a_\nu}\mathrm{res}_{z=a_\nu}\mathrm{Tr}\left(\frac{d(\delta\Theta_\nu(z))}{dz}G_\nu(z)^{-1}\frac{\partial G_\nu(z)}{\partial t}\right)$$

We conjecture that the form α is exact and the Hamiltonian is given by $\alpha = \delta H$.

Painlevé-VI equation

Painlevé-VI equation is given by

$$\frac{d^2q}{dt^2} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)\frac{dq}{dt}$$
$$+ \frac{q(q-1)(q-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{q^2} + \gamma\frac{t-1}{(q-1)^2} + \delta\frac{t(t-1)}{(q-t)^2}\right),$$

where

$$\alpha = \frac{(2\theta_\infty - 1)^2}{2}, \quad \beta = -2\theta_0^2, \quad \gamma = 2\theta_1^2, \quad \delta = \frac{1 - 4\theta_t^2}{2}.$$

It is equivalent to the Hamiltonian system with Hamiltonian

$$H = p^2\frac{q(q-1)(q-t)}{t(t-1)} + p\frac{q(q-1)}{t(t-1)} + \frac{\theta_\infty(1-\theta_\infty)q}{t(t-1)} - \frac{\theta_0^2}{q(t-1)} + \frac{\theta_1^2}{(q-1)t} - \frac{\theta_t^2(1+t)}{(q-t)t}.$$

We have the following relation

$$\ln\tau(t_1,t_2) = S(t_1,t_2) - \theta_0\frac{\partial S(t_1,t_2)}{\partial\theta_0} - \theta_1\frac{\partial S(t_1,t_2)}{\partial\theta_1} - \theta_t\frac{\partial S(t_1,t_2)}{\partial\theta_t} + \left(\frac{1-2\theta_\infty}{2}\right)\frac{\partial S(t_1,t_2)}{\partial\theta_\infty}$$
$$+ \left[\theta_0 p\frac{\partial q}{\partial\theta_0} + \theta_1 p\frac{\partial q}{\partial\theta_1} + \theta_t p\frac{\partial q}{\partial\theta_t} + \left(\frac{2\theta_\infty - 1}{2}\right)p\frac{\partial q}{\partial\theta_\infty} + \frac{1}{2}\ln\left(\frac{q-t}{t}\right)\right]\Big|_{t_1}^{t_2}.$$

Schlesinger equations

Consider the Hamiltonians for $N\times N$ matrices Q_ν and P_ν

$$H_\nu = \sum_{\mu\neq\nu}^{n}\frac{\mathrm{Tr}(Q_\mu P_\mu Q_\nu P_\nu)}{a_\nu - a_\mu}.$$

Here a_ν play role of times. Consider then the Hamiltonian system

$$\frac{dP_{\mu,jk}}{da_\nu} = -\frac{\partial H_\nu}{\partial Q_{\mu,kj}}, \quad \frac{dQ_{\mu,jk}}{da_\nu} = \frac{\partial H_\nu}{\partial P_{\mu,kj}}, \quad j,k = 1..N$$

If we take $A_\nu = Q_\nu P_\nu$, then we get the Schlesinger equations

$$\frac{dA_\mu}{da_\nu} = \frac{[A_\mu, A_\nu]}{a_\mu - a_\nu}, \quad \mu\neq\nu, \qquad \frac{dA_\nu}{da_\nu} = -\sum_{\mu\neq\nu}\frac{[A_\mu, A_\nu]}{a_\mu - a_\nu}.$$

We can introduce the classical action and the tau function

$$S(\vec{a^0}, \vec{a}) = \int_{\vec{a^0}}^{\vec{a}}\sum_{\nu=1}^{n}\mathrm{Tr}(P_\nu dQ_\nu) - H_\nu da_\nu, \quad \ln\tau(\vec{a^0}, \vec{a}) = \int_{\vec{a^0}}^{\vec{a}}\sum_{\nu=1}^{n}H_\nu da_\nu.$$

We can check that

$$\ln\tau(\vec{a^0}, \vec{a}) = S(\vec{a^0}, \vec{a}).$$

General Isomonodromic deformations

Consider the linear ODE with rational matrix coefficients.

$$\frac{d\Psi(z)}{dz} = A(z)\Psi(z).$$

Near singularities a_ν of matrix $A(z)$ there are local solutions

$$\Psi_\nu(z) \simeq G_\nu(z)e^{\Theta_\nu(z)}.$$

Denote $\vec{t}$ the isomonodromy times and $\vec{m}$ the monodromy data. Using the result from [2] we can get an analog of the formulae with action for tau function

$$\ln\tau_{JMU}(\vec{t^{(1)}}, \vec{t^{(2)}}) = \int_{\vec{t^{(1)}}}^{\vec{t^{(2)}}} -\sum_{k=1}^{L}\sum_{a_\nu}\mathrm{res}_{z=a_\nu}\mathrm{Tr}\left(\left(G_\nu(z)\right)^{-1}\frac{dG_\nu(z)}{dz}\frac{d\Theta_\nu(z)}{dt_k}\right)dt_k$$

$$= \int_{\vec{m}}^{\vec{m}_0}\sum_{k=1}^{M}\sum_{a_\nu}\mathrm{res}_{z=a_\nu}\mathrm{Tr}\left(A(z)\frac{\partial G_\nu}{\partial m_k}(z)G_\nu(z)^{-1}\right)\Big|_{\vec{t^{(1)}}}^{\vec{t^{(2)}}} dm_k.$$