Connection problem for Painlevé tau functions

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Content

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Painlevé equations

Painlevé equations

Painlevé (1902), Gambier (1910)

$$\frac{d^2u}{dx^2} = 6u^2 + x, \tag{PI}$$

$$\frac{d^2u}{dx^2} = 2u^3 + xu + \alpha, \tag{PII}$$

$$\frac{d^2u}{dx^2} = \frac{1}{u}\left(\frac{du}{dx}\right)^2 - \frac{1}{x}\left(\frac{du}{dx}\right) + \frac{\alpha u^2}{x} + \frac{\beta}{x} + \gamma u^3 + \frac{\delta}{u}, \tag{PIII}$$

$$\frac{d^2u}{dx^2} = \frac{1}{2u}\left(\frac{du}{dx}\right)^2 + \frac{3}{2}u^3 + 4xu^2 + 2(x^2-\alpha)u + \frac{\beta}{u}, \tag{PIV}$$

$$\begin{aligned}\frac{d^2u}{dx^2} = &\left(\frac{1}{2u} + \frac{1}{u-1}\right)\left(\frac{du}{dx}\right)^2 - \frac{1}{x}\left(\frac{du}{dx}\right)\\ &+ \frac{(u-1)^2}{x^2}\left(\alpha u + \frac{\beta}{u}\right) + \frac{\gamma u}{x} + \frac{\delta u(u+1)}{u-1},\end{aligned} \tag{PV}$$

$$\begin{aligned}\frac{d^2u}{dx^2} = &\frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x}\right)\left(\frac{du}{dx}\right)^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x}\right)\frac{du}{dx}\\ &+ \frac{u(u-1)(u-x)}{x^2(x-1)^2}\left(\alpha + \frac{\beta x}{u^2} + \frac{\gamma(x-1)}{(u-1)^2} + \delta\frac{x(x-1)}{(u-x)^2}\right),\end{aligned} \tag{PVI}$$

Subcases

$$\gamma\delta \neq 0, \tag{PIII(D6)}$$

$$\gamma = 0, \quad \alpha\delta \neq 0 \quad \text{or} \quad \delta = 0, \quad \beta\gamma \neq 0, \tag{PIII(D7)}$$

$$\gamma = \delta = 0, \quad \alpha\beta \neq 0, \tag{PIII(D8)}$$

$$\delta \neq 0, \tag{PV}$$

$$\delta = 0, \quad \gamma \neq 0. \tag{PV-deg}$$

Properties

- Equation of sort

$$\frac{d^2u}{dx^2} = F\left(u, \frac{du}{dx}, x\right)$$

  with right hand side rational in u, $\frac{du}{dx}$ and x.
- Painlevé property (Hinkkanen, Laine (1999), (2001), (2004)):
  - Painlevé I, II, IV – meromorphic solutions
  - Painlevé III, V – meromorphic solutions in variable $t = \ln x$
  - Painlevé VI – meromorphic solution on the covering of $\mathbb{C}\backslash\{0, 1\}$
- General solutions are transcendental (Umemura, Watanabe (1988), (1995), (1997), (1998), (1999))
- Families of rational, algebraic and special functions solutions. (see `https://dlmf.nist.gov`)
- Isomonodromic deformations of linear ODEs with rational coefficients (Jimbo, Miwa, (1980))
- Backlund transformations (see `https://dlmf.nist.gov`)
- Hamiltonian systems (Okamoto (1980))

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Applications

- gap probabilities in random matrix theory (Tracy, Widom, (1994))
- asymptotics of nonlinear PDEs (Ablowitz-Segur, (1977))
- Ising model (Barouch, Mccoy, Tracy, Wu (1976))
- conformal field theory (Gamayun, Iorgov, Lisovyy, (2012), (2013))
- quantum cohomology (Dubrovin (1996), Guzzetti (2001))
- diffusion processes (Bloomendal, Virag (2013))

Hamiltonians

Okamoto (1980)

$$H = \frac{w^2}{2} - 2u^3 - xu, \tag{PI}$$

$$H = \frac{w^2}{2} - \frac{u^4}{2} - \frac{u^2 x}{2} - u\alpha, \tag{PII}$$

$$H = \frac{w^2 u^2}{x} - \frac{\alpha u}{2} + \frac{\beta}{2u} - \frac{\gamma x u^2}{4} + \frac{\delta x}{4u^2}, \tag{PIII}$$

$$H = 2w^2 u - \frac{u^3}{8} - \frac{xu^2}{2} - \frac{u}{2}(x^2 - \alpha) + \frac{\beta}{4u}, \tag{PIV}$$

$$H = \frac{w^2 (u-1)^2 u}{x} - \frac{\alpha u}{2x} + \frac{\beta}{2ux} + \frac{\gamma}{2(u-1)} + \frac{\delta u x}{2(u-1)^2}, \tag{PV}$$

$$H = \frac{w^2 u(u-1)(u-x)}{x(x-1)} - \frac{\alpha u}{2x(x-1)} + \frac{\beta}{2u(x-1)} + \frac{\gamma}{2x(u-1)} + \frac{\delta}{2(u-x)}. \tag{PVI}$$

Hamiltonians: canonical form

$$p = w, \quad q = u, \quad t = x, \tag{PI}$$

$$p = w, \quad q = u, \quad t = x, \tag{PII}$$

$$p = 2wu, \quad q = \ln u, \quad t = \ln x, \quad x > 0 \tag{PIII}$$

$$p = 2w\sqrt{u}, \quad q = \sqrt{u}, \quad t = x, \tag{PIV}$$

$$p = 2w(u-1)\sqrt{u}, \quad q = \ln\left(\frac{\sqrt{u}-1}{\sqrt{u}+1}\right), \quad t = \ln x, \quad x > 0 \tag{PV}$$

$$q = \int\limits_0^u \frac{ds}{\sqrt{s(s-1)(s-x)}}, \quad u = x \cdot \mathrm{sn}^2\left(\frac{q}{2}, \sqrt{x}\right)$$

$$t = \ln\left(\frac{1-x}{x}\right), \quad 0 < x < 1. \tag{PVI}$$

Hamiltonians: canonical form

$$H = \frac{p^2}{2} - 2q^3 - tq, \tag{PI}$$

$$H = \frac{p^2}{2} - \frac{q^4}{2} - \frac{q^2 t}{2} - q\alpha, \tag{PII}$$

$$H = \frac{p^2}{2} - \alpha e^{t+q} + \beta e^{t-q} - \frac{\gamma}{2} e^{2t+2q} + \frac{\delta}{2} e^{2t-2q}, \tag{PIII}$$

$$H = \frac{p^2}{2} - \frac{q^6}{8} - \frac{q^4 t}{2} - \frac{q^2}{2}(t^2 - \alpha) + \frac{\beta}{4q^2}, \tag{PIV}$$

$$H = \frac{p^2}{2} - \frac{\alpha}{\sinh^2 \frac{q}{2}} - \frac{\beta}{\cosh^2 \frac{q}{2}} + \frac{\gamma}{2} e^t \cosh q + \frac{\delta}{8} e^{2t} \cosh 2q, \tag{PV}$$

$$H = \frac{p^2}{2} + \frac{1}{8} q^2 k^2 (k^2 - 1) - \alpha k^2 \mathrm{sn}^2\left(\frac{q}{2}, k\right) + \frac{\beta}{\mathrm{sn}^2\left(\frac{q}{2}, k\right)} - \frac{\gamma(k^2-1)}{\mathrm{dn}^2\left(\frac{q}{2}, k\right)} - \frac{(2\delta - 1)(k^2 - 1)}{2\mathrm{cn}^2\left(\frac{q}{2}, k\right)},$$

$$\text{where } k = \frac{1}{\sqrt{1 + e^t}} \tag{PVI}$$

Equations of motion

$$\frac{d^2q}{dt^2} = 6q^2 + t \tag{PI(F)}$$

$$\frac{d^2q}{dt^2} = 2q^3 + qt + \alpha, \tag{PII(F)}$$

$$\frac{d^2q}{dt^2} = \alpha e^{t+q} + \beta e^{t-q} + \gamma e^{2t+2q} + \delta e^{2t-2q}, \tag{PIII(F)}$$

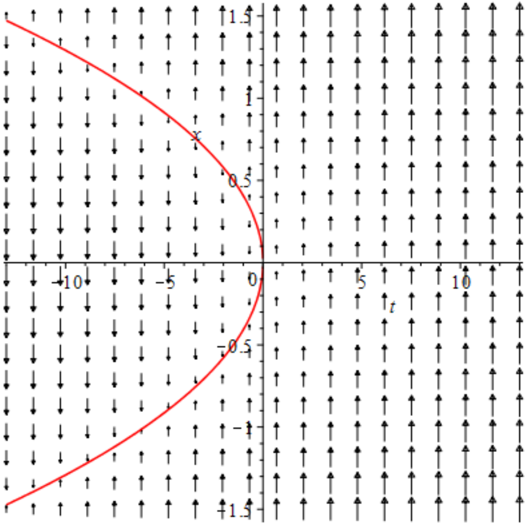
$$\frac{d^2q}{dt^2} = \frac{3q^5}{4} + 2tq^3 + q(t^2-\alpha) + \frac{\beta}{2q^3}, \tag{PIV(F)}$$

$$\frac{d^2q}{dt^2} = -\frac{\alpha\cosh(\frac{q}{2})}{\sinh^3(\frac{q}{2})} - \frac{\beta\sinh(\frac{q}{2})}{\cosh^3(\frac{q}{2})} - \frac{\gamma}{2}e^t\sinh(q) - \frac{\delta}{4}e^{2t}\sinh(2q), \tag{PV(F)}$$

$$\frac{d^2q}{dt^2} = -\frac{q}{4}k^2(k^2-1) + \alpha k^2 \mathrm{sn}\left(\frac{q}{2},k\right)\mathrm{cn}\left(\frac{q}{2},k\right)\mathrm{dn}\left(\frac{q}{2},k\right) + \frac{\beta \mathrm{cn}\left(\frac{q}{2},k\right)\mathrm{dn}\left(\frac{q}{2},k\right)}{\mathrm{sn}^3\left(\frac{q}{2},k\right)}$$
$$+ \frac{\gamma k^2(k^2-1)\mathrm{sn}\left(\frac{q}{2},k\right)\mathrm{cn}\left(\frac{q}{2},k\right)}{\mathrm{dn}^3\left(\frac{q}{2},k\right)} + \frac{(2\delta-1)(k^2-1)\mathrm{sn}\left(\frac{q}{2},k\right)\mathrm{dn}\left(\frac{q}{2},k\right)}{2\mathrm{cn}^3\left(\frac{q}{2},k\right)}, \tag{PVI(F)}$$

where $k = \frac{1}{\sqrt{1+e^t}}$

Painlevé I: force field



Painlevé II: force field for real solutions

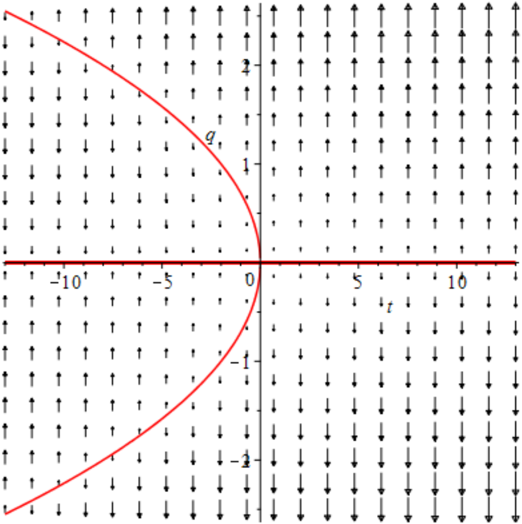


Figure: $\alpha = 0$

Painlevé II: asymptotics of real solutions

The real nonsingular solutions are parametrized by number $s_1 \in i\mathbb{R}, \quad |s_1| \leq 1$. (Kapaev, 1992) The asymptotic at $+\infty$ is given by

$$q(t) = \frac{is_1}{2\sqrt{\pi}t^{\frac{1}{4}}} e^{-\frac{2}{3}t^{\frac{3}{2}}} \left(1 + O\left(\frac{1}{t^{\frac{3}{4}}}\right)\right), \quad t \to +\infty.$$

If $|s_1| < 1$, then the asymptotics at $-\infty$ is given by Ablowitz-Segur solution

$$q(t) = \frac{d}{(-t)^{\frac{1}{4}}} \sin\left(\frac{2}{3}(-t)^{\frac{3}{2}} + \frac{3}{4}d^2 \ln(-t) + \phi\right) + O\left(\frac{1}{|t|}\right), \quad t \to -\infty,$$

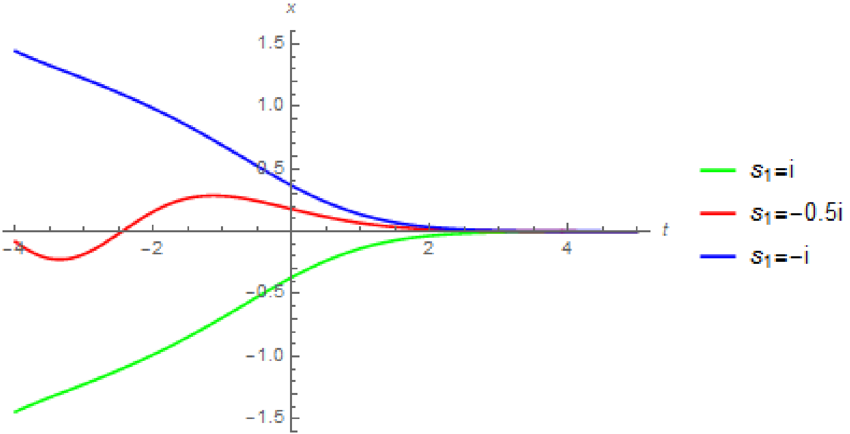
where

$$d = \sqrt{\frac{1}{\pi} \ln\left(1 - |s_1|^2\right)}, \quad \phi = -\frac{\pi}{4} + \frac{3}{2}d^2 \ln 2 - \arg\left(\Gamma\left(i\frac{d^2}{2}\right)\right) - \arg(s_1).$$

If $s_1 = \pm i$, then the asymptotics at $-\infty$ is given by Hastings-Mcleod solution

$$q(t) = is_1\sqrt{\frac{-t}{2}} + O(t^{-\frac{5}{2}}), \quad t \to -\infty.$$

Painlevé II: real solutions



Painlevé II: force field for imaginary solutions

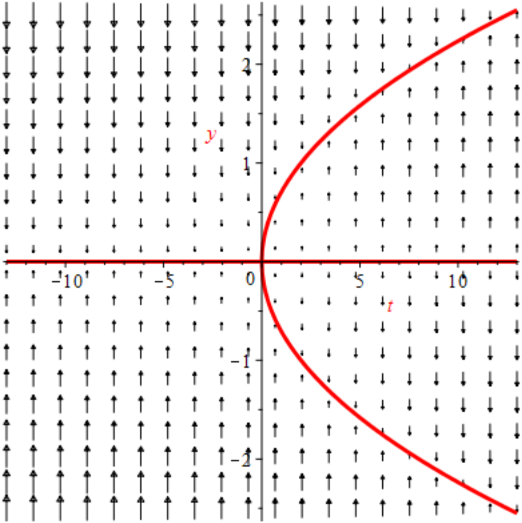


Figure: $\alpha = 0$

Painlevé II: asymptotics of imaginary solutions

All pure imaginary solutions $q = iy$ are parametrized by number $s_1 \in \mathbb{C}$. (Its, Kapaev, 1988) The asymptotic at $-\infty$ is given by

$$y(t) = \frac{d}{(-t)^{\frac{1}{4}}} \sin\left(\frac{2}{3}(-t)^{\frac{3}{2}} + \frac{3}{4}d^2 \ln(-t) + \phi\right) + O\left(\frac{1}{|t|}\right), \quad t \to -\infty,$$

$$d = \sqrt{\frac{1}{\pi}\ln\left(1+|s_1|^2\right)}, \quad \phi = -\frac{\pi}{4} + \frac{3}{2}d^2 \ln 2 - \arg\left(\Gamma\left(i\frac{d^2}{2}\right)\right) - \arg(s_1).$$

If $\mathrm{Im}\, s_1 \neq 0$ then the asymptotic at $+\infty$ is given by

$$y(t) = \sigma\sqrt{\frac{t}{2}} + \frac{\sigma\rho}{(2t)^{\frac{1}{4}}} \cos\left(\frac{2\sqrt{2}}{3}t^{\frac{3}{2}} - \frac{3}{2}\rho^2 \ln x + \theta\right) + O\left(\frac{1}{t}\right), \quad t \to +\infty.$$

$$\rho = \sqrt{\frac{1}{\pi}\ln\left(\frac{1+|s_1|^2}{2|\mathrm{Im}(s_1)|}\right)}, \quad \sigma = -\mathrm{sign}(\mathrm{Im}(s_1)),$$

$$\theta = -\frac{3\pi}{4} - \frac{7}{2}\rho^2 \ln 2 + \arg(\Gamma(i\rho^2)) + \arg(1+s_1^2).$$

If $\mathrm{Im}\, s_1 = 0$ the asymptotic at $+\infty$ is given by the Ablowitz-Segur solution,

$$y(t) = \frac{s_1}{2\sqrt{\pi}t^{\frac{1}{4}}} e^{-\frac{2}{3}t^{\frac{3}{2}}} \left(1 + O\left(\frac{1}{t^{\frac{3}{4}}}\right)\right), \quad t \to +\infty.$$

Painlevé II: imaginary solutions

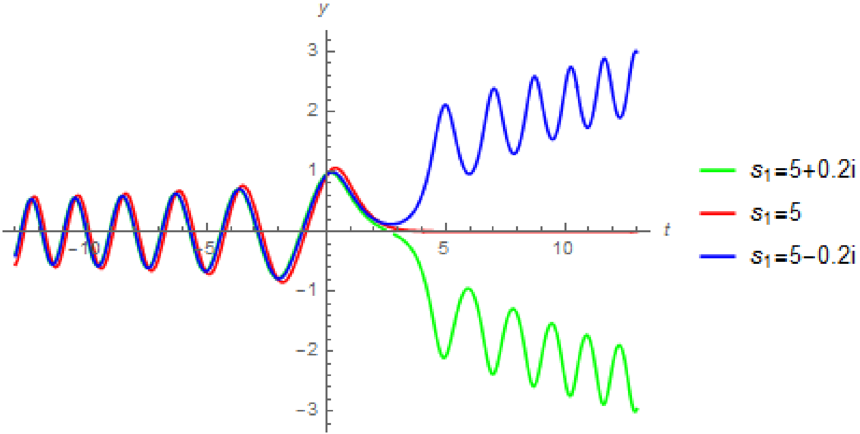


Figure: V. Vasnetsov: Knight at the crossroads.

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Painlevé III(D8): force field for imaginary solutions

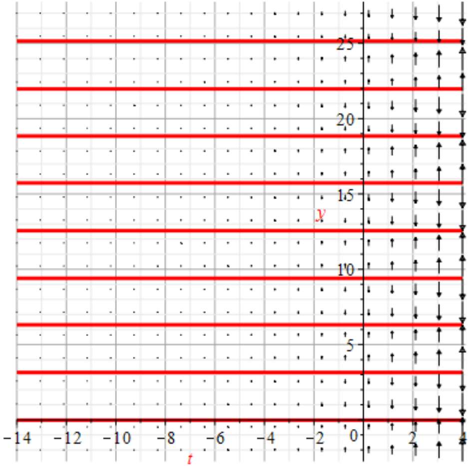

Figure: $q = -iy, \quad \alpha = -\frac{1}{8}, \quad \beta = \frac{1}{8}, \quad \gamma = \delta = 0, \quad F(y,t) = -\frac{1}{4}e^t \sin(y)$

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Painlevé III(D8): asymptotics of imaginary solutions

All pure imaginary solutions $q = -iy$ are parametrized by two imaginary numbers $a, b \in i\mathbb{R}$. (Its, Novokshenov, 1988) The asymptotic at $-\infty$ is given by

$$y(t) = iat + ib + O\left(e^{1-|\mathrm{Re}\, a|}\right), \quad t \to -\infty,$$

Introduce

$$\zeta = \frac{1}{2} + \frac{1}{2\pi}\left(ib + 6ia \ln 2\right) + \frac{i}{\pi} \ln \frac{\Gamma\left(\frac{1}{2} + \frac{a}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{a}{2}\right)}.$$

If $\zeta \notin \mathbb{Z}$ then the asymptotic at $+\infty$ is given by

$$y(t) = 2\pi k + 4\sqrt{-\nu} e^{-\frac{t}{4}} \cos\left(\exp\left(\frac{t}{2}\right) + \frac{\nu t}{2} + \phi\right) + O\left(\exp\left(\frac{3t}{4}(2|\mathrm{Im}\,\nu| - 1)\right)\right),$$

$$e^{\pi\nu} = \frac{\sin 2\pi\eta}{\sin 2\pi\sigma}, \quad k = \lfloor \mathrm{Re}\,(\zeta) \rfloor, \quad \sigma = \frac{1}{4} - \frac{a}{4}, \quad \mathrm{Re}\,\eta = \frac{1}{2}\{\mathrm{Re}\,(\zeta)\}, \quad \mathrm{Im}\,\eta = \mathrm{Im}\left(\frac{\zeta}{2}\right),$$

$$\phi = 2\nu \ln 2 + \frac{3\pi}{4} - \arg(\Gamma(i\nu)) - \arg p, \quad p = -i\frac{\sin 2\pi(\sigma + \eta)}{\sin 2\pi\eta}$$

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Painlevé III(D8): asymptotics of imaginary solutions

If $\zeta \in \mathbb{Z}$ the asymptotic at $+\infty$ is given by

$$y(t) = \pi + 2\pi k - 2\sinh\left(\frac{\pi i a}{2}\right)\sqrt{\frac{2}{\pi}}\exp\left(-\frac{t}{4}\right)\exp\left(-e^{\frac{t}{2}}\right)(1+o(1)), \quad t \to +\infty.$$

$$k = \lfloor \mathrm{Re}\,(\zeta) \rfloor$$

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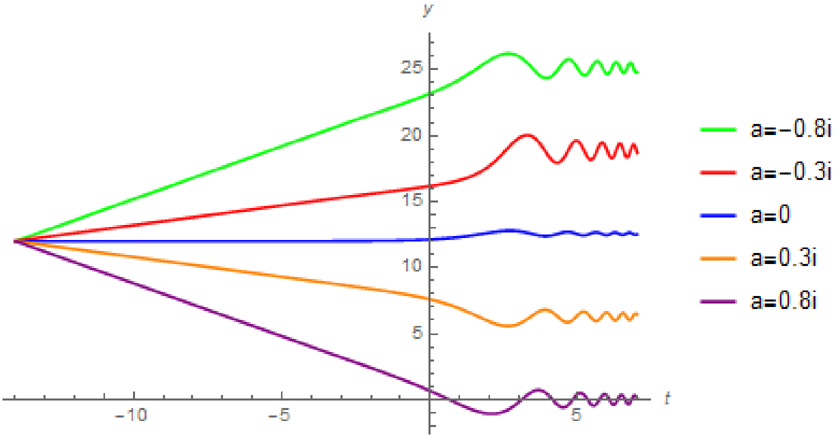
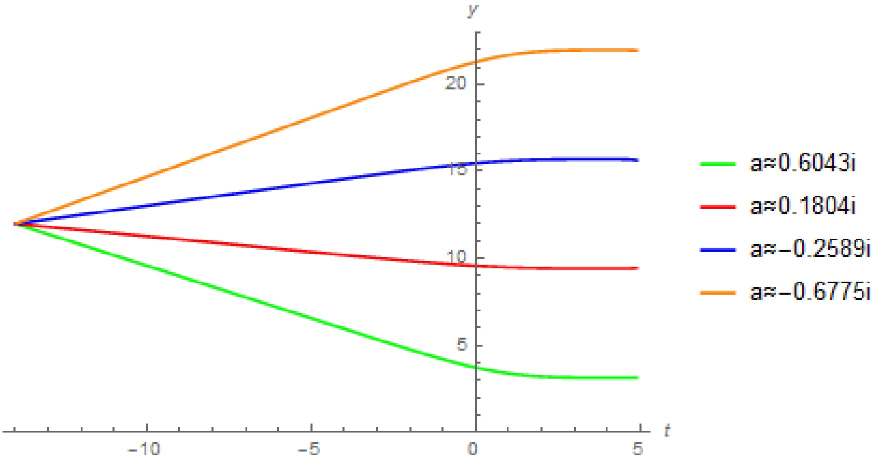
Painlevé III(D8): imaginary solutions attaining stable trajectories



Painlevé III(D8): imaginary solutions attaining unstable trajectories



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Painlevé III(D6): force field

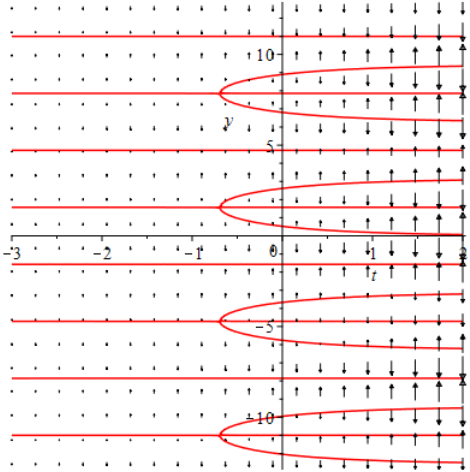

Figure: $q = iy, \quad \alpha = -\frac{i}{2}, \quad \beta = -\frac{i}{2}, \quad \gamma = \frac{1}{2}, \quad \delta = -\frac{1}{2}, \quad F(y,t) = e^t \cos(y) - e^{2t} \sin(2y)$

Painlevé IV: force field

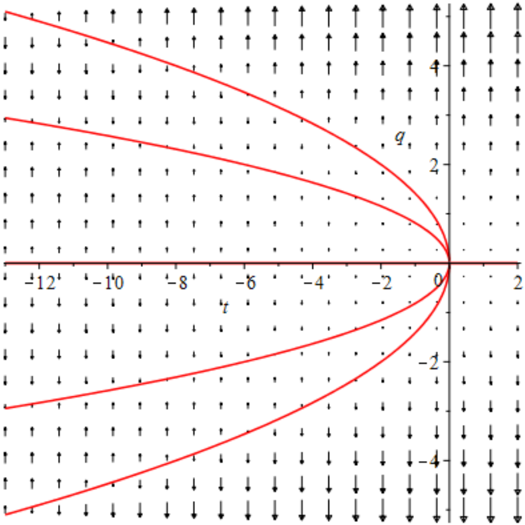

Figure: $\alpha = \beta = 0$

Painlevé V: force field

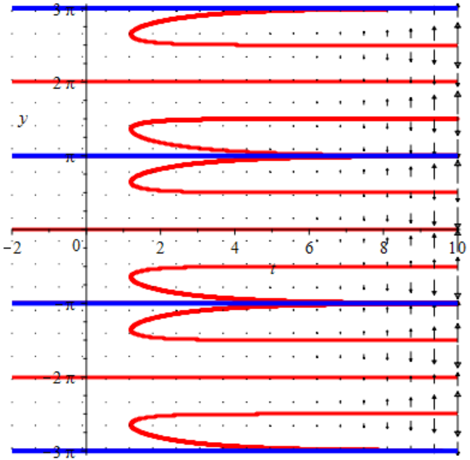

Figure: $q = iy, \quad \alpha = 0, \quad \beta = -1, \quad \gamma = -2, \quad \delta = -2, \quad F(y,t) = -\frac{\sin(\frac{y}{2})}{\cos^3(\frac{y}{2})} - e^t \sin(y) - e^{2t}\sin(2y)$

Painlevé VI: force field

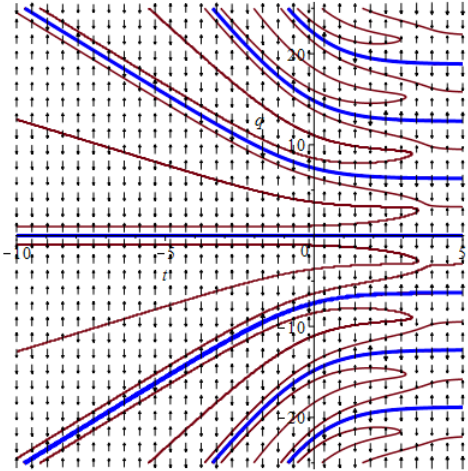

Figure: $\alpha = -20, \quad \beta = 1, \quad \gamma = 20, \quad \delta = 0$

Tau function

Tau function

Definition of tau function

$$\tau(t_1, t_2) = \exp\left(\int_{t_1}^{t_2} H(t)dt\right)$$

Connection problem for tau function

Compute the **the constant term** in the asymptotic of tau function as t_1 and t_2 approach singularities of solution.

Phase transition in Ising model: notation

The configuration σ represents the spin orientation at every point on the integer lattice.

$$\sigma : \mathbb{Z}^2 \to \{1, -1\}.$$

Energy of configuration σ restricted to $M \times N$ rectangle $\Lambda \in \mathbb{Z}^2$ is defined by the formula

$$E_\Lambda(\sigma) = -J \sum_{j,k \in \Lambda} (\sigma_{j,k}\sigma_{j,k+1} + \sigma_{j,k}\sigma_{j+1,k}), \quad J > 0.$$

Spin correlation function along the row is given by

$$\langle \sigma_{1,1}\sigma_{1,n+1} \rangle = \lim_{|\Lambda| \to \infty} \frac{\sum\limits_{\sigma} \sigma_{1,1}\sigma_{1,n+1} e^{-\frac{E_\Lambda(\sigma)}{kT}}}{\sum\limits_{\sigma} e^{-\frac{E_\Lambda(\sigma)}{kT}}}.$$

Introduce the parameter z

$$z = \tanh\left(\frac{J}{kT}\right).$$

The critical temperature T_c is described by the formula

$$z_c = \sqrt{2} - 1.$$

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Phase transition in Ising model: correlation functions behavior

The spin correlations go through transition from order to disorder.

$$T < T_c : \langle \sigma_{1,1}\sigma_{1,n+1} \rangle = O(1), \quad n \to \infty,$$

$$T = T_c : \langle \sigma_{1,1}\sigma_{1,n+1} \rangle = O(n^{-\frac{1}{4}}), \quad n \to \infty,$$

$$T > T_c : \langle \sigma_{1,1}\sigma_{1,n+1} \rangle = O\left(\frac{e^{-cn}}{\sqrt{n}}\right), \quad n \to \infty.$$

Phase transition in Ising model: Painlevé-III(D8) equation.

We introduce

$$t = \ln\left(\frac{n^2((z+1)^2-2)^2}{z(1-z^2)}\right).$$

Barouch, Mccoy, Tracy, Wu (1977) showed that

$$\lim_{\substack{n\to\infty \\ T\to T_c\pm 0}} n^{\frac{1}{4}}\langle\sigma_{1,1}\sigma_{1,n+1}\rangle = 2^{\frac{3}{8}} e^{\frac{t}{8}} \exp\left(-\int_t^{+\infty}\left(\frac{H}{4}+\frac{e^t}{16}\right)dt\right)\begin{cases}\sinh(\frac{q}{4}), & T>T_c,\\ \cosh(\frac{q}{4}), & T<T_c.\end{cases}$$

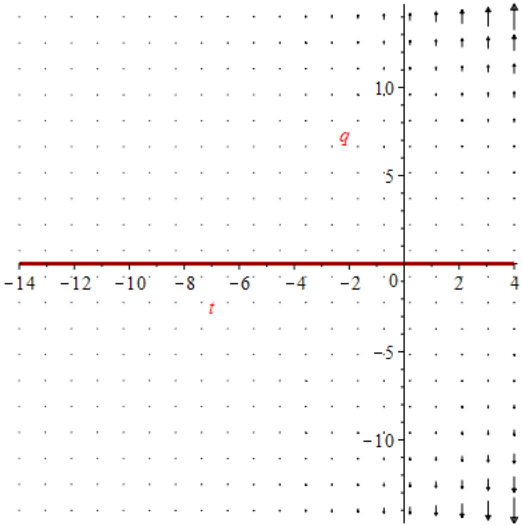
where the function $q(t)$ solves PIII(F) equation with $\alpha = \frac{1}{8}, \quad \beta = -\frac{1}{8}, \quad \gamma=\delta=0.$

$$\frac{d^2q}{dt^2} = \frac{1}{4}e^t\sinh q.$$

The Hamiltonian H is given by

$$H(p,q,t) = \frac{p^2}{2} - \frac{e^t\cosh(q)}{4}, \quad p = \frac{dq}{dt}.$$

Phase transition in Ising model: force field for Painlevé equation



Phase transition in Ising model: asymptotic of solution of Painlevé equation

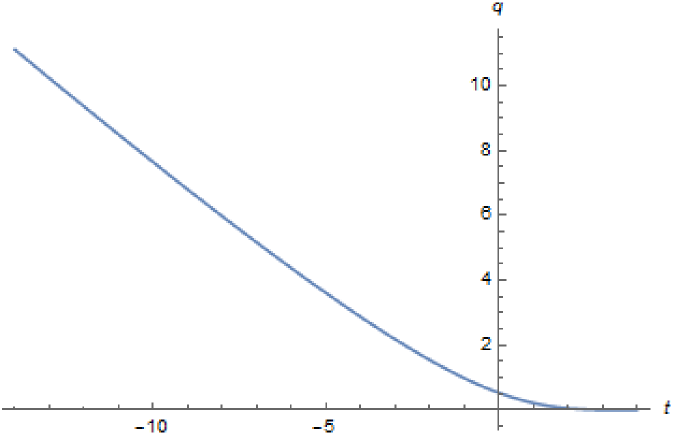
The asymptotic of solution $q(t)$ appearing above is given by

$$q(t) \simeq -t + 4\ln(2) - 2\ln(6\ln(2) - 2\gamma - t), \quad t \to -\infty,$$

$$q(t) \simeq 2\sqrt{\frac{2}{\pi}} e^{-\frac{t}{4}} e^{-e^{\frac{t}{2}}}, \quad t \to +\infty.$$

where γ is the Euler's constant.

Phase transition in Ising model: graph of solution of Painlevé equation



Phase transition in Ising model: connection problem for tau function

Tracy (1991) solved the connection problem for tau function

$$\exp\left(-\int_{t}^{+\infty}\left(\frac{H}{2}+\frac{e^{t}}{8}\right)dt\right)\simeq e^{\frac{t}{8}}A^{-3}e^{\frac{1}{4}}2^{-\frac{1}{6}},\quad t\to-\infty$$

where A is Glaisher-Kinkelin constant.
He approximated solution appearing above with the family

$$q(t)\simeq at-6a\ln(2)+2\ln\frac{\Gamma\left(\frac{1-a}{2}\right)}{\Gamma\left(\frac{1+a}{2}\right)},\quad t\to-\infty,$$

$$q(t)=-2\sin\left(\frac{\pi a}{2}\right)\sqrt{\frac{2}{\pi}}\exp\left(-\frac{t}{4}\right)\exp\left(-e^{\frac{t}{2}}\right)(1+0(1)),\quad t\to+\infty.$$

with $-1<a<0$.
The tau function for this family has Fredholm determinant representation.

Other connection problems for tau function

- Painlevé-V-deg equation, Basor, Tracy, (1991)
- Gap probability at the bulk, Sine kernel Fredholm determinant, Painlevé-V equation, Deift, Its, Krasovsky, Zhou (2007)
- Gap probability at the soft edge, Airy kernel Fredholm determinant, Painlevé-II equation, Deift, Its, Krasovsky, (2008)
- Gap probability at the hard edge, Bessel kernel Fredholm determinant, Painlevé-III equation, Deift, Krasovsky, Vasilevska, (2010)

Connection problem for tau function and conformal field theory

- Gamayun, Iorgov, Lisovyy (2012), (2013), Bershtein, Shchechkin(2015): Painlevé tau functions are Fourier transforms of conformal blocks.
- Iorgov, Lisovyy Tykhyy (2013): Painlevé VI tau function connection problem
- Its, Lisovyy, Tykhyy (2014): Painlevé III (D8) tau function connection problem

Isomonodromic tau function: recent developments on connection problem

- Definition of isomonodromic tau function Jimbo, Miwa, Ueno (1981). For Painlevé equations it coincides with integral of Hamiltonian.
- Properties of isomonodromic tau function: analiticity, meaning of zeros: Miwa (1981), Malgrange (1982), Palmer (1999).
- Monodromy dependence based on work by Malgrange: Bertola (2010,2016)
- Application of Bertola-Malgrange construction for connection problem for tau function. Its localisation. Discovery of relation between tau function and the action: Its, Prokhorov (2016).
- Further development of localised Bertola-Malgrange construction. Application to connection problems: Its, Lisovyy, Prokhorov (2018)
- Relation between action and tau function for all Painlevé equations: Its, Prokhorov (2018)
- Use of relation between tau-function and action to solve connection problem for tau function: Bothner, Its, Prokhorov (2019)
- Improvement of relation between tau-function and action: Bothner, Warner (2019)
- Derivation of relation between tau-function and action based on quasihomogeneity of Hamiltonians: Prokhorov, to appear.

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Connection problem for Painlevé-III(D8) tau function.

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Description of solutions

We consider generic complex valued family of solutions of PIII(F) with
$\alpha = -\frac{1}{8}, \quad \beta = \frac{1}{8}, \quad \gamma = \delta = 0$

$$\frac{d^2 q}{dt^2} = -\frac{1}{4} e^t \sinh q. \tag{1}$$

It is parametrised by numbers,

$$k \in \mathbb{Z},\ \sigma, \eta \in \mathbb{C}:\ 0 < \mathrm{Re}\sigma < \frac{1}{2},\ 0 < \mathrm{Re}\eta < \frac{1}{2}, \quad \left| \arg \frac{\sin 2\pi\eta}{\sin 2\pi\sigma} \right| < \frac{\pi}{2} \tag{2}$$

We specify the following behavior at $t = -\infty$,

$$q(t) = at + b + O\left(e^{1 - |\mathrm{Re}\, a|}\right), \quad t \to -\infty,$$

with

$$a = 1 - 4\sigma, \quad b = \pi i - 2\pi i k - 4\pi i\eta - (2 - 8\sigma)\ln 8 - 2\ln \frac{\Gamma(1 - 2\sigma)}{\Gamma(2\sigma)},$$

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Description of solutions

The behavior of the solution as $t \to +\infty$ was obtained in 1985 by Novokshenov. To describe it introduce parameter

$$\zeta = \frac{1}{2} + \frac{1}{2\pi}\left(ib + 6ia\ln 2\right) + \frac{i}{\pi}\ln\frac{\Gamma\left(\frac{1}{2}+\frac{a}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{a}{2}\right)}.$$

We assume that $\zeta \notin \mathbb{Z}$. Then the large t behavior of $q(t)$ is oscillatory, and it is given by the formulae,

$$q(t) = c^{+}_{0,0}\exp\left(i\exp\left(\frac{t}{2}\right) + \frac{i\nu t}{2} - \frac{t}{4}\right)\left(1 + O\left(\exp\left(-\frac{t}{2}\right)\right)\right)$$
$$+c^{-}_{0,0}\exp\left(-i\exp\left(\frac{t}{2}\right) - \frac{i\nu t}{2} - \frac{t}{4}\right)\left(1 + O\left(\exp\left(-\frac{t}{2}\right)\right)\right) +$$
$$-2\pi ik + O\left(\exp\left(\frac{3t}{4}(2|\mathrm{Im}\,\nu| - 1)\right)\right), \quad t \to \infty,$$

$$e^{\pi\nu} = \frac{\sin 2\pi\eta}{\sin 2\pi\sigma}, \quad c^{\pm}_{0,0} = ie^{\frac{\pi\nu}{2}\mp\frac{i\pi}{4}}2^{1\pm 2i\nu}\frac{1}{\sqrt{2\pi}}\Gamma(1\mp i\nu)\frac{\sin 2\pi(\sigma\mp\eta)}{\sin 2\pi\eta}, \quad \nu = \frac{1}{4}c^{+}_{0,0}c^{-}_{0,0}$$

and $\Gamma(z)$ is Euler's Gamma-function. We denoted by $\{\ldots\}$ the fractional part.

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Description of solutions

We will need more detailed asymptotics at $+\infty$

$$q(t) \simeq \sum_{l \geq k \geq 0,\ \epsilon = \pm} c_{k,l}^{\epsilon} r^{\epsilon(2k+1)} \zeta^{2l+2k+1}.$$

$$r = \exp\left(i \exp\left(\frac{t}{2}\right) + \frac{i\nu t}{2}\right), \quad \zeta = \exp\left(-\frac{t}{4}\right)$$

$$c_{0,1}^{\pm} = \pm \frac{i c_{0,0}^{\pm}}{8}(6\nu^2 \pm 4i\nu - 1) \quad c_{1,0}^{\pm} = \frac{1}{48}(c_{0,0}^{\pm})^3$$

It can be justified using nonlinear steepest descent.

Asymptotic of tau function

We remind that tau function is given by

$$\ln \tau(t_1, t_2, \sigma, \eta) = \int_{t_1}^{t_2} H(t)dt, \tag{3}$$

where the Hamiltonian is given by

$$H(p, q, t) = \frac{p^2}{2} + \frac{e^t \cosh(q)}{4}, \quad p = \frac{dq}{dt}.$$

Using the asymptotic of $q(t)$ we arrive at the following asymptotic representation of the tau function as $t_1 \to -\infty$, $t_2 \to +\infty$

$$\ln \tau(t_1, t_2, \sigma, \eta) \simeq \frac{e^{t_2}}{4} + 4\nu e^{\frac{t_2}{2}} + \frac{\nu^2 t_2}{2} - \frac{a^2 t_1}{2} + \ln \Upsilon \tag{4}$$

To solve connection problem means to evaluate constant term in the asymptotics, which we called Υ.

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Relation between tau-function and action

We remind the formula for Hamiltonian

$$H = \frac{p^2}{2} - \alpha e^{t+q} + \beta e^{t-q}.$$

It is quasihomogeneous

$$H(\lambda p, q, t + 2\ln\lambda) = \lambda^2 H(p, q, t).$$

Taking derivative with respect to λ and putting $\lambda = 1$ we get

$$p\frac{\partial H}{\partial p} + 2\frac{\partial H}{\partial t} = 2H.$$

We can rewrite it as

$$H = \left(p\frac{dq}{dt} - H\right) + 2\frac{dH}{dt}$$

or in other words

$$\ln\tau(t_1, t_2, \sigma, \eta) = 2H|_{t_1}^{t_2} + S(t_1, t_2, \sigma, \eta)$$

where

$$S(t_1, t_2, \sigma, \eta) = \int_{t_1}^{t_2} \left(p\frac{dq}{dt} - H\right) dt$$

is the classical action.

Differential identity and alternative integral representation for classical action

Differentiating with respect to parameter ρ we get

$$\frac{\partial S}{\partial \rho} = \int_{t_1}^{t_2} \left(\frac{\partial p}{\partial \rho} \frac{dq}{dt} + p \frac{d}{dt} \left(\frac{\partial q}{\partial \rho} \right) - \frac{\partial H}{\partial p} \frac{\partial p}{\partial \rho} - \frac{\partial H}{\partial q} \frac{\partial q}{\partial \rho} \right) dt$$

$$= p \frac{\partial q}{\partial \rho} \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial p}{\partial \rho} \frac{dq}{dt} - \frac{\partial q}{\partial \rho} \frac{dp}{dt} - \frac{\partial H}{\partial p} \frac{\partial p}{\partial \rho} - \frac{\partial H}{\partial q} \frac{\partial q}{\partial \rho} \right) dt = p \frac{\partial q}{\partial \rho} \Big|_{t_1}^{t_2}$$

Therefore we can write

$$S(t_1, t_2, \sigma, \eta) = S\left(t_1, t_2, \frac{1}{4}, \frac{1}{4}\right) + \int_{\left(\frac{1}{4}, \frac{1}{4}\right)}^{(\sigma, \eta)} \left(p \frac{\partial q}{\partial \sigma} d\sigma + p \frac{\partial q}{\partial \eta} d\eta \right) \Big|_{t_1}^{t_2},$$

We picked the reference point $\sigma = \eta = \frac{1}{4}$, which corresponds to solution $q(t) \equiv 0$. We have

$$S\left(t_1, t_2, \frac{1}{4}, \frac{1}{4}\right) = \frac{1}{4}(e^{t_1} - e^{t_2}).$$

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Integral representation for connection constant

As the result we have following formula

$$\ln \tau(t_1, t_2, \sigma, \eta) = 2H\Big|_{t_1}^{t_2} + \frac{1}{4}(e^{t_1} - e^{t_2}) + \int_{(\frac{1}{4},\frac{1}{4})}^{(\sigma,\eta)} \left(p\frac{\partial q}{\partial \sigma} d\sigma + p\frac{\partial q}{\partial \eta} d\eta \right)\Big|_{t_1}^{t_2}.$$

Using the asymptotic of $q(t)$ we obtain the following representation for connection constant Υ.

$$\ln \Upsilon = 2\nu^2 - a^2 - 2i\nu + \int_{(\frac{1}{4},\frac{1}{4})}^{(\sigma,\eta)} (ic_{0,0}^{+} dc_{0,0}^{-} - a db).$$

Connection constant for Painlevé-III(D8) tau function

Skipping the computation of the integral we get the formula for connection constant

Theorem 1 (Its, Prokhorov, 2016)

Let σ and η be the “monodromy” parameters of the solution $q(t)$ of Painlevé III(D8)(F) equation (1) satisfying the inequalities (2). Then the tau function (3) has the behavior (4) as $t_1 \to -\infty$, $t_2 \to +\infty$ with

$$\Upsilon = (2\pi)^{2i\nu} 2^{4\nu^2+48\sigma^2-24\sigma} e^{4\pi i(\eta^2-2\sigma\eta-\sigma^2+2\eta-\sigma)} \left(\frac{\Gamma(1-2\sigma)}{\Gamma(2\sigma)}\right)^2 \left(\frac{G(1+i\nu)G(1+2\sigma)G(1-2\sigma)\hat{G}(\sigma+\eta+\frac{1-i\nu}{2})}{\hat{G}(\sigma+\eta+\frac{1+i\nu}{2})}\right)^4 \frac{(-8i)}{\pi^2(G(\frac{1}{2}))^8}, \tag{5}$$

where ν is defined in (43), $G(z)$ is the Barnes G – function, and $\hat{G}(z) = \dfrac{G(1+z)}{G(1-z)}$,.

Isomonodromic deformations

System of ODE with rational coefficients

Consider the system of linear differential equations with rational coefficients with $n+1$ singularities at $a_1, \dots, a_n, a_\infty = \infty$ on $\hat{\mathbb{C}}$. It can be written as

$$\frac{d\Phi}{dz} = A(z)\Phi, \qquad A(z) = \sum_{\nu=1}^{n} \sum_{k=1}^{r_\nu+1} \frac{A_{\nu,-k+1}}{(z-a_\nu)^k} - \sum_{k=0}^{r_\infty-1} z^k A_{\infty,-k-1}. \tag{6}$$

We shall also assume that all highest order matrix coefficients $A_\nu \equiv A_{\nu,-r_\nu}$ are diagonalizable

$$A_{\nu,-r_\nu} = G_\nu \Theta_{\nu,-r_\nu} G_\nu^{-1}; \quad \Theta_{\nu,-r_\nu} = \operatorname{diag}\left\{\theta_{\nu,1}, \dots \theta_{\nu,N}\right\},$$

and that their eigenvalues are distinct and non-resonant:

$$\begin{cases} \theta_{\nu,\alpha} \neq \theta_{\nu,\beta} & \text{if} \quad r_\nu \geq 1, \quad \alpha \neq \beta, \\ \theta_{\nu,\alpha} \neq \theta_{\nu,\beta} \mod \mathbb{Z} & \text{if} \quad r_\nu = 0, \quad \alpha \neq \beta. \end{cases}$$

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Canonical solutions

Let a_ν be an irregular singular point of index r_ν. For $j = 1, \dots, 2r_\nu + 1$, let

$$\Omega_{j,\nu} = \left\{ z : 0 < |z - a_\nu| < \epsilon, \quad \theta_j^{(1)} < \arg(z - a_\nu) < \theta_j^{(2)}, \quad \theta_j^{(2)} - \theta_j^{(1)} = \frac{\pi}{r_\nu} + \delta \right\},$$

be the *Stokes sectors* around a_ν (see, e.g., Wasow, 1965 for more details). According to the general theory of linear systems, in each sector $\Omega_{j,\nu}$ there exists a unique *canonical solution* $\Phi_j^{(\nu)}(z)$ of (6) which satisfies the asymptotic condition

$$\Phi_j^{(\nu)}(z) \simeq \Phi_{\text{form}}^{(\nu)}(z) \qquad \text{as } z \to a_\nu, \quad z \in \Omega_{j,\nu}, \quad j = 1, \dots, 2r_\nu + 1, \tag{7}$$

where $\Phi_{\text{form}}^{(\nu)}(z)$ is the formal solution at the point a_ν

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Formal solutions

$$\Phi^{(\nu)}_{\mathrm{form}}(z) = G^{(\nu)}(z)\, e^{\Theta_\nu(z)}, \qquad G^{(\nu)}(z) = G_\nu \hat{\Phi}^{(\nu)}(z), \tag{8}$$

where

$$\hat{\Phi}^{(\nu)}(z) = \begin{cases} I + \sum_{k=1}^{\infty} g_{\nu,k}(z-a_\nu)^k, & \nu = 1,\dots,n, \\ I + \sum_{k=1}^{\infty} g_{\infty,k} z^{-k}, & \nu = \infty, \end{cases}$$

and $\Theta_\nu(z)$ are diagonal matrix-valued functions,

$$\Theta_\nu(z) = \begin{cases} \sum_{k=-r_\nu}^{-1} \dfrac{\Theta_{\nu,k}}{k}(z-a_\nu)^k + \Theta_{\nu,0}\ln(z-a_\nu), & \nu = 1,\dots,n \\ -\sum_{k=1}^{r_\infty} \dfrac{\Theta_{\infty,-k}}{k} z^k - \Theta_{\infty,0}\ln z, & \nu = \infty. \end{cases}$$

We emphasize, that in (8) we denoted constant matrices as G_ν and matrix functions as $G^{(\nu)}$.

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Stokes and connection matrices

Stokes and connection matrices relate the canonical solutions $\Phi_j^{(\nu)}(z)$ in different Stokes sectors and at different singular points:

$$\Phi_{j+1}^{(\nu)} = \Phi_j^{(\nu)} S_j^{(\nu)}, \quad j = 1, \ldots, 2r_\nu, \qquad \Phi_1^{(\nu)} = \Phi_1^{(\infty)} C_\nu, \quad \nu = 1, \ldots, n.$$

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Riemann-Hilbert correspondence

$$\mathcal{A} = \{a_\nu \in \mathbb{C},\ A_{\nu,-k+1},\ A_{\infty,-j-1},\ \Theta_{\nu,-r_\nu},\ \Theta_{\infty,-r_\infty} \in \mathfrak{sl}_N(\mathbb{C}),\ G_\nu \in SL_N(\mathbb{C}),$$
$$k = 1 \dots r_\nu,\ j = 0 \dots r_\infty - 2,\ \nu = 1 \dots n\}/\sim$$
$$\mathcal{M} = \left\{S_j^{(\nu)},\ \Theta_{\nu,0} \in \mathfrak{sl}_N(\mathbb{C}),\ C_\mu \in SL_N(\mathbb{C}) : j = 1 \dots 2r_\nu,\right.$$
$$\left.\nu = 1, \dots, n, \infty;\ \mu = 1, \dots, n\ \right\} / \sim$$
$$\mathcal{T} = \{a_\mu,\ \Theta_{\nu,k} \in \mathfrak{sl}_N(\mathbb{C}),\ k = -r_\nu, \dots, -1;\ \nu = 1, \dots, n, \infty;\ \mu = 1, \dots, n\}/\sim$$

The so-called Riemann-Hilbert correspondence states that, up to submanifolds where the inverse monodromy problem for (6) is not solvable, the space $\mathcal{A}$ can be identified with the product $\widetilde{\mathcal{T}} \times \mathcal{M}$, where $\widetilde{\mathcal{T}}$ denotes the universal covering of $\mathcal{T}$. We shall loosely write,

$$\mathcal{A} \simeq \widetilde{\mathcal{T}} \times \mathcal{M}.$$

Isomonodromic deformations

We denote by $A(z) \equiv A\left(z; \vec{t}; M\right)$ the isomonodromic family having the same set $M \in \mathcal{M}$ of monodromy data. The isomonodromy implies that the corresponding solution $\Phi\left(z\right) \equiv \Phi\left(z, \vec{t}\right)$ satisfies an overdetermined system

$$\begin{cases} \partial_z \Phi \ = A\left(z, \vec{t}\right) \Phi\left(z, \vec{t}\right), \\ d_{\mathcal{T}} \Phi = U\left(z, \vec{t}\right) \Phi\left(z, \vec{t}\right) \end{cases} \tag{9}$$

The coefficients of the matrix-valued differential form $U \equiv \sum_{k=1}^{L} U_k\left(z, \vec{t}\right) dt_k$ are rational in z. The compatibility of the system (9) implies the monodromy preserving deformation equation:

$$d_{\mathcal{T}} A = \partial_z U + [U, A]. \tag{10}$$

Isomonodromic tau function

Let us recall the standard definition of the Jimbo-Miwa-Ueno differential

$$\omega_{\mathrm{JMU}} = -\sum_{\nu=1,\dots,n,\infty} \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\hat{\Phi}^{(\nu)}(z)^{-1}\,\partial_z\hat{\Phi}^{(\nu)}(z)\; d_{\mathcal{T}}\Theta_\nu(z)\right). \tag{11}$$

It was shown by Jimbo, Miwa, Ueno (1981) that this 1-form is closed on solutions of the isomonodromy equation (10):

$$d_{\mathcal{T}}\omega_{\mathrm{JMU}} = 0.$$

Therefore one can integrate it and define Jimbo-Miwa-Ueno isomonodromic tau function by

$$\ln\tau(\vec{t_1},\vec{t_2},M) = \int_{\vec{t_1}}^{\vec{t_2}} \omega_{\mathrm{JMU}} \tag{12}$$

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Monodromy dependence of tau function

Bertola (2010):

$$d_{\mathcal{M}} \ln \tau(t_1, t_2, M) = \frac{1}{4\pi i} \int_{\Gamma} \mathrm{Tr} \left(\Psi_-^{-1} \Psi_-' \partial_{\mathcal{M}} J J^{-1} + \Psi_+^{-1} \Psi_+' J^{-1} \partial_{\mathcal{M}} J \right) dz \Big|_{t_1}^{t_2}$$

Lemma 1 (Its, Lisovyy, Prokhorov (2018))

$$d_{\mathcal{T}_2} \ln \tau(t_1, t_2, M) = \sum_{\nu=1,\dots,n,\infty} \mathrm{res}_{z=a_\nu} \mathrm{Tr} \left(G^{(\nu)}(z)^{-1} A(z)\, d_{\mathcal{T}_2} G^{(\nu)}(z) \right),$$

$$d_{\mathcal{M}} \ln \tau(t_1, t_2, M) = \sum_{\nu=1,\dots,n,\infty} \mathrm{res}_{z=a_\nu} \mathrm{Tr} \left(G^{(\nu)}(z)^{-1} A(z)\, d_{\mathcal{M}} G^{(\nu)}(z) \right) \Big|_{t_1}^{t_2}.$$

Symplectic structure on $\mathcal{M}$

Consider 1-form $\omega \in \Lambda^1\left(\widetilde{\mathcal{T}} \times \mathcal{M}\right)$ given by

$$\omega = \sum_{\nu=1,\ldots,n,\infty} \operatorname{res}_{z=a_\nu} \operatorname{Tr}\left(A(z)\; dG^{(\nu)}(z)\; G^{(\nu)}(z)^{-1}\right), \tag{13}$$

where $d = d_{\mathcal{T}} + d_{\mathcal{M}}$.

Lemma 2 (Its, Lisovyy, Prokhorov (2018))

Form $d\omega$ is closed form on $\mathcal{M}$ independent on $\mathcal{T}$.

Conjecture 1 (Its, Lisovyy, Prokhorov (2018))

Form $d\omega$ is nondegenerate form on $\mathcal{M}$.

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Relation to action

The *classical action differential* can be defined as the differential form on $\widetilde{\mathcal{T}} \times \mathcal{M}$,

$$\omega_{\text{cla}} = \sum p_j dq_j - \sum H_k dt_k \equiv \sum_k \left(\sum_j p_j \frac{\partial q_j}{\partial t_k} - H_k \right) dt_k + \sum_k \left(\sum_j p_j \frac{\partial q_j}{\partial m_k} \right) dm_k$$

$$d\omega_{\mathrm{cla}} = \sum_j d_{\mathcal{M}} p_j \wedge d_{\mathcal{M}} q_j.$$

Conjecture 2 (Its, Prokhorov (2018))

There exists a rational function $G(\vec{p}, \vec{q}, \vec{t})$ of $\vec{p}, \vec{q}, \vec{t}$ such that,

$$\omega = \omega_{\text{cla}} + dG(\vec{p}, \vec{q}, \vec{t}). \tag{14}$$

Moreover, the function $G(\vec{p}, \vec{q}, \vec{t})$ is explicitly computable.

Symplectic structure on $\mathcal{A}$

Consider the quotient space

$$\mathcal{A}_0 = \mathcal{A}/\{\mathcal{T} = \text{const}\}$$

and denote the points $f \in \mathcal{A}_0$ as

$$f = (f_1, \ldots, f_d), \quad d = \dim \mathcal{M}.$$

Introduce the differential δ on the space $\mathcal{A}_0$. Consider the form

$$\omega_a = \sum_{\nu=1,\ldots,n,\infty} \text{res}_{z=a_\nu} \operatorname{Tr}\left(G^{(\nu)}(z)^{-1} A(z)\, \delta G^{(\nu)}(z) \right). \tag{15}$$

Form $\delta\omega_a$ is closed form on $\mathcal{A}_0$.

Conjecture 3

Form $\delta\omega_a$ is nondegenerate form on $\mathcal{A}_0$.

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Hamiltonian structure

Usually

$$\frac{\partial \ln \tau \left(\vec{t}, M\right)}{\partial t_k} = H_k\big|_{A(z;\vec{t},M)}. \tag{16}$$

This identity is sensitive to the changes of coordinates and depends on the choice of symplectic form.
Let's assume that form $\delta\omega_a$ is nondegenerate and that there is a family of Hamiltonians $\{H_k\}_{k=1}^{\dim\mathcal{T}}$. The corresponding Hamiltonian vector fields X_{H_k} defined by the formula

$$\iota_{X_{H_k}}\delta\omega_a = -\delta H_k, \quad k = 1\ldots\dim\mathcal{T} \tag{17}$$

Where ι denotes the interior product. The dynamics induced by this Hamiltonians on $\mathcal{A}_0$ is described by

$$\frac{df}{dt_k} = X_{H_k}[f], \quad f \in \mathcal{A}_0, \quad k = 1\ldots\dim\mathcal{T} \tag{18}$$

Hamilonian structure

Introduce the following form

$$\Omega_k = \sum_{\nu} \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\frac{\partial A}{\partial t_k}\delta G^{(\nu)}\left(G^{(\nu)}\right)^{-1}\right) - \sum_{\nu} \mathrm{res}_{z=a_\nu} \mathrm{Tr}\left(\frac{d(\delta\Theta_\nu)}{dz}\left(G^{(\nu)}\right)^{-1}\frac{\partial G^{(\nu)}}{\partial t_k}\right) \tag{19}$$

We have the following result.

Lemma 3

Assume that dynamics (18) induced by Hamiltonians $\{H_k\}_{k=1}^{\dim \mathcal{T}}$ is isomonodromic and is described by equations (10). Then

$$\delta H_k = \Omega_k. \tag{20}$$

Conjecture 4

Form Ω_k is exact.

If this conjecture holds, it provides the formula for Hamiltonians in general case.

THANK YOU