# Algebraic Geometry I, Fall 2021
## Problem Set 1

*Due Friday, September 10, 2021 at 5 pm*

1. Familiarize yourself with the following notions from category theory: categories, functors, equivalences of categories, adjoint functors, and limits/colimits. There are many sources for this material. For instance, among the reference books for the class, you could look at
   - Appendix A of *Algebraic Geometry I* by Görtz and Wedhorn, or
   - Chapter 1 of *Foundations of Algebraic Geometry* by Vakil.

2. Prove the following result stated in class. Please feel free to use results we proved in class.

   **Theorem 1.** *Let $k$ be an algebraically closed field. Let $\mathrm{AffVar}_k$ denote the category of affine varieties over $k$, and let $\mathrm{Alg}_k^{\mathrm{rfg}}$ denote the category of reduced finitely generated $k$-algebras. Then the functor $\Gamma\colon (\mathrm{AffVar}_k)^{\mathrm{op}} \to \mathrm{Alg}_k^{\mathrm{rfg}}$ is an equivalence of categories, where $(-)^{\mathrm{op}}$ denotes the opposite category.*

   *Hint*: Use the criterion that a functor is an equivalence if and only if it is fully faithful and essentially surjective.

3. Let $X = V(y^2 - x^3) \subset \mathbf{C}^2$. Show that there exists a bijective morphism of affine varieties $\mathbf{C} \to X$, but that there does not exist an isomorphism of affine varieties $\mathbf{C} \cong X$.

4. Let $k$ be an algebraically closed field. Prove that all elements of $\mathrm{Spec}(k[x,y])$ are given as follows:
   - $(x-a, y-b)$ for $a, b \in k$.
   - $(f(x,y))$ where $f(x,y) \in k[x,y]$ is an irreducible polynomial.
   - $(0)$.

   *Hint:* One way to think about this geometrically is by studying the fibers of the map $\mathrm{Spec}(k[x,y]) \to \mathrm{Spec}(k[x])$ corresponding to the ring map $k[x] \to k[x,y]$.

5. Describe all elements of $\mathrm{Spec}(\mathbf{Z}[x])$, in the spirit of Problem 4.

6. For rings $A_1, \ldots, A_n$, prove that there is a homeomorphism
$$\mathrm{Spec}\left(\prod_{i=1}^{n} A_i\right) \cong \coprod_{i=1}^{n} \mathrm{Spec}(A_i).$$
   Prove or disprove that there is a homeomorphism
$$\mathrm{Spec}\left(\prod_{i=1}^{\infty} \mathbf{F}_2\right) \cong \coprod_{i=1}^{\infty} \mathrm{Spec}(\mathbf{F}_2).$$

7. Find the decomposition of $V(xy, xz, yz) \subset \mathrm{Spec}(\mathbf{C}[x,y,z])$ into irreducible components.