Parametrizations of k-Nonnegative Matrices

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Outline

Background

Factorizations

Cluster Algebras

Background

Introduction

In 1999, Fomin and Zelevinsky studied totally nonnegative matrices.
They explored two questions:

1. How can totally nonnegative matrices be parameterized?
2. How can we test a matrix for total positivity?

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In 1999, Fomin and Zelevinsky studied totally nonnegative matrices.
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1. How can totally nonnegative matrices be parameterized?
2. How can we test a matrix for total positivity?

We will explore the same questions for k-nonnegative and k-positive matrices.

Definition

A matrix M is *k-nonnegative* (respectively *k-positive*) if all minors of order k or less are nonnegative (respectively positive).

k-Nonnegativity

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Lemma

A matrix M is k-positive if all solid minors of order k or less are positive.

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Lemma

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Lemma

A matrix M is k-nonnegative if all column-solid minors of order k or less are nonnegative.

Factorizations

Chevalley generators

Loewner-Whitney Theorem: An invertible totally nonnegative matrix can be written as a product of e_i's, f_i's and h_i's with nonnegative entries.

$$e_i(a) = \begin{bmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ \vdots & \ddots & 0 & \dots & \dots & \vdots \\ 0 & \dots & 1 & a & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & 1 \end{bmatrix}, f_i(a) = \begin{bmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ \vdots & \ddots & 0 & \dots & \dots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & a & 1 & \dots & 0 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & 1 \end{bmatrix}$$

$$h_i(a) = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & \ddots & 0 & \ddots & \vdots \\ \vdots & \ddots & a & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix}$$

Row and Column Reductions

Lemma

If a matrix M is k-nonnegative, it can be reduced to have a $k-1$ "staircase" of 0s in its northeast and southwest corners while preserving k-nonnegativity.

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$$
\begin{array}{rcl}
 & \begin{array}{ccccccc} & & & \multicolumn{4}{c}{\overbrace{\hphantom{0 \quad 0 \quad \cdots \quad 0}}^{k-1}} \end{array} & \\
\begin{array}{r} \\ \\ \\ \\ k-1\left\{\begin{array}{l} \\ \\ \\ \end{array}\right. \end{array} &
\begin{bmatrix}
 & & & 0 & 0 & \cdots & 0 \\
 & & & & \ddots & \ddots & \vdots \\
 & & & & & \ddots & 0 \\
0 & & & & & & 0 \\
0 & \ddots & & & & & \\
\vdots & \ddots & \ddots & & & & \\
0 & \cdots & 0 & 0 & & &
\end{bmatrix}
& \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} k-1 \\ \\ \\ \\ \end{array} \\
 & \begin{array}{ccccccc} \multicolumn{4}{c}{\underbrace{\hphantom{0 \quad \cdots \quad 0 \quad 0}}_{k-1}} & & & \end{array} &
\end{array}
$$

Generators

Theorem

The semigroup of $n-1$-nonnegative invertible matrices is generated by the Chevalley generators and the $\mathcal{K}$-generators.

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The $\mathcal{K}$-generators have the following form.

$$\mathcal{K}(\vec{x},\vec{y}) = \begin{bmatrix} x_1 & x_1y_1 & \dots & \dots & \dots & \dots \\ 1 & x_2+y_1 & x_2y_2 & \dots & \dots & \dots \\ \dots & \ddots & \ddots & \ddots & \dots & \dots \\ \dots & \dots & 1 & x_{n-3}+y_{n-4} & x_{n-3}y_{n-3} & \dots \\ \dots & \dots & \dots & 1 & y_{n-3} & y_{n-2}Y \\ \dots & \dots & \dots & \dots & 1 & X \end{bmatrix},$$

$Y = y_1 \cdots y_{n-3}$

$X = x_2x_3 \cdots x_{n-3} + y_1x_3 \cdots x_{n-3} + y_1y_2x_3 \cdots x_{n-3} + \ldots + y_1 \cdots y_{n-4}.$

Relations

$$e_j(a) \cdot \mathcal{K}(\vec{x}, \vec{y}) = \mathcal{K}(\vec{u}, \vec{v}) \cdot e_{j+1}(b) \text{ where } 1 \leq j \leq n-2$$

$$e_{n-1}(a) \cdot \mathcal{K}(\vec{x}, \vec{y}) = h_n(b) \cdot \mathcal{K}(\vec{u}, \vec{v}) \cdot f_{n-1}(c)$$

$$h_{j+2}(c) \cdot f_{j+1}(a) \cdot \mathcal{K}(\vec{x}, \vec{y}) = \mathcal{K}(\vec{u}, \vec{v}) \cdot f_j(b) \cdot h_j(c) \text{ where } 1 \leq j \leq n-2$$

$$f_1(a) \cdot \mathcal{K}(\vec{x}, \vec{y}) \cdot h_1(c) = \mathcal{K}(\vec{u}, \vec{v}) \cdot e_1(c)$$

$$h_{j+1}(a) \cdot \mathcal{K}(\vec{x}, \vec{y}) = \mathcal{K}(\vec{u}, \vec{v}) \cdot h_j(a) \text{ where } 1 \leq j \leq n-2.$$

Generators

Theorem

The semigroup of $n-2$-nonnegative upper unitriangular matrices is generated by the e_i's and the $\mathcal{T}$-generators.

The $\mathcal{T}$-generators have the following form.

$$\mathcal{T}(\vec{x},\vec{y}) = \begin{bmatrix} 1 & x_1 & x_1y_1 & \dots & \dots & \dots & \dots \\ \dots & 1 & x_2+y_1 & x_2y_2 & \dots & \dots & \dots \\ \dots & \dots & \ddots & \ddots & \ddots & \dots & \dots \\ \dots & \dots & \dots & 1 & x_{n-3}+y_{n-4} & x_{n-3}y_{n-3} & \dots \\ \dots & \dots & \dots & \dots & 1 & y_{n-3} & y_{n-2}Y \\ \dots & \dots & \dots & \dots & \dots & 1 & X \\ \dots & \dots & \dots & \dots & \dots & \dots & 1 \end{bmatrix}$$

$$Y = y_1 \cdots y_{n-3}$$

$$X = x_2x_3\cdots x_{n-3} + y_1x_3\cdots x_{n-3} + y_1y_2x_3\cdots x_{n-3} + \ldots + y_1\cdots y_{n-4}.$$

Relations

$$e_j(a) \cdot \mathcal{T}(\vec{x}, \vec{y}) = \mathcal{T}(\vec{u}, \vec{v}) \cdot e_{j+2}(b) \text{ where } 1 \leq j \leq n-3$$

$$e_{n-2}(a) \cdot \mathcal{T}(\vec{x}, \vec{y}) = \mathcal{T}(\vec{u}, \vec{v}) \cdot e_1(b)$$

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Reduced Words

Alphabet $\mathcal{A} = \{1, 2, \ldots, n-1, \mathcal{T}\}$.
Let α be the word $(n-2)\ldots 1(n-1)\ldots 1$.

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The reduced words are:

$$w \in \begin{cases} w'\mathcal{T} & w'\alpha \text{ is reduced,} \\ w'(n-1)\mathcal{T} & w'\alpha \text{ is reduced,} \\ w'(n-2)\mathcal{T} & w'\alpha \text{ is reduced,} \\ w'(n-1)(n-2)\mathcal{T} & w'\alpha \text{ is reduced,} \\ w' & w' < \beta \text{ or } w' \text{ is incomparable to } \beta. \end{cases}$$

where w' does not involve $\mathcal{T}$.

Bruhat Cells

Define $V(w)$ to be the set of matrices which correspond to the reduced word w. (Then $V(w) = \{e_{w_1}(a_1)e_{w_2}(a_2)\cdots e_{w_k}(a_k)\}$.)

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Theorem

For reduced words u and w, if $u \neq w$ then $V(u) \cap V(w) = \emptyset$.

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Theorem

The poset on $\{V(w)\}$ given by the Bruhat order on reduced words $\{w\}$ is graded.

Bruhat Cells

Conjecture

The closure of a cell $\overline{V(w)}$ is the disjoint union of all cells in the interval between $\emptyset$ and $V(w)$.

Cluster Algebras

k-initial minors

Definition

A *k-initial minor* at location (i, j) of a matrix X is the maximal solid minor with (i, j) as the lower right corner which is contained in a $k \times k$ box.

The set of all k-initial minors gives a k-positivity test!

$$\begin{bmatrix} 11 & 12 & 13 & 14 & 15 & 16 \\ 21 & 22 & 23 & 24 & 25 & 26 \\ 31 & 32 & 33 & 34 & 35 & 36 \\ 41 & 42 & 43 & 44 & 45 & 46 \\ 51 & 52 & 53 & 54 & 55 & 56 \\ 61 & 62 & 63 & 64 & 65 & 66 \end{bmatrix}, \begin{bmatrix} 11 & 12 & 13 & 14 & 15 & 16 \\ 21 & 22 & 23 & 24 & 25 & 26 \\ 31 & 32 & 33 & 34 & 35 & 36 \\ 41 & 42 & 43 & 44 & 45 & 46 \\ 51 & 52 & 53 & 54 & 55 & 56 \\ 61 & 62 & 63 & 64 & 65 & 66 \end{bmatrix}$$

4-initial minors

Motivation

With total positivity tests, can "exchange" some minors for others.

Example

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Both $\{a, b, c, \det M\}$ and $\{d, b, c, \det M\}$ give total positivity tests.

Note

$$ad = bc + \det M$$

i.e. have a *subtraction-free* expression relating exchanged minors.

Definitions

Definition

A *seed* is a tuple of variables $\tilde{\mathbf{x}}$ along with some *exchange relations* of the form

$$x_i x_i' = p_i(\tilde{\mathbf{x}} \setminus x_i)$$

which allow variable x_i to be swapped for a new variable x_i'.

- *frozen variables*: not exchangeable
- *cluster variables*: are exchangeable
- *extended cluster*: entire tuple $\tilde{\mathbf{x}}$
- *cluster*: only the cluster variables

A seed (plus all seeds obtained by doing chains of exchanges) generates a *cluster algebra*. Our p_i are always *subtraction-free*.

Total Positivity Cluster Algebra

Example

Initial seed: $\tilde{\mathbf{x}}$ is minors of n-initial minors test. *Corner minors* (lower right corner on bottom or right edge) are frozen variables. There is a rule for generating the exchange relations for all other variables.

Subtraction-freeness means that any seed reachable from the initial one gives a different total positivity test.

Can we use this idea to get k-positivity tests? Yes!

k-positivity Cluster Algebras

Total positivity seed where all variables = minors.

Cluster variables:	Exchange polynomials:
X_1^1	$X_1^2 \cdot X_2^1 + X_{12,12}$
X_1^2	$X_1^3 \cdot X_{12}^{12} + X_1^1 \cdot X_{12}^{23}$
X_2^1	$X_3^1 \cdot X_{12}^{12} + X_1^1 \cdot X_{23}^{12}$
X_{12}^{12}	$X_2^1 \cdot X_1^2 \cdot \det + X_1^1 \cdot X_{23}^{12} \cdot X_{12}^{23}$

Frozen variables:

$X_1^3 \qquad X_{12}^{23} \qquad X_3^1 \qquad X_{23}^{12} \qquad \det$

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Total positivity seed where all variables = minors.

Exchange polynomial uses minor of order $> k \implies$ freeze variable.

Delete variables whose minors are "too big".

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Frozen variables:

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Getting Tests

Definition

The *test cluster* of a seed is the extended cluster, but with more minors added until we have n^2 which combined give a k-positivity test. These extra *test variables* are the same for all seeds in the cluster algebra.

Example

Restricted n-initial minors seed $+$ missing solid minors of order k $=$ the k-initial minors test.

Don't (in general) know how to choose test variables to get a valid k-positivity test. Some seeds can't be extended to give tests (of size n^2) at all!

Exchange Graph

Definition

The *exchange graph* has vertices = clusters, and edges between clusters with exchange relations connecting them.

Example

For $n = 2$ total positivity cluster algebra:

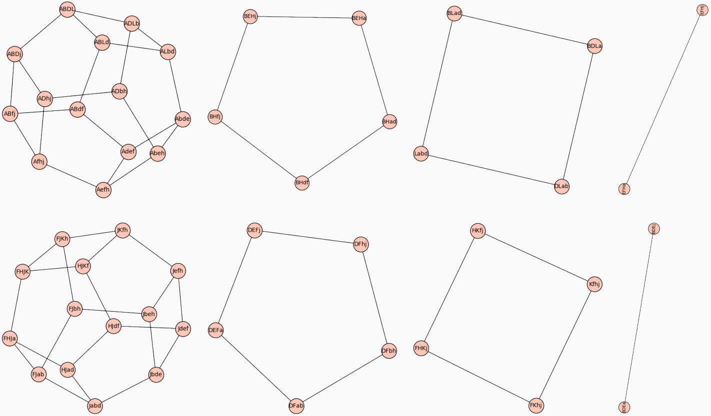
Example: $n = 3$, $k = 2$

For 3×3 matrices, when we restrict exchanges to those only involving minors of size ≤ 2, the exchange graph breaks into 8 components.

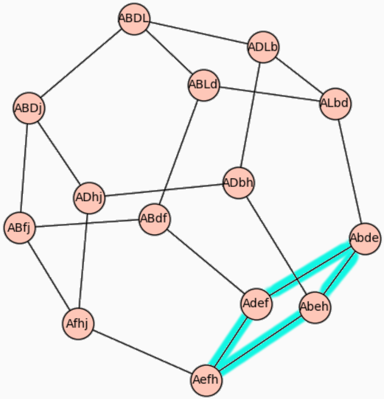
Only the two largest components provide actual 2-positivity tests.

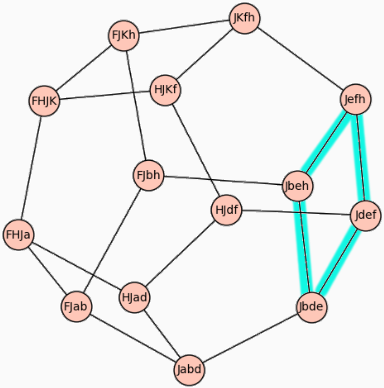
These two components share 4 vertices that correspond to different total positivity tests but restrict to the same 2-positivity tests. We say that these 4 overlapping vertices form a "bridge" between the components.

Connected Components of 2-pos test graph for 3×3 matrix



Test Components





Frozen variables: c,g,C,G,A

Test variable: J

Frozen variables: c,g,C,G,J

Test variable: A

k-essential minors

Definition

A minor is *k-essential* if there exists a matrix in which all other minors of size $\leq k$ are positive, while that minor is non-positive.

In other words, a k-essential minor is one which must be present in all k-positivity tests.

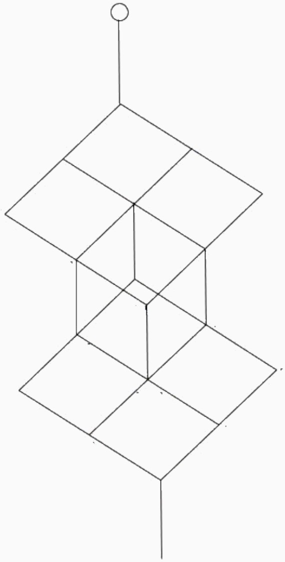
Conjecture

The k-essential minors are the corner minors of size $< k$, together with all solid k-minors.

So far, this conjecture has only been proven for the cases of $k \leq 3$.

We also observe that in all known cases, a bridge involves switching the positions of an essential minor in the extended cluster with one outside it.

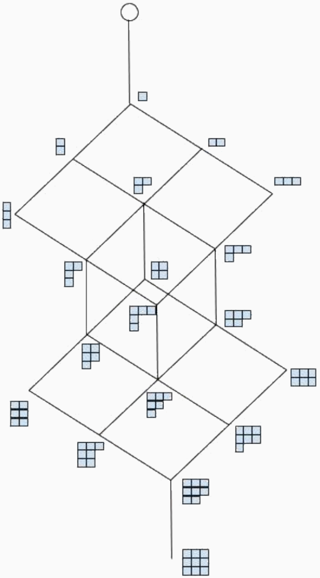
Connecting Tests



Although there are many choices to be made regarding the exact order in which some exchanges are made, we can generally speak of a natural family of paths linking the k-initial minors test to its antidiagonal flip.

If we ignore non-bridge mutations and treat each connected component as a single vertex, we get a "bridge graph".

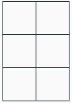
Connecting Tests



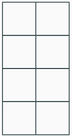
By the construction of the path, all involved bridges switch out a solid k-minor with a minor one entry down and to the left of it, yielding a total of $(n-k)^2$ distinct bridges, that we can represent as boxes in a $(n-k)\times(n-k)$ square. The components can thus be indexed by Young diagrams, with each box indicating a specific bridge that must be crossed to reach that component from the one including the k-initial minors.

$n = 5,\ k = 2$

$$\rightarrow \begin{bmatrix} 2,2 & 2,3 & 1,3 & 1,4 & 1,5 \\ 2,1 & 23,23 & 23,34 & 12,34 & 12,45 \\ 3,1 & 23,12 & 123,123 & 123,234 & 123,345 \\ 4,1 & 34,12 & 234,123 & 234,234 & 234,345 \\ 5,1 & 45,12 & 345,123 & 345,234 & 345,345 \end{bmatrix}$$



$\rightarrow$



$$\rightarrow \begin{bmatrix} 3,3 & 2,3 & 1,3 & 1,4 & 1,5 \\ 3,2 & 34,34 & 23,34 & 12,34 & 12,45 \\ 4,2 & 45,34 & 123,123 & 123,234 & 123,345 \\ 5,2 & 45,23 & 234,123 & 234,234 & 234,345 \\ 5,1 & 45,12 & 345,123 & 345,234 & 345,345 \end{bmatrix}$$

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