Algebraic Geometry I, Fall 2021

Problem Set 12

Due Friday, December 3, 2021 at 11:59 pm

1. Do Exercise 13.7.F from *Foundations of Algebraic Geometry* by Vakil, and then prove:

   If X is an integral scheme and $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, then there exists a dense open subset $U \subset X$ such that $\mathcal{F}|_U$ is finite locally free.

2. Let $\phi \colon \mathcal{F} \to \mathcal{G}$ be a morphism of coherent sheaves on a locally noetherian scheme X.

   (a) Show that if $X = \mathbf{P}^1_k$ for a field k and $\mathcal{F}$ is finite locally free, then $\ker(\phi)$ is finite locally free.

   (b) Show that if $\mathcal{F}$ and $\mathcal{G}$ are finite locally free and ϕ is surjective, then $\ker(\phi)$ is finite locally free.

   (c) Give an example where $\mathcal{F}$ and $\mathcal{G}$ are finite locally free, but $\ker(\phi)$ is not finite locally free.

   (d) Give an example where $\mathcal{F}$ is finite locally free and ϕ is surjective, but $\ker(\phi)$ is not finite locally free.

3. Read about the relative spectrum in Section 17.1 of *Foundations of Algebraic Geometry* by Vakil, and then:

   (a) Do Exercise 17.1.F.

   (b) Do the first part of Exercise 17.1.G, i.e. the part "Show that the total space of. . . ". You will need the definition of the symmetric algebra of a quasi-coherent sheaf, see Section 13.5.3 of Vakil. (Note also that by definition, for a scheme U, affine space over U is by definition the scheme $\mathbf{A}^n_U = \mathbf{A}^n_{\mathbf{Z}} \times_{\mathrm{Spec}(\mathbf{Z})} U$; in the problem, you can avoid thinking about this by taking your neighborhood U of p to be an affine scheme.)

   Together, these exercises show that for a finitely locally free sheaf $\mathcal{F}$, the total space of $\mathcal{F}$ as defined in 17.1.4 can be thought of as a "vector bundle" over X.

4. Let X be a scheme. A Cartier divisor $D \in \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$ on X is called *effective* if it can be represented by $\{(U_i, f_i)\}$ where $f_i \in \mathcal{O}_X(U_i) \cap \mathcal{K}_X^\times(U_i)$. (Recall that $\mathcal{O}_X \hookrightarrow \mathcal{K}_X$, so we can consider the intersection $\mathcal{O}_X(U_i) \cap \mathcal{K}_X^\times(U_i)$ in $\mathcal{K}_X(U_i)$.)

   (a) Show that if D is an effective Cartier divisor, then $\mathcal{O}_X(-D)$ is a quasi-coherent ideal sheaf on X, and if $i \colon Y \hookrightarrow X$ denotes the corresponding closed subscheme $V(\mathcal{O}_X(-D))$ of X, then there is an exact sequence of $\mathcal{O}_X$-modules

   $$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0.$$

   (b) Conversely, suppose that $Y \hookrightarrow X$ is a closed subscheme such that there exists an affine cover $X = \bigcup U_i$ and non-zerodivisors $f_i \in \mathcal{O}_X(U_i)$ such that $Y \cap U_i = V(f_i)$ as a closed subscheme of U_i. Show that $\{(U_i, f_i)\}$ represents an effective Cartier divisor.

5. In class we defined the line bundle $\mathcal{O}_X(d)$ on $X = \mathbf{P}^1_k$, where k is a field and $d \in \mathbf{Z}$. Construct explicit data $\{(U_i, f_i)\}$ representing a Cartier divisor D such that there is an isomorphism $\mathcal{O}_X(D) \cong \mathcal{O}_X(d)$ of line bundles.