Counting the points in the Hilbert scheme

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Outline

1. Background
   - Modules
   - Hilbert schemes
2. Recursion
   - Motivation
   - Going down
   - Coming up
   - Formula

Definition

Definition

For a ring R, an R-*module* M is an additive abelian group with an operation $\cdot : R \times M \to M$ such that for all $r_1, r_2 \in R$, $m_1, m_2 \in M$, we have

- $r \cdot (m_1 + m_2) = r \cdot m_1 + r \cdot m_2$
- $(r_1 + r_2) \cdot m = r_1 \cdot m + r_2 \cdot m$
- $1_R \cdot m = m$
- $r_1 \cdot (r_2 \cdot m) = (r_1 r_2) \cdot m$.

Examples:

- $\mathbb{R}^n$ and $\mathbb{Z}^n$ are $\mathbb{Z}$-modules using usual multiplication.
- Any ring R is an R-module over itself.

Torsion

Definition

Let M be an R-module, for R a ring. Then $m \neq 0 \in M$ is *torsion* if there exists some $r \neq 0 \in R$ such that $rm = 0$. M is called a *torsion module* if every $m \in M$ is torsion. If no $m \in M$ is torsion, then M is *torsion-free*.

Examples:

- $\mathbb{R}^n$ is a torsion-free $\mathbb{R}$-module, since $a \cdot \vec{v} = \vec{0}$ implies $a = 0$ or $\vec{b} = \vec{0}$ for any $a \in \mathbb{R}$ and $\vec{b} \in \mathbb{R}^n$.
- $\mathbb{Z}/\mathbb{Z}_n$ is a torsion $\mathbb{Z}$-module since for any $a \in \mathbb{Z}/\mathbb{Z}_n$, $n \cdot a = na = 0 \in \mathbb{Z}/\mathbb{Z}_n$.

Definition

Definition

Let $k = \mathbb{F}_q$ be a finite field with q elements, and $R = k[y]$. The punctual Hilbert scheme of type (m_0, m_1) is defined as

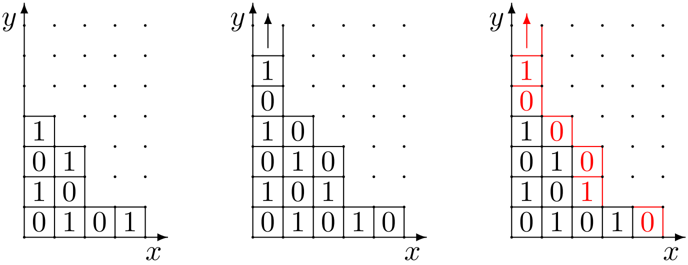
$$\mathrm{Hilb}_0^{(m_0,m_1)} k^2 = \{I \subseteq k[x,y] \mid k[x,y]/I \simeq m_0\rho_0 + m_1\rho_1, \quad V(I) = 0\}.$$

The stratified version is defined as

$$\mathrm{Hilb}_0^{(m_0,m_1)(d_0,d_1)} k^2 = \{I \in \mathrm{Hilb}_0^{(m_0,m_1)} k^2 \mid I|_l \simeq F_I \oplus T_I, \quad T_I \simeq d_0\rho_0 + d_1\rho_1\}.$$

where F_I is a torsion-free R-module, T_I is a torsion R-module, and $I|_l = I/x \cdot I$.

Example



I $\quad$ $x \cdot I$ $\quad$ $I/(x \cdot I)$

- $I = \langle y^4, xy^3, x^2y, x^4 \rangle \in \mathrm{Hilb}_0^{(4,5)} k^2$
- $x \cdot I = \langle xy^4, x^2y^3, x^3y, x^5 \rangle$
- $I/(x \cdot I) \simeq Ry^5 \oplus kxy^3 \oplus kx^2y^2 \oplus kx^2y \oplus kx^4$
- $T_I = kxy^3 \oplus kx^2y^2 \oplus kx^2y \oplus kx^4 \simeq 3\rho_0 + \rho_1$
- $F_I = Ry^5$

Outline of our goal

- Find the generating function for the Hilbert scheme of points, which has the form

$$\sum_{m_0,m_1\geq 0}\left(\#\mathrm{Hilb}^{(m_0,m_1)}k^2\right)\cdot t_0^{m_0}t_1^{m_1}$$

  where $k=\mathbb{F}_q$.
- Need to count the number of points in $\mathrm{Hilb}_0^{(m_0,m_1)}k^2$.
- Do this by counting points in the stratified version.
  - $\mathrm{Hilb}_0^{(m_0,m_1)}k^2=\bigcup_{d_0,d_1\geq 0}\mathrm{Hilb}_0^{(m_0,m_1)(d_0,d_1)}k^2$, so $\#\mathrm{Hilb}_0^{(m_0,m_1)}k^2=\sum_{d_0,d_1\geq 0}\#\mathrm{Hilb}_0^{(m_0,m_1)(d_0,d_1)}k^2$
- Specifically, want a recursion giving the number of points in stratified Hilbert scheme in terms of number of points in smaller Hilbert scheme

Getting I'

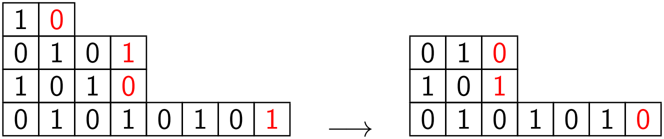
For any ideal I, define $x \cdot I'$ to be the kernel of the map $I \to I|_l \to F_I$. Exact commutative diagram shows uniqueness.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & T_I & \longrightarrow & I|_l & \longrightarrow & F_I & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & x \cdot I' & \longrightarrow & I & \longrightarrow & F_I & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & x \cdot I & = & x \cdot I & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

For monomial ideals, get I' by deleting the first column of the Young diagram, so if $I \in \mathrm{Hilb}_0^{(m_0,m_1)(d_0,d_1)} k^2$, then $I' \in \mathrm{Hilb}_0^{(m_0-d_1,m_1-d_0)(d_0',d_1')} k^2$.

Why $(m_0 - d_1, m_1 - d_0)$?

- Recurse by "chopping off" first column and sliding diagram over, then counting which ideals give same diagram.
- Same as removing last block in each row
- 0 in torsion part $\Rightarrow$ 1 in last box.
- 1 in torsion part $\Rightarrow$ 0 in last box.
- Also requires $d_0' \leq d_1$ and $d_1' \leq d_0$ in smaller scheme.



Recover I from I'

Fix a torsion module T and map $\varphi\colon I' \to T$, and define $I = \ker\varphi$. Exact commutative diagram shows if $F = \ker I'|_l \to T$ is torsion-free, then $I|_l \simeq F \oplus x \cdot T$.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & F & \longrightarrow & I'|_l & \longrightarrow & T & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & I & \longrightarrow & I' & \longrightarrow & T & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & x \cdot I' & = & x \cdot I' & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Choosing torsion-free

For any $I' \in \mathrm{Hilb}_0^{(m_0-d_1,m_1-d_0)(d_0',d_1')}k^2$, the number of possible I it came from is the number of F such that F is a rank 1 torsion-free submodule of $I'|_l$ with $I'|_l/F \simeq T \simeq d_1\rho_0 + d_0\rho_1$.

- Rank 1 since the torsion-free part is always the first column above the Young diagram, generated by single element y^a.
- $d_1\rho_0 + d_0\rho_1$ since we require $x \cdot T \simeq d_0\rho_0 + d_1\rho_1$ and multiplying by x switches the parity of basis elements.

Or, number of $F \subseteq F_{I'}$, rank 1 and torsion free, with $F_{I'}/F \simeq (d_1 - d_0')\rho_0 + (d_0 - d_1')\rho_1$ times number of ways to embed into $I'|_l$.

How many F?

From [1], at most one $F \subseteq F_{I'}$ which works. If I' is a monomial ideal, then $F_{I'} = Ry^{d_0'+d_1'}$ and $F = Ry^{d_0+d_1}$. Basis for $F_{I'}/F$ is $\{y^j \mid d_0' + d_1' \le j < d_0 + d_1\}$. Since $F_{I'}/F \simeq (d_1 - d_0')\rho_0 + (d_0 - d_1')\rho_1$, must have $d_1 - d_0'$ even degree basis elements and $d_0 - d_1'$ odd degree ones. Three cases to check:

1. If $d_0' + d_1' + d_0 + d_1 \equiv 0 \mod 2$, then $d_0 - d_1 = d_1' - d_0'$.
2. If $d_0' + d_1' + d_0 + d_1 \equiv 1 \mod 2$ and $d_0' + d_1' \equiv 0 \mod 2$, then $1 + d_0 - d_1 = d_1' - d_0'$.
3. If $d_0' + d_1' + d_0 + d_1 \equiv 1 \mod 2$ and $d_0' + d_1' \equiv 1 \mod 2$, then $d_0 - d_1 - 1 = d_1' - d_0'$.

so $d_1' - d_0' = d_0 - d_1 + (-1)^{d_0'+d_1'}((d_0' + d_1' + d_0 + d_1)\%2)$.

How many ways to embed?

Suppose $F = Ry^{d_0+d_1}$, $b_1, \ldots, b_{d_0'}$ are basis for trivial torsion elements, and $c_1, \ldots, c_{d_1'}$ basis for non-trivial torsion elements. If we don't care about type, then can embed F as

$$\widetilde{F} := R(y^{d_0+d_1}, \sum_{i=1}^{d_0'} \beta_i b_i + \sum_{j=1}^{d_1'} \gamma_j c_j)$$

for any $\beta_i, \gamma_j \in k$. q choices for each $\Rightarrow q^{d_0'+d_1'}$ possible $\widetilde{F}$.
We do care about type, so can only use torsion elements of same type as y^a. Therefore q^r possible $\widetilde{F}$, where

$$r = \begin{cases} d_0' & \text{if } d_0 + d_1 \equiv 0 \mod 2 \\ d_1' & \text{if } d_0 + d_1 \equiv 1 \mod 2 \end{cases}$$

Re-CURSE-ion

$$\#\mathrm{Hilb}_0^{(m_0,m_1)(d_0,d_1)} k^2 = \sum_{\substack{0\le d_0'\le d_1 \\ 0\le d_1'\le d_0 \\ d_1'-d_0'=d_0-d_1+(-1)^{(d_0'+d_1')}((d_0'+d_1'+d_0+d_1)\%2)}} q^r \cdot \#\mathrm{Hilb}_0^{(m_0-d_1,m_1-d_0)(d_0',d_1')} k^2$$

where

$$r = \begin{cases} d_0' & \text{if } d_0+d_1 \equiv 0 \mod 2 \\ d_1' & \text{if } d_0+d_1 \equiv 1 \mod 2 \end{cases}$$

Let $a, b, c, d \in \mathbb{Z}_{\geq 0}$. The base cases are

$$\#\mathrm{Hilb}_0^{(0,b>0),(c,d)} k^2 = 0 \qquad \#\mathrm{Hilb}_0^{(a,b),(c>b,d)} k^2 = 0$$

$$\#\mathrm{Hilb}_0^{(a,b),(c,d>a)} k^2 = 0 \qquad \#\mathrm{Hilb}_0^{(a\neq 0,b),(0,0)} k^2 = 0$$

$$\#\mathrm{Hilb}_0^{(0,0),(0,0)} k^2 = 1$$

Summary

- Found recursion for number of points in stratified Hilbert scheme!
- Hard to work with, so unsuccessful in finding a closed form with this method.
- Still interesting, especially because of special case closed formulas.
- In Future
  - Look at more special cases.
  - Pursue abacus method.

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