Connection problem for Painlevé tau functions

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Content

Airy equation

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Airy equation

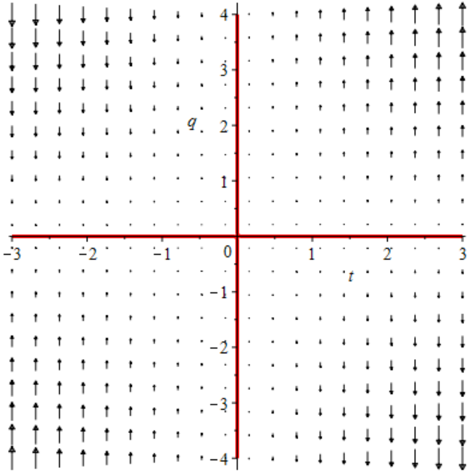
$$q'' = tq$$

$$q(t) = \alpha \mathrm{Ai}(t) + \beta \mathrm{Bi}(t)$$

$$\mathrm{Ai}(t) = \int_{e^{-\frac{i\pi}{3}}\infty}^{e^{\frac{i\pi}{3}}\infty} e^{\frac{z^3}{3} - tz} \frac{dz}{2\pi}$$

$$\mathrm{Bi}(t) = \int_{-\infty}^{e^{-\frac{i\pi}{3}}\infty} e^{\frac{z^3}{3} - tz} \frac{dz}{2\pi} + \int_{-\infty}^{e^{-\frac{i\pi}{3}}\infty} e^{\frac{z^3}{3} - tz} \frac{dz}{2\pi}$$

Force field



$$F(q,t) = tq$$

Asymptotics

$$\mathrm{Ai}(t) \simeq \frac{e^{-\frac{2}{3}t^{\frac{3}{2}}}}{2\sqrt{\pi}t^{\frac{1}{4}}}, \quad t \to +\infty$$

$$\mathrm{Bi}(t) \simeq \frac{e^{\frac{2}{3}t^{\frac{3}{2}}}}{\sqrt{\pi}t^{\frac{1}{4}}}, \quad t \to +\infty$$

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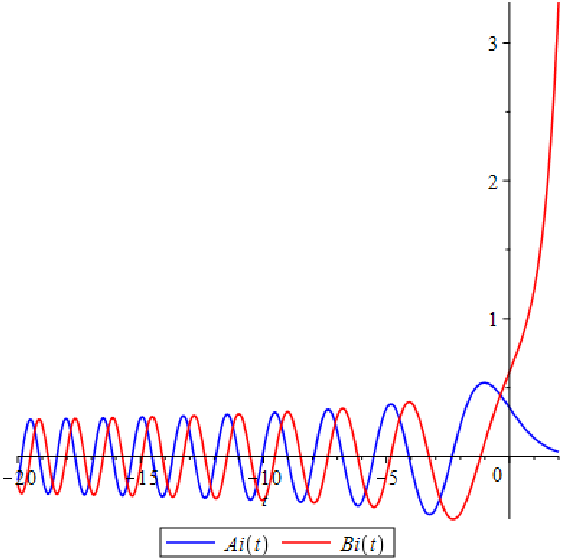
$$\mathrm{Ai}(t) \simeq \frac{1}{\sqrt{\pi}(-t)^{\frac{1}{4}}} \left(\cos\left(\frac{2}{3}(-t)^{\frac{3}{2}} - \frac{\pi}{4}\right)\right), \quad t \to -\infty$$

$$\mathrm{Bi}(t) \simeq -\frac{1}{\sqrt{\pi}(-t)^{\frac{1}{4}}} \left(\sin\left(\frac{2}{3}(-t)^{\frac{3}{2}} - \frac{\pi}{4}\right)\right), \quad t \to -\infty$$

Airy functions graph



Tau function

$$H(p,q,t)=\frac{p^2}{2}-\frac{tq^2}{2},\quad \begin{cases} \dfrac{dq}{dt}=\dfrac{\partial H}{\partial p}, \\ \dfrac{dp}{dt}=-\dfrac{\partial H}{\partial q}, \end{cases}$$

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$$H(t) = -\frac{\beta^2}{4\pi t} e^{\frac{4}{3} t^{\frac{3}{2}}} (1 + O(t^{-\frac{3}{2}})) + \frac{\alpha^2}{16\pi t} e^{-\frac{4}{3} t^{\frac{3}{2}}} (1 + O(t^{-\frac{3}{2}})), \quad t \to +\infty$$

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$$H(t)=\frac{\alpha^2+\beta^2}{2\pi}\sqrt{-t}+O(t^{-1}),\quad t\to-\infty$$

$$\ln\tau(t_1,t_2,\alpha,\beta)=-\frac{\beta^2}{8\pi t_2^{\frac{3}{2}}}e^{\frac{4}{3}t_2^{\frac{3}{2}}}(1+O(t_2^{-\frac{3}{2}}))-\frac{\alpha^2}{32\pi t_2^{\frac{3}{2}}}e^{-\frac{4}{3}t_2^{\frac{3}{2}}}(1+O(t_2^{-\frac{3}{2}}))$$

$$+\frac{\alpha^2+\beta^2}{3\pi}(-t_1)^{\frac{3}{2}}+c_0+O((-t_1)^{-\frac{3}{2}}),\quad t_2\to+\infty,\quad t_1\to-\infty.$$

Quasihomogeneous Hamiltonian

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Classical action

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$$S(t_1, t_2, \alpha, \beta) = \frac{\alpha^2}{2} \mathrm{Ai}'(t)\mathrm{Ai}(t) + \alpha\beta \mathrm{Ai}'(t)\mathrm{Bi}(t) + \frac{\beta^2}{2} \mathrm{Bi}'(t)\mathrm{Bi}(t)\bigg|_{t_1}^{t_2}$$

Constant

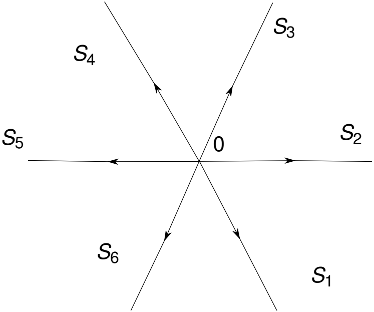
Theorem

$$c_0 = \frac{\alpha\beta}{6\pi}.$$

Painlevé II equation

Painlevé II equation

$$q'' = tq + 2q^3$$



$$\Psi_+(\lambda, t) = \Psi_-(\lambda, t) S_j$$

$$S_{2k} = \begin{pmatrix} 1 & 0 \\ s_{2k} e^{-\frac{z^3}{3} + tz} & 1 \end{pmatrix},$$

$$S_{2k+1} = \begin{pmatrix} 1 & s_{2k+1} e^{\frac{z^3}{3} - tz} \\ 0 & 1 \end{pmatrix}$$

$$s_{j+3} = s_j, \quad s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0$$

$$\lim_{\lambda \to \infty} \Psi(\lambda, t) = I$$

Painlevé II equation

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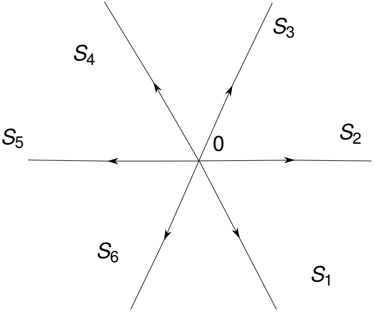

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$$q(t) = -\lim_{\lambda \to \infty} \Psi_{12}(\lambda, t)$$

Asymptotics at $-\infty$

Its, Kapaev (1988) , Kapaev (1992)

- special behavior for $1 - s_1 s_3 = 0$

$$q(t) \simeq \sigma\sqrt{\frac{-t}{2}}\sum_{n=0}^{\infty} b_n(-t)^{-\frac{3n}{2}} - \frac{s_1+s_2}{\sqrt{\pi}2^{\frac{7}{4}}(-t)^{\frac{1}{4}}}\exp\left(-\frac{2\sqrt{2}}{3}(-t)^{\frac{3}{2}}\right)(1+O((-t)^{-\frac{1}{4}})),\ t\to-\infty$$

where $s_1 = -i\sigma, \quad \sigma = \pm 1.$

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where $s_1 = -i\sigma, \quad \sigma = \pm 1.$

- singular behavior for $1 - s_1 s_3 < 0$

$$q(t) = \frac{2\sqrt{-t}}{ae^{ig} + be^{-ig} + O((-t)^{-\frac{3}{10}})}$$

where

$$a = \frac{\sqrt{2\pi}e^{\frac{\pi\beta}{2}}}{s_1\Gamma\left(\frac{1}{2}+i\beta\right)}, \quad b = \frac{\sqrt{2\pi}e^{\frac{\pi\beta}{2}}}{s_3\Gamma\left(\frac{1}{2}-i\beta\right)}, \quad ab = 1,$$

$$g = \frac{2}{3}(-t)^{\frac{3}{2}} + \frac{3\beta}{2}\ln(-t) + 3\beta\ln 2 - \frac{\pi}{2}, \quad \beta = \frac{1}{2\pi}\ln(s_1 s_3 - 1).$$

Asymptotics at $-\infty$

- generic behavior for $|\arg(1-s_1s_3)|<\pi$

$$q(t)=a_{0,0}^{+}e^{\frac{2i}{3}(-t)^{\frac{3}{2}}}(-t)^{\frac{3\mu}{2}-\frac{1}{4}}+a_{0,0}^{-}e^{-\frac{2i}{3}(-t)^{\frac{3}{2}}}(-t)^{-\frac{3\mu}{2}-\frac{1}{4}}+O\left(|t|^{\frac{9|\mathrm{Re}\mu|}{2}-\frac{7}{4}}\right),\quad t\to-\infty,$$

$$\begin{aligned}\mu&=-\frac{\ln(1-s_1s_3)}{2\pi i}, & a_{0,0}^{+}a_{0,0}^{-}&=\frac{i\mu}{2},\\ a_{0,0}^{+}&=\frac{\sqrt{\pi}\,2^{3\mu}e^{-\frac{i\pi\mu}{2}-\frac{i\pi}{4}}}{s_1\Gamma(\mu)}, & a_{0,0}^{-}&=\frac{\sqrt{\pi}\,2^{-3\mu}e^{-\frac{i\pi\mu}{2}+\frac{i\pi}{4}}}{s_3\Gamma(-\mu)},\end{aligned}\tag{1}$$

Asymptotics at $+\infty$

- special behavior for $s_2 = 0$

$$q(t) \simeq \frac{is_1}{2\sqrt{\pi}t^{\frac{1}{4}}} \exp\left(-\frac{2}{3}t^{\frac{3}{2}}\right)(1 + O(t^{-\frac{3}{4}})),\ t \to -\infty.$$

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- singular behavior for $s_2 \in \mathbb{R}, \quad s_2 \neq 0$

$$q(t) = \frac{i\varepsilon}{\sqrt{2}}\left(\frac{ce^{ih} - 1 + O(t^{-\frac{3}{4}})}{ce^{ih} + 1 + O(t^{-\frac{3}{4}})}\right) + O(t^{-\frac{3}{2}})$$

where

$$c = \frac{\sqrt{2\pi}e^{\frac{\pi\gamma}{2}}}{(1 + s_2 s_3)\Gamma\left(\frac{1}{2} + i\gamma\right)}, \quad h = \frac{2\sqrt{2}}{3}t^{\frac{3}{2}} + \frac{3\gamma}{2}\ln t + \frac{7\gamma}{2}\ln 2, \quad \gamma = \frac{1}{\pi}\ln(\varepsilon s_2), \quad \varepsilon = \operatorname{sign} s_2.$$

Asymptotics at $+\infty$

- generic behavior for

$$|\arg(i\sigma s_2)| < \frac{\pi}{2}, \quad \sigma = \operatorname{sign} \mathrm{Re}\,(is_2) = \pm 1 \tag{2}$$

$$\sigma q(t) = i\sqrt{\frac{t}{2}} + b_{1,1}^{+} e^{\frac{2i\sqrt{2}}{3} t^{\frac{3}{2}}} t^{-\frac{3\nu}{2} - \frac{1}{4}} + b_{1,1}^{-} e^{-\frac{2i\sqrt{2}}{3} t^{\frac{3}{2}}} t^{\frac{3\nu}{2} - \frac{1}{4}} + O\left(t^{3|\mathrm{Re}\nu| - 1}\right), \qquad t \to +\infty,$$

$$\nu = \frac{\ln(i\sigma s_2)}{\pi i}, \qquad b_{1,1}^{+} b_{1,1}^{-} = \frac{i\nu}{4\sqrt{2}},$$
$$b_{1,1}^{+} = \frac{\sqrt{\pi}\, 2^{-\frac{7\nu}{2} - \frac{3}{4}} e^{\frac{i\pi\nu}{2} - \frac{i\pi}{4}}}{(1 + s_2 s_3)\Gamma(-\nu)}, \qquad b_{1,1}^{-} = -\frac{\sqrt{\pi}\, 2^{\frac{7\nu}{2} - \frac{3}{4}} e^{\frac{i\pi\nu}{2} + \frac{i\pi}{4}}}{(1 + s_1 s_2)\Gamma(\nu)}. \tag{3}$$

Painlevé II: force field for real solutions

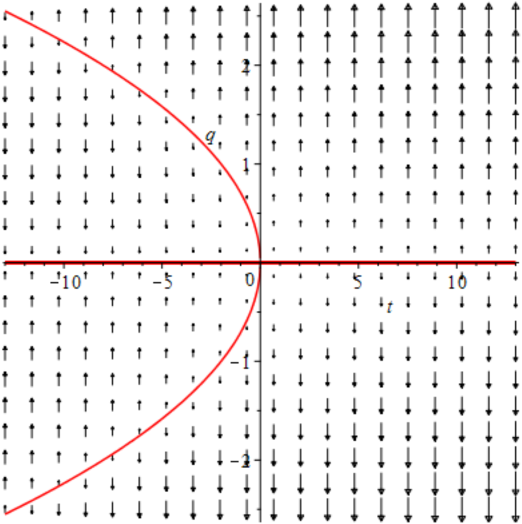

Figure: $F(q, t) = tq + 2q^3$

Painlevé II: asymptotics of real solutions

The real nonsingular solutions are parametrized by number $s_1 \in i\mathbb{R}, \quad |s_1| \leq 1$.
(Kapaev, 1992) The asymptotic at $+\infty$ is given by

$$q(t) = \frac{is_1}{2\sqrt{\pi}t^{\frac{1}{4}}} e^{-\frac{2}{3}t^{\frac{3}{2}}} \left(1 + O\left(\frac{1}{t^{\frac{3}{4}}}\right)\right), \quad t \to +\infty.$$

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If $|s_1| < 1$, then the asymptotics at $-\infty$ is given by Ablowitz-Segur solution

$$q(t) = \frac{d}{(-t)^{\frac{1}{4}}} \sin\left(\frac{2}{3}(-t)^{\frac{3}{2}} + \frac{3}{4}d^2 \ln(-t) + \phi\right) + O\left(\frac{1}{|t|}\right), \quad t \to -\infty,$$

where

$$d = \sqrt{\frac{1}{\pi} \ln\left(1 - |s_1|^2\right)}, \quad \phi = -\frac{\pi}{4} + \frac{3}{2}d^2 \ln 2 - \arg\left(\Gamma\left(i\frac{d^2}{2}\right)\right) - \arg(s_1).$$

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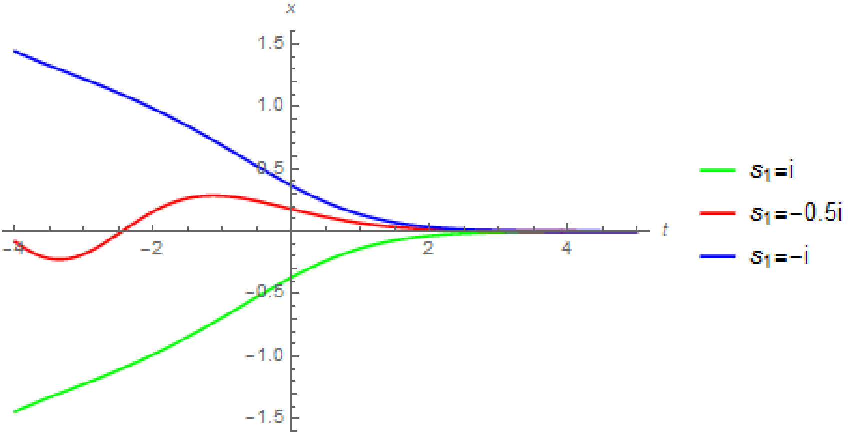
If $s_1 = \pm i$, then the asymptotics at $-\infty$ is given by Hastings-Mcleod solution

$$q(t) = is_1 \sqrt{\frac{-t}{2}} + O(t^{-\frac{5}{2}}), \quad t \to -\infty.$$

Painlevé II: real solutions



Painlevé II: force field for imaginary solutions

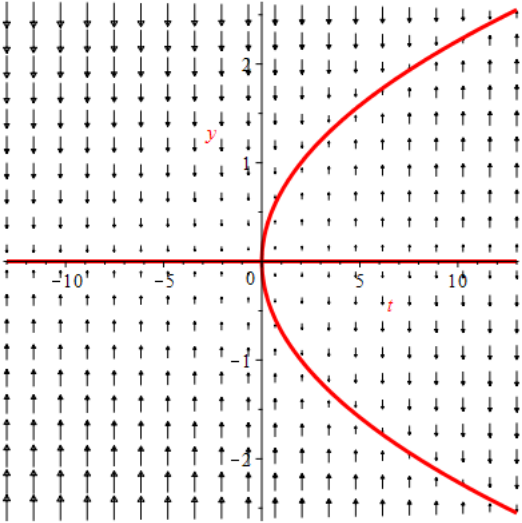


Figure: $F(y,t) = ty - 2y^3$

Painlevé II: asymptotics of imaginary solutions

All pure imaginary solutions $q = iy$ are parametrized by number $s_1 \in \mathbb{C}$. (Its, Kapaev, 1988) The asymptotic at $-\infty$ is given by

$$y(t) = \frac{d}{(-t)^{\frac{1}{4}}} \sin\left(\frac{2}{3}(-t)^{\frac{3}{2}} + \frac{3}{4}d^2 \ln(-t) + \phi\right) + O\left(\frac{1}{|t|}\right), \quad t \to -\infty,$$

$$d = \sqrt{\frac{1}{\pi} \ln\left(1 + |s_1|^2\right)}, \quad \phi = -\frac{\pi}{4} + \frac{3}{2}d^2 \ln 2 - \arg\left(\Gamma\left(i\frac{d^2}{2}\right)\right) - \arg(s_1).$$

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If $\mathrm{Im}\, s_1 \neq 0$ then the asymptotic at $+\infty$ is given by

$$y(t) = \sigma\sqrt{\frac{t}{2}} + \frac{\sigma\rho}{(2t)^{\frac{1}{4}}} \cos\left(\frac{2\sqrt{2}}{3}t^{\frac{3}{2}} - \frac{3}{2}\rho^2 \ln x + \theta\right) + O\left(\frac{1}{t}\right), \quad t \to +\infty.$$

$$\rho = \sqrt{\frac{1}{\pi} \ln\left(\frac{1 + |s_1|^2}{2|\mathrm{Im}(s_1)|}\right)}, \quad \sigma = -\mathrm{sign}(\mathrm{Im}(s_1)),$$

$$\theta = -\frac{3\pi}{4} - \frac{7}{2}\rho^2 \ln 2 + \arg(\Gamma(i\rho^2)) + \arg(1 + s_1^2).$$

Painlevé II: asymptotics of imaginary solutions

All pure imaginary solutions $q = iy$ are parametrized by number $s_1 \in \mathbb{C}$. (Its, Kapaev, 1988) The asymptotic at $-\infty$ is given by

$$y(t) = \frac{d}{(-t)^{\frac{1}{4}}} \sin\left(\frac{2}{3}(-t)^{\frac{3}{2}} + \frac{3}{4}d^2 \ln(-t) + \phi\right) + O\left(\frac{1}{|t|}\right), \quad t \to -\infty,$$

$$d = \sqrt{\frac{1}{\pi} \ln\left(1 + |s_1|^2\right)}, \quad \phi = -\frac{\pi}{4} + \frac{3}{2}d^2 \ln 2 - \arg\left(\Gamma\left(i\frac{d^2}{2}\right)\right) - \arg(s_1).$$

If $\mathrm{Im}\, s_1 \neq 0$ then the asymptotic at $+\infty$ is given by

$$y(t) = \sigma\sqrt{\frac{t}{2}} + \frac{\sigma\rho}{(2t)^{\frac{1}{4}}} \cos\left(\frac{2\sqrt{2}}{3}t^{\frac{3}{2}} - \frac{3}{2}\rho^2 \ln x + \theta\right) + O\left(\frac{1}{t}\right), \quad t \to +\infty.$$

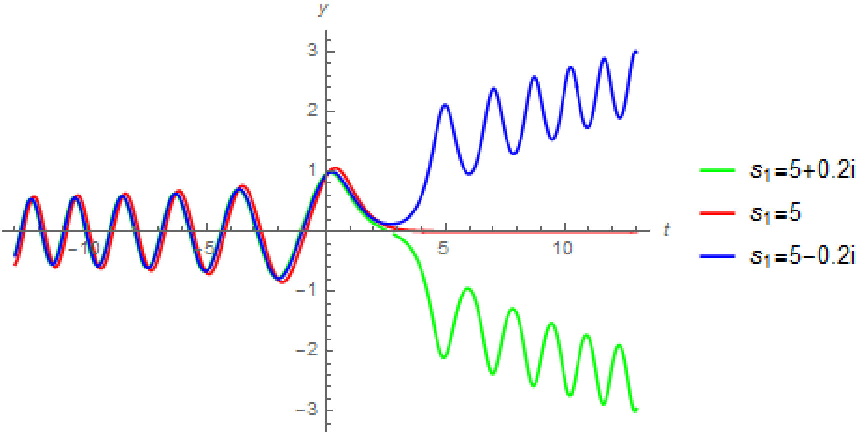
$$\rho = \sqrt{\frac{1}{\pi} \ln\left(\frac{1 + |s_1|^2}{2|\mathrm{Im}(s_1)|}\right)}, \quad \sigma = -\mathrm{sign}(\mathrm{Im}(s_1)),$$

$$\theta = -\frac{3\pi}{4} - \frac{7}{2}\rho^2 \ln 2 + \arg(\Gamma(i\rho^2)) + \arg(1 + s_1^2).$$

If $\mathrm{Im}\, s_1 = 0$ the asymptotic at $+\infty$ is given by the Ablowitz-Segur solution,

$$y(t) = \frac{s_1}{2\sqrt{\pi}t^{\frac{1}{4}}} e^{-\frac{2}{3}t^{\frac{3}{2}}} \left(1 + O\left(\frac{1}{t^{\frac{3}{4}}}\right)\right), \quad t \to +\infty.$$

Painlevé II: imaginary solutions



Quasihomogeneous Hamiltonian and Classical action

$$H = \frac{p^2}{2} - \frac{tq^2}{2} - \frac{q^4}{2}$$

$$\ln \tau(t_1, t_2, s_1, s_2) \simeq \frac{t_2^3}{24} + \frac{i\sqrt{2}}{3}\nu t_2^{\frac{3}{2}} - \frac{(6\nu^2+1)}{16}\ln t_2 + \frac{2i\mu}{3}(-t_1)^{\frac{3}{2}} + \frac{3\mu^2}{4}\ln(-t_1) + \ln \Upsilon$$

for generic behaviours at both infinities.

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Classical action

$$\frac{\partial S}{\partial \mathrm{s}_1} = p\frac{\partial q}{\partial s_1}\bigg|_{t_1}^{t_2}$$

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$$\frac{\partial}{\partial s_j}\ln\tau(t_1, t_2, s_1, s_2) = \frac{\partial}{\partial s_j}\left(\frac{2tH}{3} - \frac{pq}{3}\right)\Big|_{t_1}^{t_2} + p\frac{\partial q}{\partial s_j}\Big|_{t_1}^{t_2}.$$

$$\ln\tau(t_1, t_2, s_1, s_2) = \ln\tau(t_1, t_2, 0, -i) + \left(\left(\frac{2tH}{3} - \frac{pq}{3}\right)\Big|_{t_1}^{t_2}\right)\Big|_{(0,-i)}^{(s_1,s_2)}$$

$$+ \int_{(0,-i)}^{(s_1,s_2)} p\frac{\partial q}{\partial s_1}\Big|_{t_1}^{t_2} ds_1 + p\frac{\partial q}{\partial s_2}\Big|_{t_1}^{t_2} ds_2$$

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$$\ln\tau(t_1,t_2,0,-i) \sim \frac{t_2^3}{24} - \frac{1}{16}\ln t_2 + \ln\Upsilon_0 \tag{5}$$

Numerical constant

$$\ln \tau(t_1, t_2, -i, 0) \sim -\frac{t_1^3}{24} - \frac{1}{16}\ln(-t_1) + \ln \Upsilon_{HM}, \quad t_1 \to -\infty, \quad t_2 \to +\infty$$

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$$q(t; 0, -i) = e^{\frac{2\pi i}{3}} q\left(te^{\frac{2\pi i}{3}}; -i, 0\right).$$

Results

Theorem

Its, Lisovyy, P., 2018

$$\Upsilon=\Upsilon_0 2^{\frac{3}{2}\mu^2-\frac{7\nu^2}{8}}(2\pi)^{-\frac{\mu}{2}-\frac{\nu}{4}}e^{\frac{\pi i}{8}\left(\eta^2+2\mu^2+2\eta\nu-8\mu\eta\right)}\frac{\sqrt{G(1-\nu)\,\hat{G}(\eta)}}{G(1-\mu)\,\hat{G}\left(\frac{\eta-\nu}{2}\right)},$$

$$\Upsilon_0=2^{\frac{1}{48}}e^{\frac{\zeta'(-1)}{2}+\frac{i\pi}{48}},$$

where $\zeta(z)$ *- Riemann Zeta function,* $G(z)$ *- Barnes G-function,* $\hat{G}(z)=\dfrac{G(1+z)}{G(1-z)}$*, and* $\mu,\nu,\sigma,\eta,\Upsilon,\Upsilon_0$ *are described by* (1)*,* (2)*,* (3)*,*(4)*,*(5) *and* (6)

$$s_3^{-1}=e^{i\pi\eta}e^{i\pi\frac{\sigma}{2}}. \tag{6}$$

Last slide

THANK YOU