Monodromy map under the confluence PIII(D_6) $\to$PIII(D_8).

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Overview of isomonodromic deformations

Matrix linear ODEs with rational coefficients

- Consider the system of linear differential equations with rational coefficients with $n+1$ singularities at $a_1, \dots, a_n, a_\infty = \infty$ on $\hat{\mathbb{C}}$. It can be written as

$$\frac{d\Phi}{dz} = A(z)\Phi, \qquad A(z) = \sum_{\nu=1}^{n} \sum_{k=1}^{r_\nu+1} \frac{A_{\nu,-k+1}}{(z-a_\nu)^k} + \sum_{k=0}^{r_\infty-1} z^k A_{\infty,-k-1}.$$

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- r_ν is called Poincaré rank at the point a_ν.

Space of coefficients

- We can notice that the transformation $\Phi(z) \to c(z)\Phi(z)$ with scalar function $c(z)$ results in map $A(z) \to A(z) + c'(z)c^{-1}(z)$(exercise). Choosing proper $c(z)$ we can guarantee (exercise)

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- **For start** we shall also assume that all highest order matrix coefficients $A_\nu \equiv A_{\nu,-r_\nu}$ are diagonalizable

$$A_{\nu,-r_\nu} = G_\nu \Theta_{\nu,-r_\nu} G_\nu^{-1}; \quad \Theta_{\nu,-r_\nu} = \mathrm{diag}\left\{\theta_{\nu,1},\ldots\theta_{\nu,N}\right\},$$

and that their eigenvalues are distinct and non-resonant:

$$\begin{cases} \theta_{\nu,\alpha} \neq \theta_{\nu,\beta} & \text{if} \quad r_\nu \geq 1, \quad \alpha \neq \beta, \\ \theta_{\nu,\alpha} \neq \theta_{\nu,\beta} \mod \mathbb{Z} & \text{if} \quad r_\nu = 0, \quad \alpha \neq \beta. \end{cases}$$

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- If $r_\infty = 0$, then we define

$$A_{\infty,0} = -\sum_{\nu=1}^{n} A_{\nu,0}.$$

and make it diagonal.

Space of coefficients

- We introduce the space $\mathcal{A}$ of coefficients.

$$\mathcal{A} = \{a_\nu \in \mathbb{C},\ A_{\nu,-k+1},\ A_{\infty,-j-1},\ \Theta_{\nu,-r_\nu},\ \Theta_{\infty,-r_\infty} \in \mathfrak{sl}_N(\mathbb{C}),$$

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- As the result we have the following formula for dimension of $\mathcal{A}$ (exercise)

$$\dim \mathcal{A} = n + (N^2 - 1)\left(\sum_{\nu=1}^{n} r_\nu + r_\infty - 1\right) + (N-1)(n+1)$$
$$+ n(N^2 - 1) - 2$$

Formal solutions

- The differential equation has formal solutions with the asymptotic

$$\Phi^{(\nu)}_{\mathrm{form}}(z) \simeq G_\nu \hat{\Phi}^{(\nu)}(z)\, e^{\Theta_\nu(z)} \quad \text{as } z \to a_\nu,$$

where

$$\hat{\Phi}^{(\nu)}(z) = \begin{cases} I + \sum_{k=1}^{\infty} g_{\nu,k}(z-a_\nu)^k, & \nu = 1,\dots,n, \\ I + \sum_{k=1}^{\infty} g_{\infty,k} z^{-k}, & \nu = \infty, \end{cases}$$

and $\Theta_\nu(z)$ are diagonal matrix-valued functions,

$$\Theta_\nu(z) = \begin{cases} \sum_{k=-r_\nu}^{-1} \frac{\Theta_{\nu,k}}{k}(z-a_\nu)^k + \Theta_{\nu,0}\ln(z-a_\nu), & \nu = 1,\dots,n \\ \sum_{k=1}^{r_\infty} \frac{\Theta_{\infty,-k}}{k} z^k + \Theta_{\infty,0}\ln z, & \nu = \infty. \end{cases}$$

Examples

- One finite point a of rank $r > 0$.

$$\frac{d\Phi}{dz} = \frac{A}{(z-a)^{r+1}}\Phi$$

Given $A = G\Theta G^{-1}$ we have for arbitrary constant matrix C(exercise)

$$\Phi(z) = G\exp\left(\frac{\Theta}{-r}(z-a)^{-r}\right)C$$

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Stokes rays

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- More precisely we denote them as

$$\ell^j_{i,k} = \{z : 0 < |z-a_\nu| < \epsilon,$$

$$\arg(z-a_\nu) = \frac{1}{r_\nu}\arg((\Theta_{\nu,-r_\nu})_{ii} - (\Theta_{\nu,-r_\nu})_{kk}) - \frac{\pi}{2r_\nu} + \frac{\pi}{r_\nu} j\Big\}$$

$$j = 1, \dots 2r_\nu$$

Stokes sectors

- For $j = 1, \ldots, 2r_\nu$, let

$$\Omega_{j,\nu} = \Big\{ z : 0 < |z - a_\nu| < \epsilon,\ \theta_j^{(1)} < \arg (z - a_\nu) < \theta_j^{(2)},$$
$$\theta_j^{(2)} - \theta_j^{(1)} = \frac{\pi}{r_\nu} + \delta \Big\},$$

be the *Stokes sectors* around a_ν

- The angles $\theta_j^{(1)}, \theta_j^{(2)}$ can be chosen in such a way that Stokes sector contains **exactly one** Stokes ray $\ell_{i,k}^{j}$ for each pair (i, k). (exercise)

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  be the *Stokes sectors* around a_ν
- The angles $\theta_j^{(1)}, \theta_j^{(2)}$ can be chosen in such a way that Stokes sector contains **exactly one** Stokes ray $\ell_{i,k}^j$ for each pair (i, k). (exercise)
- Parameter δ can be chosen small enough so the intersection $\Omega_{j,\nu} \cap \Omega_{j+1,\nu}$ does not contain any Stokes sectors

Solutions in the Stokes sectors

- It can be shown that in each Stokes sector $\Omega_{j,\nu}$ there is a canonical solution $\Phi_j^{(\nu)}(z)$ with asymptotic $\Phi_j^{(\nu)}(z) \simeq \Phi_{\text{form}}^{(\nu)}(z)$.

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- Let's show that it is **unique**. Assume there are two solutions $\Phi_j^{(\nu)}(z)$ and $\widetilde{\Phi}_j^{(\nu)}(z)$. We have

$$\left(\widetilde{\Phi}_j^{(\nu)}(z)\right)^{-1} \Phi_j^{(\nu)}(z) = e^{\frac{\Theta_{\nu,-r_\nu}}{r_\nu}(z-a_\nu)^{-r_\nu}} (I+O(z-a_\nu)) e^{-\frac{\Theta_{\nu,-r_\nu}}{r_\nu}(z-a_\nu)^{-r_\nu}}$$

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- Since the Stokes sector contains Stokes ray, we can chose direction in which we compute asymptotic for different entries in such a way that

$$\mathrm{Re}(((\Theta_{\nu,-r_\nu})_{ii} - (\Theta_{\nu,-r_\nu})_{kk})(z-a_\nu)^{-r_\nu}) < 0$$

for all i, k.(exercise)

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- As the result

$$\left(\widetilde{\Phi}_j^{(\nu)}(z)\right)^{-1}\Phi_j^{(\nu)}(z) = I$$

Stokes and connection matrices

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- Stokes and connection matrices relate the canonical solutions $\Phi_j^{(\nu)}(z)$ in different Stokes sectors and at different singular points:

$$\Phi_{j+1}^{(\nu)} = \Phi_j^{(\nu)} S_j^{(\nu)}, \quad j = 1, \dots, 2r_\nu, \qquad \Phi_1^{(\nu)} = \Phi_1^{(\infty)} C_\nu, \quad \nu = 1, \dots, n.$$

Stokes and connection matrices

- For Stokes matrices we have the formula

$$S_j^{(\nu)} = \left(\Phi_j^{(\nu)}(z)\right)^{-1} \Phi_{j+1}^{(\nu)}(z)$$

$$= e^{\frac{\Theta_{\nu,-r_\nu}}{r_\nu}(z-a_\nu)^{-r_\nu}} (I + O(z-a_\nu)) e^{-\frac{\Theta_{\nu,-r_\nu}}{r_\nu}(z-a_\nu)^{-r_\nu}}$$

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  for $z \in \Omega_{j,\nu} \cap \Omega_{j+1,\nu}$

- Since there is no Stokes rays in the intersection $\Omega_{j,\nu} \cap \Omega_{j+1,\nu}$, the expression $\mathrm{Re}(((\Theta_{\nu,-r_\nu})_{ii} - (\Theta_{\nu,-r_\nu})_{kk})(z - a_\nu)^{-r_\nu})$ does not change sign in it.
- That means that we can take limit $z \to a_\nu$ for $\frac{N(N-1)}{2}$ entries of $S_j^{(\nu)}$ and obtain zero.

Properties of solution

- Plugging the asymptotic formula in the differential equation we get (exercise)

$$A(z) = G_\nu \hat{\Phi}^{(\nu)}(z) \frac{d\Theta_\nu(z)}{dz} \left(\hat{\Phi}^{(\nu)}(z)\right)^{-1} G_\nu^{-1}$$

$$+ \begin{cases} O(1), & \nu = 1, \ldots, n, \\ O\left(z^{-2}\right), & \nu = \infty. \end{cases}$$

- The property $\mathrm{Tr}(A(z)) = 0$, implies that $\mathrm{Tr}(\Theta_\nu(z)) = 0$.

Properties of solution

- Using the Liouville's formula

$$\det(\Phi(z)) = \det(\Phi(z_0)) \exp\left(\int_{z_0}^{z} \mathrm{Tr}\,(A(s))\, ds \right)$$

  and identities

$$\det(G_\nu) = 1, \quad \mathrm{Tr}\,(A(z)) = 0$$

  we deduce that $\det(\Phi(z)) = 1$.
- Property $\det(\Phi(z)) = 1$ implies $\det(C_\nu)) = 1$.

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- To remove the extra argument around infinity we multiply the obtained earlier solution by

$$e^{-2\pi i \Theta_{\infty,0}}$$

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- To remove the argument coming from finite point a_1 we multiply the obtained earlier solution by

$$C_1 e^{2\pi i \Theta_{1,0}} \left(S^{(1)}_{2r_1}\right)^{-1} \left(S^{(1)}_{2r_1-1}\right)^{-1} \dots \left(S^{(1)}_{1}\right)^{-1} C_1^{-1}$$

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- After removing argument around each of the finite point we get the identity called cyclic relation.

$$\begin{aligned} & S_1^{(\infty)} S_2^{(\infty)} \dots S_{2r_\infty}^{(\infty)} e^{-2\pi i \Theta_{\infty,0}} C_1 e^{2\pi i \Theta_{1,0}} \left(S_{2r_1}^{(1)}\right)^{-1} \left(S_{2r_1-1}^{(1)}\right)^{-1} \\ & \dots \left(S_1^{(1)}\right)^{-1} C_1^{-1} C_2 e^{2\pi i \Theta_{2,0}} \left(S_{2r_2}^{(2)}\right)^{-1} \left(S_{2r_2-1}^{(2)}\right)^{-1} \dots \left(S_1^{(2)}\right)^{-1} C_2^{-1} \dots \\ & \times C_n e^{2\pi i \Theta_{n,0}} \left(S_{2r_n}^{(n)}\right)^{-1} \left(S_{2r_n-1}^{(n)}\right)^{-1} \dots \left(S_1^{(n)}\right)^{-1} C_n^{-1} = I \end{aligned}$$

Monodromy data

- We introduce monodromy data

$$\mathcal{M} = \Big\{ S_j^{(\nu)},\, \Theta_{\nu,0} \in \mathfrak{sl}_N\left(\mathbb{C}\right),\, C_\mu \in SL_N(\mathbb{C}) : j = 1 \dots 2r_\nu,$$
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$$\left.\nu=1,\ldots,n,\infty;\ \mu=1,\ldots,n\ \right\}/\sim$$

- We compute its dimension (exercise)

$$\dim\mathcal{M}=\left(\sum_{\nu=1}^{n}2r_\nu+2r_\infty\right)\frac{N(N-1)}{2}+(n+1)(N-1)$$
$$+n(N^2-1)-(N^2-1)$$

Isomonodromic times

- Introduce now the set of times

$$\mathcal{T} = \{a_\mu,\, \Theta_{\nu,k} \in \mathfrak{sl}_N(\mathbb{C}),\, k = -r_\nu, \dots, -1;\, \nu = 1, \dots, n,$$

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$$\infty;\ \mu = 1, \dots, n\}/\sim$$

- We put two constraints on this set using change of variable $z \to \alpha z + \beta$
- We have the following formula for the dimension (exercise)

$$\dim \mathcal{T} = n + \left(\sum_{\nu=1}^{n} r_\nu + r_\infty \right) (N-1) - 2$$

Riemann-Hilbert correspondence

- The so-called Riemann-Hilbert correspondence states that, up to the points where the inverse monodromy problem is not solvable, the space $\mathcal{A}$ can be identified with the product $\widetilde{\mathcal{T}} \times \mathcal{M}$, where $\widetilde{\mathcal{T}}$ denotes the universal covering of $\mathcal{T}$.

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- In particular (exercise)

$$\dim \mathcal{A} = \dim \mathcal{T} + \dim \mathcal{M}$$

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    - $\Phi_+(z) = \Phi_-(z) S_j^{(\nu)}$ or $\Phi_+(z) = \Phi_-(z) C_\nu$ or $\Phi_+(z) = \Phi_-(z) e^{2\pi i \Theta_{\nu,0}}$ on different parts of contour Γ. Here $+$ denotes left side of the contour, while $-$ denotes the right side of the contour.
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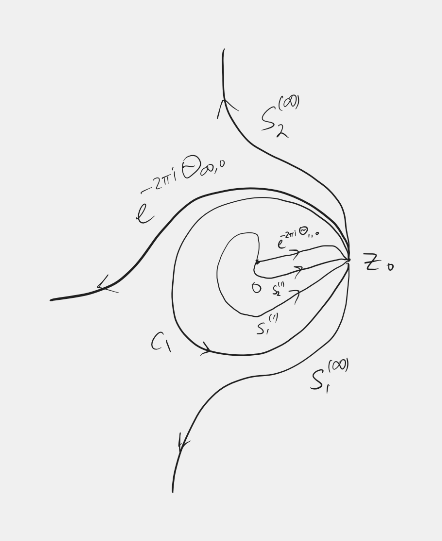
  - $\Phi(z) \simeq \Phi^{(\nu)}_{\mathrm{form}}(z), \quad z \to a_\nu.$
- Solution of Riemann-Hilbert problem is unique. Actually, given two different soltions $\Phi(z)$ and $\widetilde{\Phi}(z)$ we can notice that $\Phi(z)\widetilde{\Phi}^{-1}(z)$ is analytic on the whole plane, and equal to identity by Liouville's theorem. (exercise)

Example



$$\Phi(z) \simeq G_0\hat{\Phi}^{(0)}(z)\, e^{-ixz^{-1}\sigma_3/2} z^{\Theta_0\sigma_3/2},$$

$$z \to 0.$$

$$r_0 = 1$$

$$\Phi(z) \simeq \hat{\Phi}^{(\infty)}(z)\, e^{ixz\sigma_3/2} z^{(-\Theta_\infty - 1)\sigma_3/2},$$

$$z \to \infty$$

$$r_\infty = 1$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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- The function $\Phi\left(z\right) \equiv \Phi\left(z, \vec{t}\right)$ satisfies an overdetermined system

$$\begin{cases} \frac{\partial \Phi}{\partial z} \ = A\left(z, \vec{t}\right)\Phi\left(z, \vec{t}\right), \\ \frac{\partial \Phi}{\partial t_i} = U_i\left(z, \vec{t}\right)\Phi\left(z, \vec{t}\right), \quad i = 1, \ldots \dim(\mathcal{T}) \end{cases}$$

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- The compatibility of this system implies the monodromy preserving deformation equation:

$$\frac{\partial A}{\partial t_i} = \frac{\partial U_i}{\partial z} + [U_i, A], \quad i = 1, \dots \dim(\mathcal{T})$$

Painlevé equations

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$$
\begin{aligned}
&n = 0, \quad r_\infty = 3, && \text{(Painlevé II (JM))}\\
&n = 1, \quad r_1 = 1, \quad r_\infty = 1, && \text{(Painlevé III}(D_6)\text{)}\\
&n = 1, \quad r_1 = 0, \quad r_\infty = 2, && \text{(Painlevé IV)}\\
&n = 2, \quad r_1 = r_2 = 0, \quad r_\infty = 1, && \text{(Painlevé V)}\\
&n = 3, \quad r_1 = r_2 = r_3 = r_\infty = 0, && \text{(Painlevé VI)}
\end{aligned}
$$

Painlevé equations

- We can also write the structure of corresponding matrix linear ODEs.

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$$A(z) = A_{\infty,-3}z^2 + A_{\infty,-2}z + A_{\infty,-1}, \qquad \text{(Painlevé II (JM))}$$

$$A(z) = \frac{A_{0,1}}{z^2} + \frac{A_{0,0}}{z} + A_{\infty,-1} \qquad \text{(Painlevé III}(D_6))$$

$$A(z) = \frac{A_{0,0}}{z} + A_{\infty,-1} + A_{\infty,-2}z \qquad \text{(Painlevé IV)}$$

$$A(z) = \frac{A_{1,0}}{z-1} + \frac{A_{0,0}}{z} + A_{\infty,-1}, \qquad \text{(Painlevé V)}$$

$$A(z) = \frac{A_{3,0}}{z-x} + \frac{A_{1,0}}{z-1} + \frac{A_{0,0}}{z}, \qquad \text{(Painlevé VI)}$$

Resonant cases

- Consider $N = 2$. If the leading coefficient of $A(z)$ at a_ν is nilpotent then corresponding rank is substracted by $\frac{1}{2}$.

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- The isomonodromic deformations in resonant cases produce the following Painlevé equations.

$$n = 0, \quad r_\infty = \frac{5}{2}, \qquad \text{(Painlevé I)}$$

$$n = 0, \quad r_\infty = \frac{3}{2}, \qquad \text{(Painlevé II (FN))}$$

$$n = 1, \quad r_1 = 1, \quad r_\infty = \frac{1}{2}, \qquad \text{(Painlevé III}(D_7)\text{)}$$

$$n = 1, \quad r_1 = \frac{1}{2}, \quad r_\infty = \frac{1}{2}, \qquad \text{(Painlevé III}(D_8)\text{)}$$

$$n = 2, \quad r_1 = r_2 = 0, \quad r_\infty = \frac{1}{2}, \qquad \text{(Painlevé V-deg)}$$

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$$A(z) = A_{\infty,-3}z^2 + A_{\infty,-2}z + A_{\infty,-1}, \quad A_{\infty,-3}^2 = 0, \qquad \text{(Painlevé I)}$$

$$A(z) = A_{\infty,-2}z + A_{\infty,-1} + \frac{A_{0,0}}{z}, \quad A_{\infty,-2}^2 = 0, \qquad \text{(Painlevé II (FN))}$$

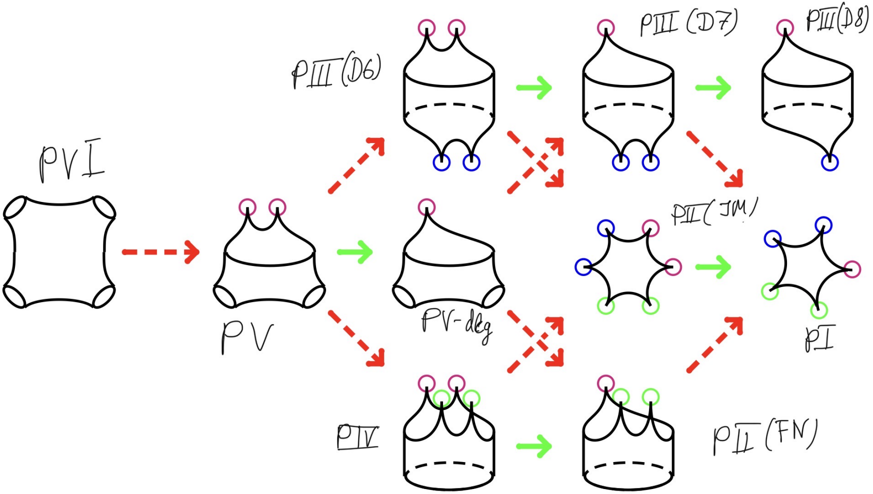
$$A(z) = \frac{A_{0,1}}{z^2} + \frac{A_{0,0}}{z} + A_{\infty,-1}, \quad A_{\infty,-1}^2 = 0, \qquad \text{(Painlevé III}(D_7))$$

$$A(z) = \frac{A_{0,1}}{z^2} + \frac{A_{0,0}}{z} + A_{\infty,-1}, \quad A_{\infty,-1}^2 = A_{0,1}^2 = 0 \qquad \text{(Painlevé III}(D_8))$$

$$A(z) = \frac{A_{1,0}}{z-1} + \frac{A_{0,0}}{z} + A_{\infty,-1}, \quad A_{\infty,-1}^2 = 0 \qquad \text{(Painlevé V-deg)}$$

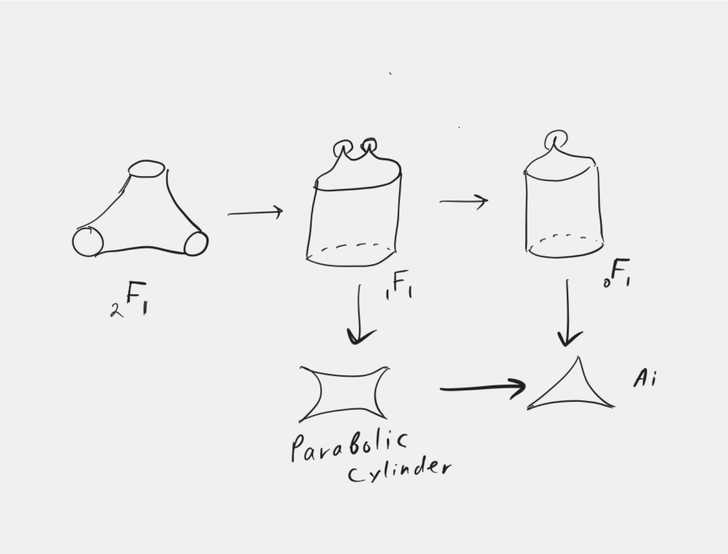
Confluence diagram

Confluence diagram for Painlevé equations



Drawing from the paper by Chekhov, Mazzocco, Rubtsov (2016).

Confluence diagram for classical special functions



Panlevé III(D6) equation

- The Painlevé-III (D6) equation is given by

$$u'' = \frac{(u')^2}{u} - \frac{u'}{x} + \frac{4\Theta_0 u^2}{x} + \frac{4\Theta_\infty}{x} + 4u^3 - \frac{4}{u}$$

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- The confluence PIII(D_6) $\to$ PIII(D_8) is described by the following limiting procedure (exercise)

$$x \to \varepsilon x, \quad \Theta_0 \to \Theta_0 + \varepsilon^{-1}, \quad \Theta_\infty \to \Theta_\infty - \varepsilon^{-1}, \quad \varepsilon \to 0.$$

where PIII(D_8) is given by

$$u'' = \frac{(u')^2}{u} - \frac{u'}{x} + \frac{4u^2}{x} + \frac{4}{x}.$$

Bäcklund transformation

Bäcklund transformation

- Introduce the following Bäcklund transformation with

$$B : (a, b, u) \to \left(a + 1, b - 1, \frac{xu' + 2xu^2 + 2bu - u + 2x}{u\left(xu' + 2xu^2 + 2au + u + 2x\right)}\right)$$

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$$(\Theta_0,\Theta_\infty)\to(\Theta_0+n,\Theta_\infty-n)$$

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- We expect that sequence of functions $u_n\left(\frac{x}{n}\right)$ models confluence PIII$(D_6)\to$ PIII(D_8).
- **General idea:** Bäcklund transformations are expected to model all other confluence maps as well.

Main results

Behavior at zero of generic solutions of $PIII(D_6)$.

- Assume that

$$u(x) \simeq \alpha_0 x^{\beta_0}$$

.

Behavior at zero of generic solutions of $PIII(D_6)$.

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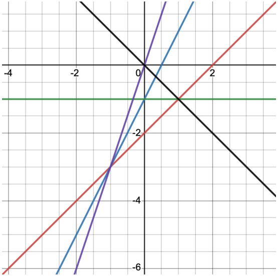
.
- Plugging it in the equation we can obtain three terms of the type $x^{\beta_0 - 2}$, and terms $x^{2\beta_0 - 1}$, $x^{3\beta_0}$, $x^{-\beta_0}$, x^{-1}
- Plotting the powers we can see that we can have cancellation of leading terms only for $-1 < \beta_0 < 1$ (exercise!)



Behavior at zero of generic solutions of $PIII(D_6)$.

Theorem (Barhoumi, Lisovyy, Miller, P.)

The behavior at zero of generic solutions of $PIII(D_6)$ is described by

$$u(x) \simeq e^{i\pi(\Theta_\infty - \Theta_0 + 2\eta)} \frac{\Gamma(1-2\mu)^2 \Gamma\left(\mu - \frac{\Theta_0}{2}\right) \Gamma\left(-\frac{\Theta_\infty}{2} + \frac{1}{2} + \mu\right)}{\Gamma(2\mu)^2 \Gamma\left(-\mu - \frac{\Theta_0}{2} + 1\right) \Gamma\left(-\frac{\Theta_\infty}{2} + \frac{1}{2} - \mu\right)} x^{4\mu - 1}$$

where $0 < \mathrm{Re}(\mu) < \frac{1}{2}$, $-\frac{1}{2} < \mathrm{Re}(\eta) < \frac{1}{2}$, $x \to 0$.

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- For $\mu = \frac{1}{4}$, $\eta = 0$, $\Theta_\infty = \Theta_0 = m$ we get $u(x) = 1$, and $u_n(x)$ become the rational solutions discussed in previous talk.

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- For $\mu = \frac{1}{4}$, $\eta = 0$, $\Theta_\infty = \Theta_0 = m$ we get $u(x) = 1$, and $u_n(x)$ become the rational solutions discussed in previous talk.
- Variables μ, η, Θ_0, Θ_∞ parametrize the **monodromy data** for $PIII(D_6)$ equation.

Bäcklund iterates

Theorem (Barhoumi, Lisovyy, Miller, P.)

The behavior at zero of Bäcklund iterates $u_n(x)$ is described by

$$u_n(x) \simeq e^{i\pi(\Theta_\infty - \Theta_0 + 2\eta_n)}$$

$$\times \frac{\Gamma(1-2|\mu_n|)^2\Gamma\left(-\frac{n}{2}+|\mu_n|-\frac{\Theta_0}{2}\right)\Gamma\left(\frac{n}{2}-\frac{\Theta_\infty}{2}+\frac{1}{2}+|\mu_n|\right)}{\Gamma(2|\mu_n|)^2\Gamma\left(-\frac{n}{2}-|\mu_n|-\frac{\Theta_0}{2}+1\right)\Gamma\left(\frac{n}{2}-\frac{\Theta_\infty}{2}+\frac{1}{2}-|\mu_n|\right)} x^{4|\mu_n|-1}$$

$$\eta_n = \begin{cases} \eta, & n \in 2\mathbb{Z}, \\ \eta+1, & n+1 \in 2\mathbb{Z} \end{cases} \quad \textit{and} \quad \mu_n = \begin{cases} \mu, & n \in 2\mathbb{Z}, \\ \mu - \frac{1}{2}, & n+1 \in 2\mathbb{Z} \end{cases}$$

- Frobenius method does not work immediately, since solution has branching at zero. (difficulty compared to rational solutions case). We use Riemann-Hilbert method.

Limiting solution

Theorem (Barhoumi, Lisovyy, Miller, P.)

We have

$$u_{2n}\left(\frac{x}{2n}\right) \to w_0(x), \quad u_{2n+1}\left(\frac{x}{2n+1}\right) \to w_1(x)$$

where $w_j(x)$ *solves the PIII*(D_8) *equation*

$$w_j'' = \frac{(w_j')^2}{w_j} - \frac{w_j'}{x} + \frac{4w_j^2}{x} + \frac{4}{x}.$$

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$$w_j'' = \frac{(w_j')^2}{w_j} - \frac{w_j'}{x} + \frac{4w_j^2}{x} + \frac{4}{x}.$$

and

$$w_j(x) \simeq e^{i\pi(\Theta_\infty - \Theta_0 + 2\eta_j)\mathrm{sign}(\mu_j)}$$
$$\times \frac{2^{1-4|\mu_j|}\Gamma(1-2|\mu_j|)^2 \sin\left(\frac{1}{2}\pi(\Theta_0 + 2|\mu_j|) + \frac{\pi j}{2}\right)}{\Gamma(2|\mu_j|)^2 \sin\left(\frac{1}{2}\pi(\Theta_0 - 2|\mu_j|) + \frac{\pi j}{2}\right)} x^{4|\mu_j|-1}$$

Further questions

- The Backlund transformation corresponding to $(\Theta_0, \Theta_\infty) \to (\Theta_0 + n, \Theta_\infty + n)$ describes confluence PIII(D_6) $\to$PII.

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- The Backlund transformation corresponding to $(\Theta_0, \Theta_\infty) \to (\Theta_0 + 2n, \Theta_\infty)$ describes confluence PIII(D_6) $\to$PIII(D_7).

Thank you!