Algebraic Geometry I, Fall 2021

Problem Set 10

Due Friday, November 12, 2021 at 11:59 pm

1. Let $f\colon X \to S$ be a universally closed morphism of schemes. Suppose given a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{u} & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & \xrightarrow{v} & Y \end{array}$$

   where A is a discrete valuation ring and $K = \mathrm{Frac}(A)$. Show that a lift of v exists, i.e. there exists a morphism $\tilde{v}\colon \mathrm{Spec}(A) \to X$ such that $f \circ \tilde{v} = v$, by the following steps:

   (a) Show that it is enough to prove the case where $Y = \mathrm{Spec}(A)$.

   (b) From now on assume $Y = \mathrm{Spec}(A)$. Let $Z \subset X$ be the closure of the image point $p \in X$ of $u\colon \mathrm{Spec}(K) \to X$. Show that Z, with the reduced induced structure, is an integral scheme with function field isomorphic to K.

   (c) Show that there exists a point $q \in Z$ such that the local ring $\mathcal{O}_{Z,q}$ is isomorphic to A, and use this to show that the desired lift $\tilde{v}$ exists.

   Hint: Recall that if B is a local ring such that $A \subset B \subset K$ where $\mathfrak{m}_A = A \cap \mathfrak{m}_B$, then $A = B$. In fact, this maximality property is one of the characterizations of a valuation ring A, and this problem goes through more generally for any valuation ring instead of a DVR.

2. Let $f\colon X \to Y$ be a finite type morphism of schemes with Y locally noetherian. Suppose that f satisfies the existence part of the valuative criterion, i.e. for any commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{u} & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & \xrightarrow{v} & Y \end{array}$$

   where A is a discrete valuation ring and $K = \mathrm{Frac}(A)$, there exists a lift $\tilde{v}\colon \mathrm{Spec}(A) \to X$ of v. Show that f is a closed map. (This is the main step in the proof of the valuative criterion of properness.)

   You may use that a constructible subset of a scheme is closed if and only if it is stable under specialization, see `https://stacks.math.columbia.edu/tag/0542`.

3. Let R be a ring and A a local ring. Show that there is an identification

$$\mathrm{Hom}_{\mathrm{Spec}(R)}(\mathrm{Spec}(A), \mathbf{P}^n_R) \cong \frac{\{(x_0, \ldots, x_n) \in A^{n+1} \mid x_i \in A^\times \text{ for some } i\}}{A^\times}$$

   where $A^\times$ acts on the set of tuples by multiplication.

4. Let X be a proper variety over a field k. Let $p \in \mathbf{A}_k^1$ be the origin (corresponding to the ideal $(t) \subset k[t]$). Show that if $f\colon \mathbf{A}_k^1 \backslash \{p\} \to X$ is a morphism of k-varieties (i.e. a morphism of Spec(k)-schemes), then there exists a unique morphism of k-varieties $\widetilde{f}\colon \mathbf{A}_k^1 \to X$ such that $\widetilde{f}|_{\mathbf{A}_k^1 \backslash \{p\}} = f$.

5. Read about the definition of dimension of schemes in Section 11.1 of *Foundations of Algebraic Geometry* by Vakil. Prove that if $f\colon X \to Y$ is a finite surjective morphism of schemes, then $\dim X = \dim Y$.