Connection problem for the tau-function of Sine-Gordon reduction of Painlevé-III equation via Riemann-Hilbert approach.

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Equation

We study the Sine-Gordon reduction of Painlevé-III equation

$$u_{xx}+\frac{u_x}{x}+\sin u=0. \qquad (1)$$

Function $u(x)$ is a nonlinear analogue of Bessel function $J_0(x)$. After substitution $w(x)=e^{iu(x)}$ equation (1) becomes Painlevé-III equation.

Hamiltonian structure

Equation (1) can be written as a non-autonomous Hamiltonian system,

$$\frac{du}{dx}=\frac{\partial\mathscr{H}}{\partial v},\qquad \frac{dv}{dx}=-\frac{\partial\mathscr{H}}{\partial u}.$$

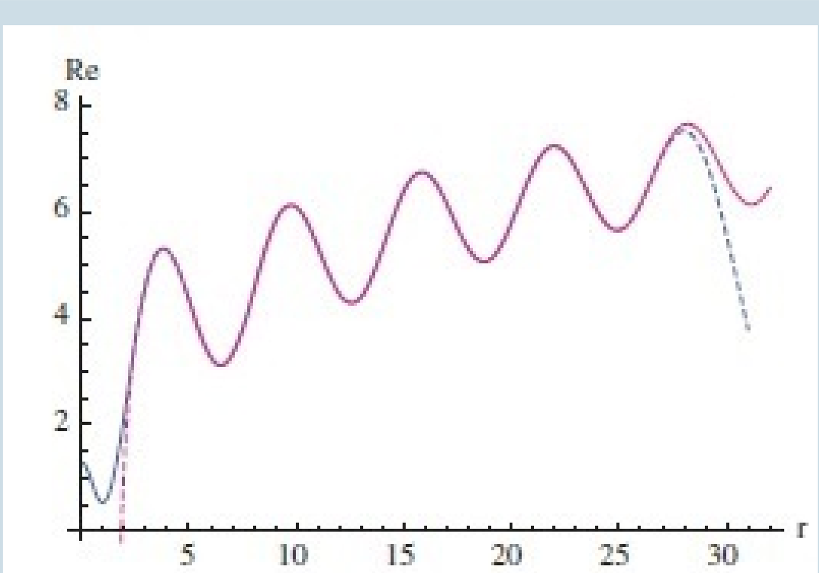
The Hamiltonian $\mathscr{H}$ is given by the formula

$$\mathscr{H}=\frac{v^2}{2x}-x\cos u.$$

Tau- function is defined by

$$\frac{d\ln\tau}{dx}=-\frac{1}{4}\mathscr{H}. \qquad (2)$$

Numerical simulation from [1]



$\sigma=0.12-0.25i$
$\eta=0.23+0.42i$
Slightly different definition of $\tau(x)$ was used in [1].

Connection formula for u(x)

The connection formula describes behaviour of u(x) at infinity in terms of its behaviour at zero. Here are the formulae (see [3])

$$u(x)=\alpha\ln x+\beta+O\left(x^{2-|\mathrm{Im}\alpha|}\right),\quad x\to 0, \qquad (3)$$

$$u(x)=b_+e^{ix}x^{i\nu-1/2}+b\_e^{-ix}x^{-i\nu-1/2}++O\left(x^{3|\mathrm{Im}\nu|-3/2}\right),\quad x\to\infty, \qquad (4)$$

$$\nu=\frac{1}{2\pi}\ln\left(\frac{\sin^2 2\pi\eta}{\sin^2 2\pi\sigma}\right),\quad b_\pm=-e^{\frac{\pi\nu}{2}\mp\frac{i\pi}{4}}2^{1\pm2i\nu}\frac{1}{\sqrt{2\pi}}\Gamma(1\mp i\nu)\frac{\sin 2\pi(\sigma\mp\eta)}{\sin 2\pi\eta},$$

$$\sigma=\frac{1}{4}+\frac{i}{8}\alpha,\quad \eta=\frac{1}{4}+\frac{1}{4\pi}\left(\beta+\alpha\ln 8\right)+\frac{i}{2\pi}\ln\left(\frac{\Gamma\left(\frac{1}{2}-\frac{i\alpha}{4}\right)}{\Gamma\left(\frac{1}{2}+\frac{i\alpha}{4}\right)}\right),$$

$\Gamma(z)$ is Euler's Gamma-function. The formulae are correct under restrictions $|\mathrm{Im}\alpha|<2, |\mathrm{Im}\nu|<1/2$. In particular, they are true for real α and β.

Connection formula for tau-function

For tau-function formulae (3),(4) imply

$$\tau(x)=C_0x^{-\frac{\alpha^2}{8}}(1+o(1)),\quad x\to 0,$$

$$\tau(x)=C_\infty x^{\nu^2}e^{\frac{x^2}{8}+2\nu x}(1+o(1)),\quad x\to\infty.$$

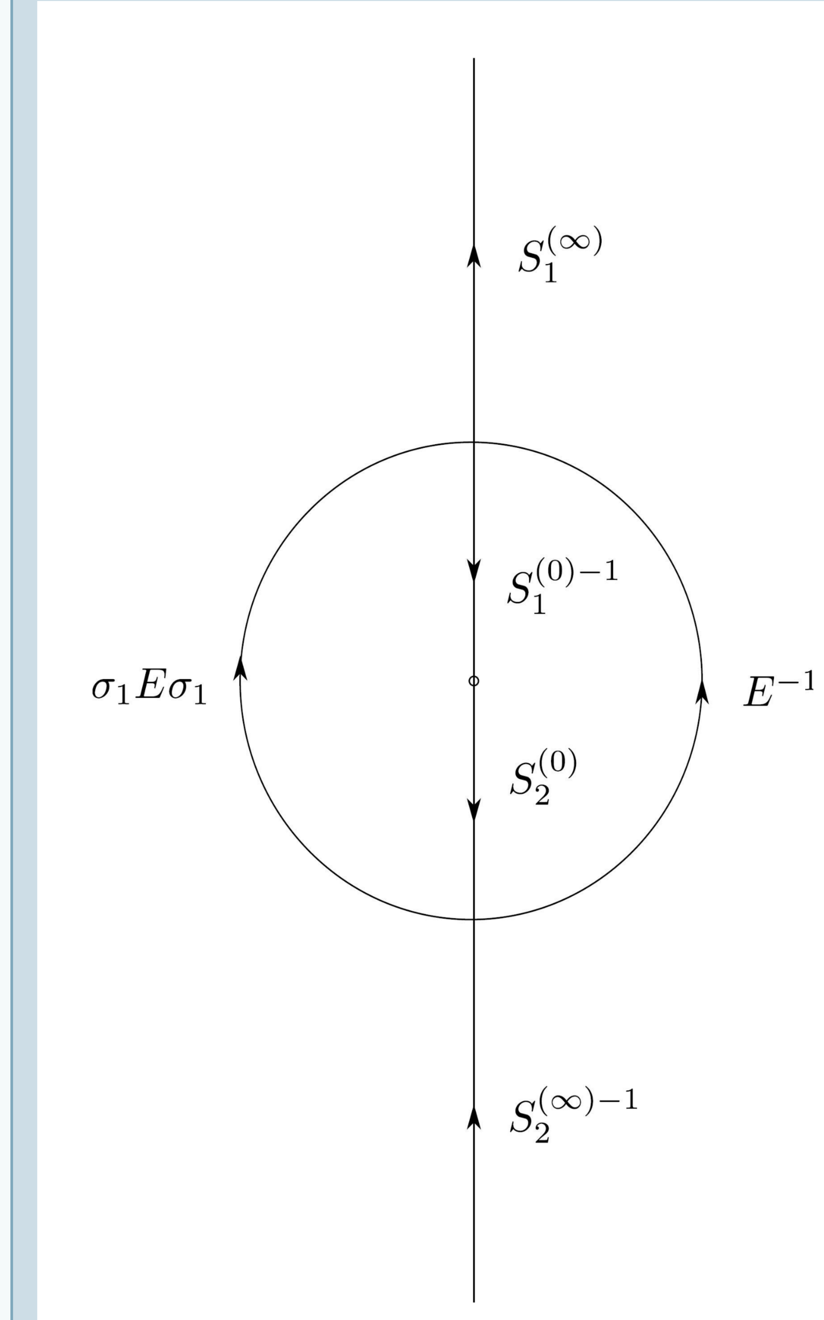
Formula (2) defines tau-function up to multiplicative constant. Nevertheless, question about evaluating the ratio C_∞/C_0 is well-posed. In [1] the formula for this ratio was determined, but proof was not finished. The authors used recently discovered representation of tau-function as Fourier transform of the irregular c=1 Virasoro conformal block. The result is the following formula

$$\frac{C_\infty}{C_0}=\frac{2^{\frac{3}{2}}e^{-i\frac{\pi}{4}}}{\pi(G(\frac{1}{2}))^4}(2\pi)^{i\nu}2^{2\nu^2+\sigma^2 24-12\sigma}e^{2\pi i(\eta^2-2\sigma\eta-\sigma^2+2\eta-\sigma)}\frac{\Gamma(1-2\sigma)}{\Gamma(2\sigma)}$$

$$\times\left(\frac{G(1+i\nu)G(1+2\sigma)G(1-2\sigma)G(1+\sigma+\eta+\frac{1-i\nu}{2})G(\frac{1-i\nu}{2}-\sigma-\eta)}{G(1+\sigma+\eta+\frac{1+i\nu}{2})G(\frac{1+i\nu}{2}-\sigma-\eta)}\right)^2.$$

Here $G(z)$ is the Barnes G - function. We give arguments for this expression using Riemann-Hilbert approach.

Riemann-Hilbert setting



Contour Γ

Matrix-valued function $\Psi(\lambda)$ is supposed to be analytic outside the contour Γ. On contour Γ it has continious limits which satisfy jump condition $\Psi_+(\lambda)=\Psi_-(\lambda)S(\lambda)$. Here " + " denotes the boundary values from the left side of the contour and " − " denotes the boundary values from the right side of the contour. Jump matrix $S(\lambda)$ is determined piecewise by expressions

$$S_1^{(\infty)}=S_2^{(0)}=\begin{pmatrix}1 & 0\\ (p+q)e^{\frac{ix^2\lambda}{8}+\frac{2i}{\lambda}} & 1\end{pmatrix},$$

$$S_2^{(\infty)}=S_1^{(0)}=\begin{pmatrix}1 & (p+q)e^{-\frac{ix^2\lambda}{8}-\frac{2i}{\lambda}}\\ 0 & 1\end{pmatrix},$$

$$E=\frac{1}{\sqrt{1+pq}}\begin{pmatrix}1 & pe^{-\frac{ix^2\lambda}{8}-\frac{2i}{\lambda}}\\ -qe^{\frac{ix^2\lambda}{8}+\frac{2i}{\lambda}} & 1\end{pmatrix}.$$

Function $\Psi(\lambda)$ satisfies the following conditions at zero and infinity

$$\Psi(\lambda)=P_0(I+O(\lambda)),\quad \lambda\to 0,$$

$$\Psi(\lambda)=(I+\frac{m_1^{(\infty)}}{\lambda}+\frac{m_2^{(\infty)}}{\lambda^2}+O(\frac{1}{\lambda^3})),\quad \lambda\to\infty.$$

$P_0, m_j^{(\infty)}$ here are some constant matrices. If we put

$$u(x)=-i\ln(1+\frac{1}{8}\{(m_2^{(\infty)})_{12}+(m_1^{(\infty)})_{12}[(m_1^{(\infty)})_{11}+(m_1^{(\infty)})_{12}-ix]\}+\frac{i}{x}[(m_1^{(\infty)})_{11}+(m_1^{(\infty)})_{12}])+2\pi n,$$

then $u(x)$ satisfies (1). Parameters $p,q\in\mathbb{C}$ are related with the parameters of asymptotic of $u(x)$ via

$$p=-i\frac{\sin 2\pi(\sigma+\eta)}{\sin 2\pi\eta},\quad q=i\frac{\sin 2\pi(\sigma-\eta)}{\sin 2\pi\eta}.$$

Differential form

We introduce the differential form considered in [2]: $\Omega=\int_\Gamma \mathrm{Tr}(\Psi_-^{-1}\Psi_-'(dS)S^{-1})\frac{d\lambda}{2\pi i}+\frac{1}{2}\int_\Gamma \mathrm{Tr}(S'S^{-1}(dS)S^{-1})\frac{d\lambda}{2\pi i}+\frac{x}{4}dx+\frac{\alpha d\beta}{4}$.
Prime here denotes the derivative with respect to λ. This form acts on vector fields in the space of parameters x,p,q and it is closed. If we define $\tau=e^{\int\Omega}$, then τ will satisfy equation (2). Actually one can express Ω in terms of $u(x)$. So we have asymptotics for $\tau(x)$

$\ln\tau(x)=-\frac{\alpha^2}{8}\ln x-\frac{\alpha^2}{8}+c_1+o(1),\quad x\to 0,$

$\ln\tau(x)=\frac{x^2}{8}+2\nu x+\nu^2\ln x+\nu^2-\frac{i}{4}\int(b_+db_- - b_-db_+)+\int\frac{\alpha d\beta}{4}+c_2+o(1),\quad x\to\infty.$

Here c_1 and c_2 do not depend on x,p,q. This formulae allow us to compute the ratio C_∞/C_0.

References

[1] A. Its, O. Lisovyy, Y. Tykhyy *Connection Problem for the Sine-Gordon/Painlevé III Tau Function and Irregular Conformal Blocks*, International Mathematics Research Notices, 22 pages, (2014).

[2] M. Bertola, *The dependence on the monodromy data of the isomonodromic tau function*, Commun. Math. Phys. 294, 539–579 (2010).

[3] A. S. Fokas, A. R. Its, A. A. Kapaev, V. Yu. Novokshenov, *Painlevé transcendents: the Riemann-Hilbert approach*, Mathematical Surveys and Monographs 128, AMS, Providence, RI, (2006).