Algebraic Geometry I, Fall 2021
Problem Set 13

Due Friday, December 10, 2021 at 11:59 pm

In this problem set, "Vakil" refers to *Foundations of Algebraic Geometry* by Vakil.

You may freely use that for an integral variety X over a field k and an irreducible closed subscheme $Z \subset X$, we have $\dim(Z) + \operatorname{codim}_X(Z) = \dim(X)$ (see Vakil 11.2.9).

1. Do Vakil 5.4.H. Also read and understand Vakil 5.4.F and 5.4.J, but you don't need to write up the solutions to these two.

2. Let $X = V(xy - z^2) \subset \mathbf{A}^3_k = \operatorname{Spec} k[x, y, z]$ with k a field of characteristic not equal to 2. This is a normal and integral variety of dimension 2, cf. problem 5 below, hence satisfies all of the hypotheses from our discussion of Weil divisors in class.

   (a) Consider the prime divisor $Z = V(x, z) \subset X$. Show that Z is not in the image of the map $\operatorname{CaDiv}(X) \to \operatorname{WDiv}(X)$. In words: Z is a Weil divisor which is not Cartier. (In particular, this shows that Z is not a principal Weil divisor, and hence gives a nonzero element of $\operatorname{Cl}(X)$.)

   Hint: First, argue that if Z were the image of a Cartier divisor D, then D would necessarily be an effective Cartier divisor; for this, you will need to use the commutative algebra fact that if A is an integrally closed noetherian domain, then A is equal to the intersection inside $\operatorname{Frac}(A)$ of $A_\mathfrak{p}$ over all prime ideals $\mathfrak{p} \subset A$ such that $V(\mathfrak{p}) \subset \operatorname{Spec}(A)$ is codimension 1. Second, show that Z is locally cut out by a single equation, i.e. for any point $p \in X$ there exists an open affine neighborhood $U \subset X$ of p and $f \in \mathcal{O}_X(U)$ such that $Z \cap U = V(f)$ as a closed subscheme of U; you may want to use the same commutative algebra fact again. Finally, by taking p to be the origin, obtain a contradiction — see Vakil 12.1.3.

   (b) Deduce that X is not locally factorial.

   Remark: The normality of X in fact implies that the map $\operatorname{CaCl}(X) \to \operatorname{Cl}(X)$ is injective, so in this example we find $\operatorname{CaCl}(X) = 0$.

3. Let X be an affine scheme, and let D be an effective Cartier divisor on X. Let $Y \subset X$ be the closed subscheme defined by the ideal sheaf $\mathcal{O}_X(-D)$ as in Problem Set 12, problem 4(a). Show that the open subset $U := X \setminus Y \subset X$ is an affine scheme.

   Hint: Show that the morphism $U \to X$ is affine.

4. Let $X = V(xy - wz) \subset \mathbf{A}^4_k = \operatorname{Spec} k[x, y, z, w]$ with k a field of characteristic not equal to 2. This is a normal and integral variety of dimension 3, cf. problem 5 below, hence satisfies all of the hypotheses from our discussion of Weil divisors in class.

   Consider the prime divisor $Z = V(x, w) \subset X$. Show that for any integer $n \neq 0$, the Weil divisor $n[Z]$ is not in the image of the map $\operatorname{CaDiv}(X) \to \operatorname{WDiv}(X)$.

   Hint: If $n[Z]$ were the image of a Cartier divisor D, first show as in the hint to Problem 2 that D would necessarily be an effective Cartier divisor. Then use Problem 3 to obtain a contradiction.

5. Let k be an algebraically closed field of characteristic not equal to 2. Let

$$X = V(x_0^2 + x_1^2 + \cdots + x_n^2) \subset \mathbf{A}_k^{n+1} = \operatorname{Spec} k[x_0, \ldots, x_n].$$

   (a) Show that X is normal and integral if $n \geq 2$.

   (b) Prove that

$$\operatorname{Cl}(X) \cong \begin{cases} \mathbf{Z}/2 & \text{if } n = 2 \\ \mathbf{Z} & \text{if } n = 3 \\ 0 & \text{if } n \geq 4. \end{cases} \tag{1}$$

   Hint: Up to a change of variables, the equation of X is $x_0 x_1 - x_2^2 - x_3^2 - \cdots - x_n^2 = 0$. For $n = 2$ we will calculate this class group in lecture, and for $n \geq 4$ you can imitate our argument. For $n = 3$, change variables instead to the equation $xy - wz = 0$, and then use Problem 4.

6. Read the definition of a smooth scheme from Vakil 12.2.6. Smooth varieties over a field k are locally factorial, by a combination of Vakil 12.2.10 and 12.8.5.