Connection problem for the isomonodromic tau-function of the Sine-Gordon reduction of Painlevé-III equation

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Outline

- Painlevé-III equation
- Main result
- Malgrange-Bertola form
- Classical action
- Conformal block structure

Linear system

Consider 2×2 system of linear ODE with rational coefficients

$$\frac{d\Psi}{d\lambda} = A(\lambda)\Psi(\lambda)$$

We suppose that it has **two irregular singularities of Poincaré rank 1** and also matrix $A(\lambda)$ satisfies condition

$$\sigma_1 A(-\lambda)\sigma_1 = -A(\lambda).$$

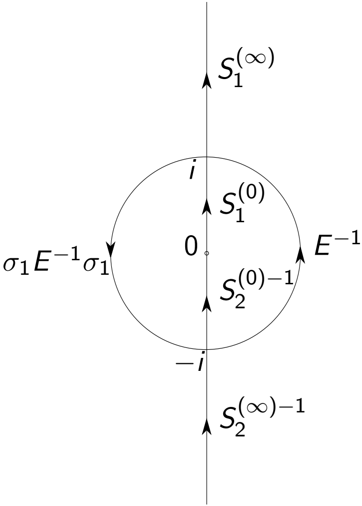
Linear system

After scaling and gauge transformations the matrix of coefficents can be written as

$$A(\lambda) = -\frac{ix^2\sigma_3}{16} + \frac{w\sigma_1}{\lambda} + \frac{i\cos(u)\sigma_3 - i\sin(u)\sigma_2}{\lambda^2},$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Riemann-Hilbert problem



Contour Γ

$$\Psi_+(\lambda) = \Psi_-(\lambda) S(\lambda).$$

$$S_1^{(\infty)} = S_2^{(0)} = \begin{pmatrix} 1 & 0 \\ p+q & 1 \end{pmatrix},$$

$$S_2^{(\infty)} = S_1^{(0)} = \begin{pmatrix} 1 & p+q \\ 0 & 1 \end{pmatrix},$$

$$E = \frac{1}{\sqrt{1+pq}} \begin{pmatrix} 1 & p \\ -q & 1 \end{pmatrix},$$

$$\Psi = P_0(I + O(\lambda)) e^{-\frac{i}{\lambda}\sigma_3}, \lambda \to 0,$$

$$\Psi = \left(I + O\left(\frac{1}{\lambda}\right)\right) e^{-\frac{ix^2\lambda}{16}\sigma_3}, \lambda \to \infty.$$

Isomonodromic deformation

Consider isomonodromic deformations, x is isomonodromic time.

$$\begin{cases} p(u,w,x) = const \\ q(u,w,x) = const \end{cases} \Rightarrow \begin{cases} u = u(x) \\ w = w(x) \end{cases}.$$

$\frac{d\Psi}{dx}\Psi^{-1}$ is rational, hence we have Lax pair

$$\begin{cases} \frac{d\Psi}{d\lambda} = A(\lambda,x)\Psi(\lambda,x) \\ \frac{d\Psi}{dx} = U(\lambda,x)\Psi(\lambda,x) \end{cases},$$

$$U(\lambda,x) = -\frac{i\lambda x\sigma_3}{8} + \frac{2w\sigma_1}{x}.$$

Isomonodromic deformation

Compatibility condition gives equation of isomonodromic deformations

$$\frac{dA}{dx} = \frac{dU}{d\lambda} + [U, A] \Leftrightarrow \begin{cases} u_{xx} + \frac{u_x}{x} + \sin u = 0 \\ w = -i\frac{xu_x}{4} \end{cases}.$$

This equation is called Sine-Gordon reduction of Painlevé-III equation. $f(x) = exp\left(\frac{iu(2x)}{2}\right)$ satisfies

$$f'' = \frac{(f')^2}{f} - \frac{f'}{x} + \frac{\alpha f^2 + \beta}{x} + \gamma f^3 + \frac{\delta}{f}$$

with

$$\alpha = \beta = 0, \quad \gamma = -1, \quad \delta = 1.$$

Painlevé function

From the formula

$$\Psi(0,x) = \begin{pmatrix} \cos\left(\frac{u}{2}\right) & -i\sin\left(\frac{u}{2}\right) \\ -i\sin\left(\frac{u}{2}\right) & \cos\left(\frac{u}{2}\right) \end{pmatrix} = exp\left(\frac{-iu\sigma_1}{2}\right)$$

we have representation for solution of Painlevé equation

$$u(x) = 2\arccos\Big(\Psi(0,x)\Big)_{11}.$$

Asymptotics

$p, q \in \mathbb{C}$ play role of initial conditions. For generic choice

$$p + q \notin (-i\infty, -2i] \cup [2i, +i\infty) \quad \text{and} \quad pq \notin (-\infty, -1],$$

function $u(x)$ exhibits behaviour

$$u(x) = \alpha \ln x + \beta + O\left(x^{2-|\Im\alpha|}\right), \quad x \to 0,$$

$$u(x) = b_+ e^{ix} x^{i\nu - 1/2}\left(1 + O\left(\frac{1}{x}\right)\right) + b_- e^{-ix} x^{-i\nu-1/2}\left(1 + O\left(\frac{1}{x}\right)\right)$$
$$+ O\left(x^{3|\Im\nu|-3/2}\right) \pmod{2\pi}, \quad x \to \infty.$$

Connection formulae for Painlevé function

Parameters on the previous slide are related by formulae

$$p = -i\frac{\sin 2\pi(\sigma+\eta)}{\sin 2\pi\eta}, \quad q = i\frac{\sin 2\pi(\sigma-\eta)}{\sin 2\pi\eta}.$$

$$\alpha = i(2-8\sigma), \quad \beta = -\pi + 4\pi\eta - i(2-8\sigma)\ln 8 - 2i\ln\left(\frac{\Gamma(1-2\sigma)}{\Gamma(2\sigma)}\right),$$

$$e^{2\pi\nu} = \frac{\sin^2(2\pi\eta)}{\sin^2(2\pi\sigma)}, \quad b_\pm = -e^{\frac{\pi\nu}{2}\mp\frac{i\pi}{4}}2^{1\pm 2i\nu}\frac{1}{\sqrt{2\pi}}\Gamma(1\mp i\nu)\frac{\sin 2\pi(\sigma\mp\eta)}{\sin 2\pi\eta}.$$

M. Jimbo, 1982; V. Novokshenov, 1985.

Jimbo-Miwa-Ueno tau-function

Denote

$$\Psi(\lambda) = \hat{\Psi}^{(\infty)}(\lambda) e^{-\frac{ix^2\lambda}{16}\sigma_3}.$$

Jimbo-Miwa-Ueno (1981) differential form is then equal to

$$\omega_{JMU} = -\operatorname*{res}_{\lambda=\infty} \mathrm{Tr}\left[\left(\hat{\Psi}^{(\infty)}(\lambda)\right)^{-1}\left(\hat{\Psi}^{(\infty)}(\lambda)\right)'_{\lambda}\left(-\frac{i\lambda x}{8}\sigma_3\right)\right] dx.$$

And isomonodromic tau-function is defined as

$$\ln \tau(x) = \int \omega_{JMU}.$$

Hamiltonian formulation

Introduce Hamiltonian

$$H = \frac{v^2}{2x} - x \cos u.$$

Here $v = xu_x$ plays role of the momentum and u plays role of the coordinate. We have

$$\begin{cases} \dfrac{du}{dx} = \dfrac{\partial H}{\partial v}, \\ \dfrac{dv}{dx} = -\dfrac{\partial H}{\partial u}, \end{cases} \Leftrightarrow u_{xx} + \frac{u_x}{x} + \sin u = 0$$

$$\ln \tau(x) = -\frac{1}{4} \int H dx$$

Asymptotics

Plugging asymptotics of $u(x)$ in formula for $\tau(x)$ we get

$$\tau(x) = C_0 x^{-\frac{\alpha^2}{8}}(1 + o(1)), \quad x \to 0,$$

$$\tau(x) = C_\infty x^{\nu^2} e^{\frac{x^2}{8}+2\nu x}(1 + o(1)), \quad x \to \infty.$$

Problem: Determine $\frac{C_\infty}{C_0}(p, q)$.

History

- The Strong Szegő Theorem (Szegő; Onsager & Kaufman, 1952, special PVI)
- The constant in the scaling theory of the Ising model (Tracy, 1991, special PIII)
- Random matrices, Dyson's constant (Dyson, 1976; Krasovsky, 2004; Ehrhardt, 2006, special PV)
- Toeplitz determinants and Fredholm determinants arising in statistical mechanics and random matrices (Widom; Basor & Widom; Basor& Tracy; Budylin&Buslaev; Deift, Krasovsky, Zhou, I; Baik, Buckingham, DiFranco)

Ising model

Consider Ising model on $\mathbb{Z}^2$. For square $\Lambda \in \mathbb{Z}^2$ energy of configuration σ is given by formula

$$E_\Lambda(\sigma) = -J \sum_{j,k\in\Lambda} (\sigma_{j,k}\sigma_{j,k+1} + \sigma_{j,k}\sigma_{j+1,k}).$$

Introduce correlation function

$$\langle \sigma_{00}\sigma_{MN} \rangle = \lim_{|\Lambda|\to\infty} \frac{\sum_\sigma \sigma_{00}\sigma_{MN} e^{-\frac{E_\Lambda(\sigma)}{kT}}}{\sum_\sigma e^{-\frac{E_\Lambda(\sigma)}{kT}}}.$$

Ising model

Correlation function is related to solution of Painlevé-III with particular choice of monodromy data $\eta = \sigma = 0$.

$$u(x) - \pi = 2i \ln \left[\frac{x}{2} \left(\gamma + \ln \frac{x}{8} \right) \right] + O\left(x \ln^{-1}(x) \right), \quad x \to 0,$$

$$u(x) - \pi = -2i \sqrt{\frac{2}{\pi}} x^{-1/2} e^{-x} (1 + o(1)) (\mathrm{mod}\, 2\pi), \quad x \to \infty.$$

γ is Euler constant.

McCoy, Tracy, Wu (1977); H. Widom(2008); D. Niles(2009)

Ising model

Introduce

$$z = \tanh(\frac{J}{kT}),\ \zeta(T) = \frac{\sqrt{z(1-z^2)}}{z^2+2z-1},\ \sinh(\frac{2J}{kT_c}) = 1,$$

$$\zeta(T) \stackrel{T\to T_c}{\longrightarrow} \infty,\ R^2 = M^2 + N^2 \to \infty, \quad t = \frac{R}{\zeta} - \text{fixed}.$$

The following formula holds (McCoy,Tracy,Wu,Barough, 1976)

$$\lim_{R\to\infty, T\to T_c} R^{\frac{1}{4}} \langle \sigma_{00}\sigma_{MN} \rangle = 2^{\frac{3}{8}} t^{\frac{1}{4}} e^{\frac{t^2}{16}} \sqrt{\tau(t)} \begin{cases} i\sin(\frac{u+\pi}{4}),\ T > T_c, \\ \cos(\frac{u+\pi}{4}),\ T < T_c. \end{cases}$$

$\tau(t)$ is tau-function and it is fixed by

$$\tau(t) = e^{-\frac{t^2}{8}}(1 + o(1)), \quad t \to \infty.$$

Result for different monodromy

C. Tracy (1991) solved connection problem for tau-function for monodromy data

$$\eta = 0,\, 0 < \sigma < \frac{1}{4}.$$

The connection matrix has in this case form $E = i\sigma_1$. The Painlevé function has following behaviour at infinity

$$u(x) - \pi = -2i\cos(2\pi\sigma)\sqrt{\frac{2}{\pi}}x^{-1/2}e^{-x}(1+o(1))(\mathrm{mod}\,2\pi), \quad x \to \infty.$$

$$\tau(x) = C_\infty e^{-\frac{x^2}{8}}(1 + o(1)), \quad x \to \infty.$$

The answer for connection constant is

$$\frac{C_\infty}{C_0} = \frac{2^{24\sigma^2}\Big(G(1+2\sigma)G(1-2\sigma)\Big)^2}{2^{\frac{3}{2}}\pi\Big(G(\frac{1}{2})\Big)^4}$$

$G(z)$ - Barnes G -function.

Main result

We consider other generic set of monodromy data

$$\cos(2\pi\sigma) \notin (-\infty, -1] \cup [1, +\infty),$$

$$\left| \arg \frac{\sin 2\pi\eta}{\sin 2\pi\sigma} \right| < \pi/2, \quad \sin(2\pi\eta) \neq 0.$$

Main result

Theorem

$$\frac{C_\infty}{C_0} = \frac{2^{\frac{3}{2}} e^{-i\frac{\pi}{4}}}{\pi(G(\frac{1}{2}))^4}(2\pi)^{i\nu} 2^{2\nu^2+\sigma^2 24-12\sigma} e^{2\pi i(\eta^2-2\sigma\eta-\sigma^2+2\eta-\sigma)}$$

$$\times \frac{\Gamma(1-2\sigma)}{\Gamma(2\sigma)} \left(\frac{G(1+i\nu)G(1+2\sigma)G(1-2\sigma)}{G(\frac{1+i\nu}{2}-\sigma-\eta)} \right)^2$$

$$\times \left(\frac{G(1+\sigma+\eta+\frac{1-i\nu}{2})G(\frac{1-i\nu}{2}-\sigma-\eta)}{G(1+\sigma+\eta+\frac{1+i\nu}{2})} \right)^2,$$

$G(z)$ - Barnes G -function.

O. Lisovyy, Yu. Tykhyy, A. Its - conjecture, 2014;
A. Prokhorov, A. Its - proven, 2015.

Proof

M. Bertola (2009), developing the earlier ideas of B. Malgrange (1983), considered the following extension of Jimbo-Miwa-Ueno form. Put $Y(\lambda) = \Psi(\lambda)e^{(\frac{ix^2\lambda}{16}+\frac{i}{\lambda})\sigma_3}$.

$$Y(\lambda) = P_0(I + O(\lambda)),\ \lambda \to 0,$$

$$Y(\lambda) = \left(I + O\left(\frac{1}{\lambda}\right)\right),\ \lambda \to \infty.$$

Denote $M(\lambda)$ the jump matrix for $Y(\lambda)$. *Malgrange-Bertola differential form* is defined by the formula

$$\omega_{MB}[\partial] := \frac{1}{2}\int_\Gamma \mathrm{Tr}(Y_-^{-1}Y_-'(\partial M)M^{-1} + Y_+^{-1}Y_+'M^{-1}(\partial M))\frac{d\lambda}{2\pi i}.$$

∂ are derivatives with respect to x, p and q

Malgrange-Bertola differential form

Lemma

Malgrange-Bertola differential form, restricted to isomonodromic deformations, coincide with Jimbo-Miwa-Ueno form up to elementary term

$$\omega_{MB}[\partial_x] = \omega_{JMU}[\partial_x] - \frac{x}{4}.$$

Localization

It is possible to localize ω_{MB}. The proof of this Lemma was simplifed significantly by M. Bertola.
$\partial Y_+(\lambda)Y_+^{-1}(\lambda) = \partial Y_-(\lambda)Y_-^{-1}(\lambda) + Y_-(\lambda)\partial M(\lambda)Y_+^{-1}(\lambda),$

Lemma

$$\omega_{MB}[\partial] = \int_\Gamma \mathrm{Tr}\Big(A(\lambda)Y_-(\lambda)\partial M(\lambda)Y_+^{-1}(\lambda)\Big)\frac{d\lambda}{2\pi i}$$

$$= \int_\Gamma \mathrm{Tr}\,\Big(A(\lambda)\Big(\partial Y_+(\lambda)Y_+^{-1}(\lambda) - \partial Y_-(\lambda)Y_-^{-1}(\lambda)\Big)\Big)\frac{d\lambda}{2\pi i}$$

$$= \sum_{\textit{poles of } A(\lambda)} \mathrm{res}\ \mathrm{Tr}\,\Big(A(\lambda)\partial Y(\lambda)Y^{-1}(\lambda)\Big).$$

This formula is very similar to the original definition of Jimbo-Miwa-Ueno form.

Localization

As result of localization we get

$$\omega_{MB} = \left(-\frac{xu_x^2}{8} + \frac{x}{4}(\cos u - 1)\right) dx$$

$$- \left(\frac{x^2}{4}u_p \sin u + \frac{x^2}{4}u_x u_{px} + \frac{xu_x u_p}{4}\right) dp$$

$$- \left(\frac{x^2}{4}u_q \sin u + \frac{x^2}{4}u_x u_{qx} + \frac{xu_x u_q}{4}\right) dq.$$

Closedness

$$d\omega_{MB} = \frac{v_p u_q - v_q u_p}{4} dq \wedge dp = \frac{dv \wedge du}{4},$$

where $v = xu_x$. It is symplectic form. We have that

$$\frac{d}{dx}(v_p u_q - v_q u_p) = 0,$$

and hence

$$d\omega_{MB} = \lim_{x \to 0} d\omega_{MB} = \frac{d\beta \wedge d\alpha}{4}.$$

Therefore, if we define

$$\omega = \omega_{MB} + \frac{x}{4} dx + \frac{\alpha d\beta}{4},$$

then ω is closed and $\omega[\partial_x] = \omega_{JMU}[\partial_x]$. This means we can put,

$$\ln \tau(x, p, q) := \int \omega.$$

Asymptotics

$$\omega = -d\left(\frac{\alpha^2}{8}\ln x + \frac{\alpha^2}{8}\right) + o(1), \quad x \to 0,$$

and

$$\omega = d\left(2\nu x + \nu^2 \ln x + \nu^2 + \frac{x^2}{8}\right) - \frac{i}{4}(b_+ db_- - b_- db_+) + \frac{\alpha d\beta}{4} + o(1),$$

$$x \to \infty.$$

We remind

$$\tau(x) = C_0 x^{-\frac{\alpha^2}{8}}(1 + o(1)), \quad x \to 0,$$

$$\tau(x) = C_\infty x^{\nu^2} e^{\frac{x^2}{8} + 2\nu x}(1 + o(1)), \quad x \to \infty.$$

Answer

$$\ln \frac{C_\infty}{C_0} = \nu^2 + \frac{\alpha^2}{8} - i\nu + \frac{1}{4}\int(\alpha d\beta - 2ib_+ db_-) + c,$$

Essentially $\ln \frac{C_\infty}{C_0}$ is generating function for canonical transformation from variables (α, β) to $(2ib_+, b_-)$. It was computed in the work by A. Its, O. Lisovyy, Y. Tykhyy. Then choosing (σ, ν) as independent variables, one can express generating function in terms of Barnes G-function.

To find numerical constant c one can use the fact that for $\sigma = \eta = \frac{1}{4}$ solution is $u \equiv 0 \mod 2\pi$, and $C_\infty = C_0$.

Hamiltonian meaning of Malgrange-Bertola form

One can check directly that

$$xH_x + vu_x = H.$$

Using this formula we get

$$\omega_{MB} = -\frac{1}{4}d\left(\frac{x^2}{2} + xH\right) - \frac{1}{4}\Big(vdu - Hdx\Big).$$

Here differential is taken with respect to all variables x, p, q. So Jimbo-Miwa-Ueno tau-function is almost classical action

$$\ln \tau(x) = -\frac{xH}{4} - \frac{1}{4}\int (vu_x - H)dx.$$

Hamiltonian meaning of Malgrange-Bertola form

The action integral is suitable for differentiation

$$\partial\Big(\int_{t_0}^{t_1} vu_x - Hdx\Big) = \int_{t_0}^{t_1} (u_x\partial v + v\partial u_x - H_v\partial v - H_u\partial u)dx = v\partial u\Big|_{t_0}^{t_1}.$$

Hamiltonian meaning of Malgrange-Bertola form

So there is alternative way to get the formula for connection constant

$$\ln \frac{C_\infty}{C_0} = \lim_{t_0\to 0} \lim_{t_1\to +\infty} \left(\int_{t_0}^{t_1} \frac{\mathcal{H} - vu_x}{4} dx - x\mathcal{H}\Big|_{t_0}^{t_1} - \frac{t_1^2}{8} - 2\nu t_1 - \nu^2 \ln t_1 \right.$$

$$\left. - \frac{\alpha^2}{8} \ln t_0 \right).$$

$$\partial\left(\ln \frac{C_\infty}{C_0}\right) = \lim_{t_0\to 0} \lim_{t_1\to +\infty} \left[-\left(\frac{v\partial u}{4} + x\partial\mathcal{H}\right)\Big|_{t_0}^{t_1} - \partial\left(\frac{t_1^2}{8} + 2\nu t_1 + \nu^2 \ln t_1 \right.\right.$$

$$\left.\left. + \frac{\alpha^2}{8} \ln t_0 \right) \right].$$

Conformal block approach

A. Its, O. Lizovyy, Yu. Tykhyy (2014)

Make change of variable and gauge in tau-function

$$\ln \tau_m\left(2^{-12}x^4\right) = \frac{1}{2}\ln\tau(x) + \frac{1}{4}\ln(x) + \frac{iu(x)}{4}.$$

For generic monodromy data we have the following asymptotic at zero

$$\tau_m(t) = \text{const}\,\Bigg[t^{\sigma^2}\left(1 + \frac{t}{2\sigma^2} + \frac{8\sigma^2+1}{4\sigma^2(4\sigma^2-1)}t^2 + \ldots\right)$$
$$-e^{4\pi i\eta}\frac{\Gamma^2(-1-2\sigma)}{\Gamma^2(1+2\sigma)}t^{(\sigma+1)^2}\left(1 + \frac{t}{2(\sigma+1)^2} + \{\sigma\to\sigma+1\} + \ldots\right)$$
$$-e^{-4\pi i\eta}\frac{\Gamma^2(-1+2\sigma)}{\Gamma^2(1-2\sigma)}t^{(\sigma-1)^2}\left(1 + \frac{t}{2(\sigma-1)^2} + \{\sigma\to\sigma-1\} + \ldots\right)$$
$$+\ldots\Bigg], \quad t\to 0.$$

Choose

$$\text{const} := \frac{1}{G(1+2\sigma)G(1-2\sigma)}.$$

Since $G(z+1) = \Gamma(z)G(z)$, the series get the following structure

$$\tau_m(t) = \frac{1}{G(1+2\sigma)G(1-2\sigma)} t^{\sigma^2} \left(1 + \frac{t}{2\sigma^2} + \dots\right)$$

$$-e^{4\pi i\eta} \frac{1}{G(1+2\sigma+2)G(1-2\sigma-2)} t^{(\sigma+1)^2} \left(1 + \frac{t}{2(\sigma+1)^2} + \dots\right)$$

$$-e^{-4\pi i\eta} \frac{1}{G(1+2\sigma-2)G(1-2\sigma+2)} t^{(\sigma-1)^2} \left(1 + \frac{t}{2(\sigma-1)^2} + \dots\right)$$

$$+\dots, \quad t \to 0$$

$$\tau_m(t) = \sum_{n \in \mathbb{Z}} e^{4\pi i n \eta} F(\sigma + n, t),$$

$$F(\sigma, t) = \frac{t^{\sigma^2}}{G(1+2\sigma)G(1-2\sigma)} B(\sigma, t)$$

$$B(\sigma, t) = 1 + \sum_{k=1}^{\infty} B_k(\sigma) t^k.$$

- Both the series are convergent
- $F(\sigma, t)$ is the irregular $c = 1$ Virasoro conformal block
- AGT duality $\to$ $F(\sigma, t)$ = the partition function of the $N = 2$ supersymmetric pure SU(2) gauge theory $\to$

$$B(\sigma, t) = \sum_{\lambda,\mu\in\mathbb{Y}} \left(\frac{\dim\lambda \dim\mu}{|\lambda|!|\mu|!} \right)^2 \frac{t^{|\lambda|+|\mu|}}{|b_{\lambda,\mu}(\sigma)|^2},$$

$$b_{\lambda,\mu}(\sigma) = \prod_{(k,l)\in\lambda} (\lambda'_l - k + \mu_k - l + 1 + 2\sigma) \prod_{(k,l)\in\mu} (\mu'_l - k + \lambda_k - l + 1 - 2\sigma)$$

(Nekrasov instanton sum)
O. Gamayun, N. Iorgov, O. Lisovyy, A. Shchechkin, Yu. Tykhyy, J. Teschner (2012,2013,2014)

- We have the folowing asymptotic at infinity

$$\tau_m\left(2^{-12}x^4\right) = \text{const } x^{\frac{1}{4}} e^{\frac{x^2}{16}} \left[x^{\frac{\nu^2}{2}} e^{\nu x} \left(1 + \frac{\nu(2\nu^2+1)}{8x} + \ldots\right)\right.$$

$$+\frac{ib_+}{4} x^{\frac{(\nu+i)^2}{2}} e^{(\nu+i)x} \left(1 + \frac{(\nu+i)(2(\nu+i)^2+1)}{8x} + \ldots\right)$$

$$+\frac{ib_-}{4} x^{\frac{(\nu-i)^2}{2}} e^{(\nu-i)x} \left(1 + \frac{(\nu-i)(2(\nu-i)^2+1)}{8x} + \ldots\right)$$

$$\left. + \ldots \right], \quad x \to \infty,$$

Choose

$$\text{const} := e^{\frac{i\pi\nu^2}{4}} 2^{\nu^2} (2\pi)^{-\frac{i\nu}{2}} G(1+i\nu)$$

and take into account that,

$$\frac{ib_\pm}{4} e^{\frac{i\pi\nu^2}{4}} 2^{\nu^2} (2\pi)^{-\frac{i\nu}{2}} G(1+i\nu)$$

$$= e^{\frac{i\pi(\nu\pm i)^2}{4}} 2^{(\nu\pm i)^2} (2\pi)^{-\frac{i(\nu\pm i)}{2}} G(1+i(\nu\pm i)) e^{\pm 4\pi i\rho},$$

where

$$e^{4\pi i\rho} = \frac{\sin 2\pi\eta}{\sin 2\pi(\sigma+\eta)}.$$

$$\tau_m\left(2^{-12}x^4\right) = \chi(\sigma,\nu)\sum_{n\in\mathbb{Z}} e^{4\pi i n\rho} J(\nu+in,x),$$

$$J(\nu,x) = e^{\frac{i\pi\nu^2}{4}} 2^{\nu^2} (2\pi)^{-\frac{i\nu}{2}} G(1+i\nu) x^{\frac{1}{4}+\frac{\nu^2}{2}} e^{\frac{x^2}{16}+\nu x} D(\nu,x),$$

$$D(\nu,x) \sim 1 + \sum_{k=1}^{\infty} D_k(\nu) x^{-k},$$

• *Conjecture*: The Fourier series for $\tau_m\left(2^{-12}x^4\right)$ is convergent. The series for $D(\nu, x)$ is an asymptotic series

• *Open question*: Conformal block interpretation of $J(\nu, x)$ and the Nekrasov type combinatorial formula for $D(\nu, x)$.

Functional relation

$$\tau_m(t) = \sum_{n\in\mathbb{Z}} e^{4\pi i n\eta} F(\sigma+n,t), \quad t\to 0.$$

$$\tau_m\left(2^{-12}x^4\right) = \chi(\sigma,\nu)\sum_{n\in\mathbb{Z}} e^{4\pi i n\rho} J(\nu+in,x), \quad x\to\infty.$$

These relations imply that constant prefactor for modified tau-function is quasiperiodic

$$\begin{cases} \chi(\sigma+1,\nu) = e^{-4\pi i\eta}\chi(\sigma,\nu), \\ \chi(\sigma,\nu+i) = e^{4\pi i\rho}\chi(\sigma,\nu). \end{cases}$$

A. Its, O. Lisovyy and Y. Tykhyy using the expression for generating function managed to come up with expression, which satisfies this relation. So they got the answer up to periodic function.

THANK YOU!