Cartier core map for Cartier algebras

Anna Brosowsky

University of Michigan

Introduction

In char p commutative algebra, Frobenius split rings are nice, but strongly F-regular rings are even nicer (e.g., Cohen-Macaulay & normal). The splitting prime [1], F-pure centers [2], and Cartier cores [3] give obstructions to being strongly F-regular. Strong F-regularity and Frobenius splitting can be generalized to divisor pairs $(\operatorname{Spec} R, \Delta)$, or even more broadly, to Cartier algebra pairs $(R, \mathcal{D})$. In this most general setting, the F-splitting prime [4] and $\mathcal{D}$-compatible ideals are obstructions to strong F-regularity of the pair $(R, \mathcal{D})$.

Motivation: Given $(R, \mathcal{D})$ and prime P, how "far" is $(R_P, \mathcal{D}_P)$ from being strongly F-regular? Given J, how "far" is it from being $\mathcal{D}$-compatible?

We define the Cartier core of an ideal J with respect to a Cartier subalgebra $\mathcal{D}$ and prove some properties of it as a map on $\operatorname{Spec} R$. As an application, we give an exact description of this map for Stanley-Reisner rings.

Notation & Assumptions

- All rings R have prime char p & are Noetherian.
- $F^e_* R$ is R as an R-module over e-th iterated Frobenius map, i.e., $sF^e_* r = F^e_*(s^{p^e} r)$
- All rings R are F-*finite*, i.e., $F_* R$ is finitely-generated R-module

Cartier algebras

Give the group $\oplus_{e\geq 0} \operatorname{Hom}_R(F^e_* R, R)$ a graded non-commutative ring structure: for maps

$$\phi \in \operatorname{Hom}_R(F^e_* R, R), \quad \psi \in \operatorname{Hom}_R(F^d_* R, R),$$

define $\phi \cdot \psi \in \operatorname{Hom}_R(F^{e+d}_* R, R)$ where

$$(\phi \cdot \psi)(F^{e+d}_* r) = \phi(F^e_*(\psi(F^d_* r))).$$

Call $\mathcal{C}^R := \oplus_e \operatorname{Hom}_R(F^e_* R, R)$ the *full Cartier algebra*. Any graded subring $\mathcal{D} \subset \mathcal{C}^R$ with $\mathcal{D}_0 = R$ is a *Cartier subalgebra*.

F-Singularities

Fix pair $(R, \mathcal{D})$, with $\mathcal{D} \subset \mathcal{C}^R$ a Cartier subalgebra

- $(R, \mathcal{D})$ is *Frobenius split* (or F-*split*) if $\exists e > 0$, $\phi \in \mathcal{D}_e$ with $\phi(F^e_* 1) = 1$.
- $(R, \mathcal{D})$ is *strongly F-regular* if $\forall r$ not in minimal primes of R, $\exists e > 0$, $\phi \in \mathcal{D}_e$ with $\phi(F^e_* r) = 1$.
- Ideal $J \subset R$ is $\mathcal{D}$-compatible if $\forall e > 0$, $\phi \in \mathcal{D}_e$, have $\phi(F^e_* J) \subset J$. Equivalently, for the quotient ring R/J, ϕ induces a map in $\mathcal{C}^{R/J}_e$.

Cartier core

For $J \subset R$ and $\mathcal{D} \subset \mathcal{C}^R$, the *Cartier core* of J with respect to $\mathcal{D}$ is

$$\mathrm{C}_{\mathcal{D}}(J) := \bigcap_{e>0} \{ r \in R \mid \phi(F^e_* r) \in J \ \forall \phi \in \mathcal{D}_e \}.$$

Cartier core map

R an F-finite Noetherian ring; $\mathcal{D}$ a Cartier subalgebra; $\mathcal{U}_{\mathcal{D}}$ the F-split locus of $(R, \mathcal{D})$. We prove:

- Cartier core gives map $\mathrm{C}_{\mathcal{D}} : \mathcal{U}_{\mathcal{D}} \to \mathcal{U}_{\mathcal{D}}$ which is **continuous** and **preserves containment**.
- The image of $\mathrm{C}_{\mathcal{D}}$ is the set of $\mathcal{D}$-compatible ideals in $\mathcal{U}_{\mathcal{D}}$, and these are **fixed** by $\mathrm{C}_{\mathcal{D}}$.
- The image is the set of minimal primes of R precisely when the pair $(R, \mathcal{D})$ is strongly F-regular.

Key Properties: General

Fix pair $(R, \mathcal{D})$ with $\mathcal{D} \subset \mathcal{C}^R$.

- **Localization:** if J ideal, W multiplicative set avoiding primes in $\operatorname{Ass}(J)$, then
$$\mathrm{C}_{\mathcal{D}}(J) = C_{W^{-1}\mathcal{D}}(JW^{-1}R) \cap R$$
$$\mathrm{C}_{\mathcal{D}}(J)W^{-1}R = C_{W^{-1}\mathcal{D}}(JW^{-1}R)$$
- **Lattice:** the set of Cartier cores forms a lattice under $+$ and $\cap$

Application: Stanley-Reisner

Let $R = k[x_1, \ldots, x_n]/I$ for k an F-finite field and I a square-free monomial ideal. We work with full Cartier algebra, so write $\mathrm{C}_R := \mathrm{C}_{\mathcal{C}^R}$. We prove:

- For Q prime ideal,
$$C_R(Q) = \sum_{\substack{P \in \operatorname{Min}(R) \\ P \subseteq Q}} P$$
- For J any ideal,
$$\mathrm{C}_R(J) = \sum_{\substack{\mathcal{Q} \subset \operatorname{Min}(R) \\ \left(\bigcap_{P\in\mathcal{Q}} P\right) \subset J}} \left(\bigcap_{P \in \mathcal{Q}} P \right)$$

Key Properties: Quotients

Assume $R = S/I$ is a quotient of regular ring S.

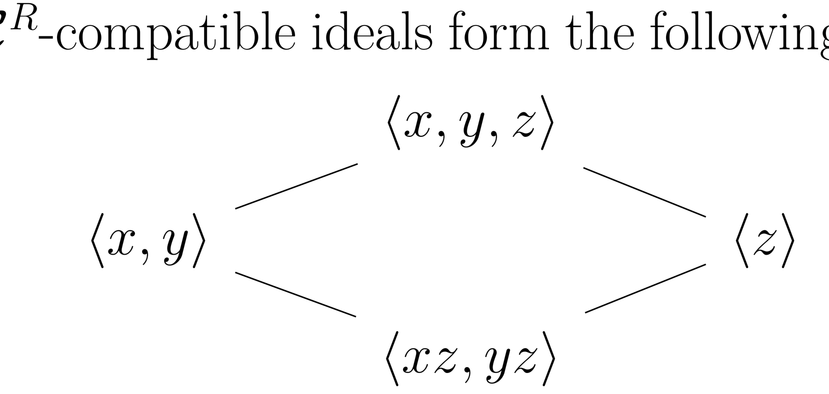
- **Fedder/Glassbrenner-like description:**
$$\mathrm{C}_R(J) = \left(\bigcap_{e \geq 1} J^{[p^e]} :_S (I^{[p^e]} :_S I) \right) / I$$
- **Adjoining variables:** For J' an ideal of $R[x]$ with $JR[x] \subseteq J' \subseteq JR[x] + \langle x \rangle$, get
$$\mathrm{C}_{R[x]}(J') = \mathrm{C}_R(J)R[x]$$
$$\mathrm{C}_R(J) = \mathrm{C}_{R[x]}(J') \cap R$$
- **Homogenization:** For S a polynomial ring, I homogeneous, ^{h} the minimal homogenization in $R[t]$, and $\delta : R[t] \to R$ via $\delta(t) = 1$ the corresponding dehomogenization, get
$$(\mathrm{C}_R(J))^h = \mathrm{C}_{R[t]}(J^h)$$
$$\mathrm{C}_R(J) = \delta(\mathrm{C}_{R[t]}(J^h))$$

Example: $k[x, y, z]/\langle xz, yz \rangle$

If $R = k[x, y, z]/\langle xz, yz \rangle$, then

$$\operatorname{Min}(R) = \{ \langle x, y \rangle, \ \langle z \rangle \}.$$

The $\mathcal{C}^R$-compatible ideals form the following lattice.

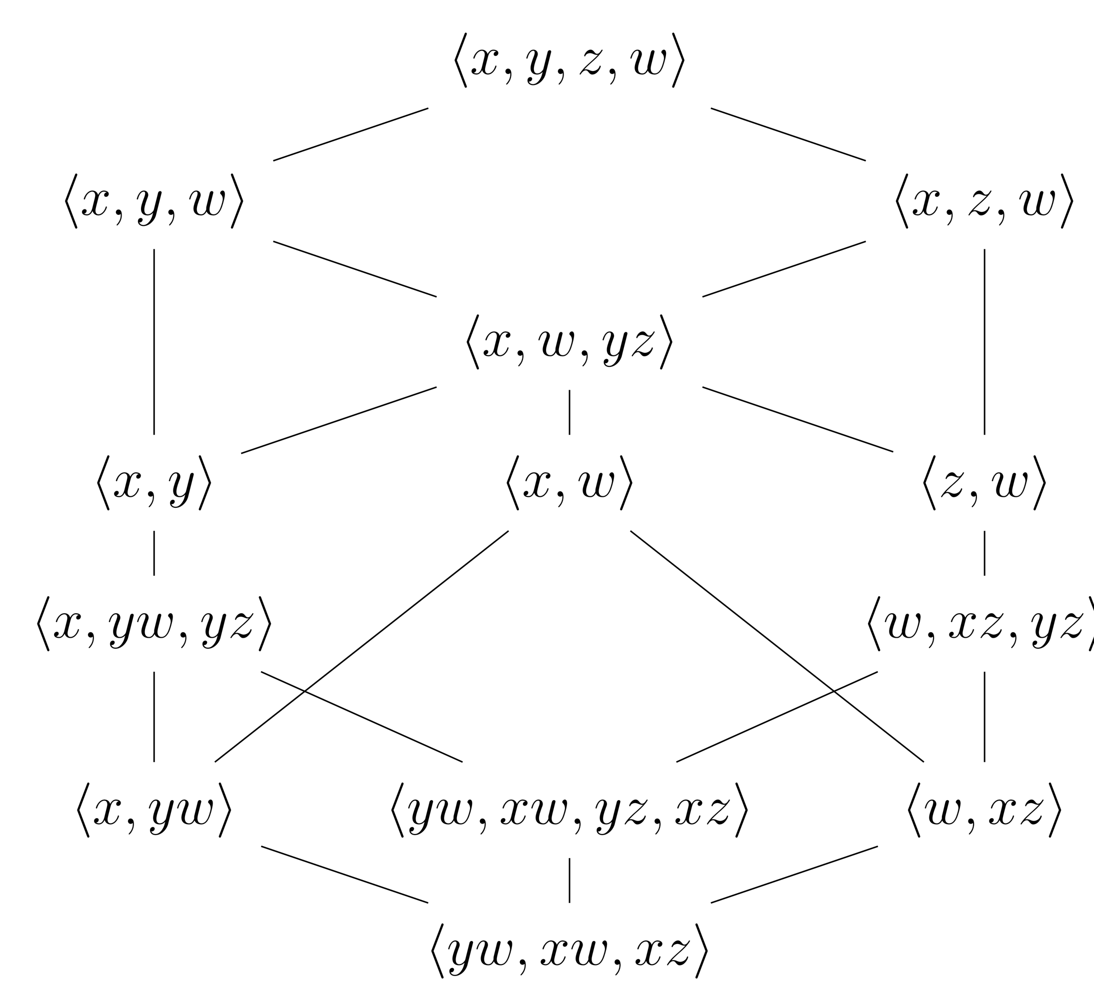


Example: $k[x, y, z, w]/\langle yw, xw, xz \rangle$

If $R = k[x, y, z, w]/\langle yw, xw, xz \rangle$ then

$$\operatorname{Min}(R) = \{ \langle x, y \rangle, \ \langle x, w \rangle, \ \langle z, w \rangle \}.$$

The $\mathcal{C}^R$-compatible ideals, i.e., the image of the map C_R, form the following lattice.



References

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