Algebraic Geometry I, Fall 2021

Problem Set 5

Due Friday, October 8, 2021 at 5 pm

1. Let X be a scheme. Let A be a local ring with maximal ideal $\mathfrak{m} \subset A$.

   (a) Let $f\colon \operatorname{Spec}(A) \to X$ be a morphism of schemes, and let $p = f(\mathfrak{m}) \in X$. Show that any neighborhood $U \subset X$ of p contains $f(\operatorname{Spec}(A)) \subset X$.

   (b) Show that there is a bijection between the set of morphisms $f\colon \operatorname{Spec}(A) \to X$ and the set of pairs (p, ϕ) where $p \in X$ and $\phi\colon \mathcal{O}_{X,p} \to A$ is a local homomorphism, which takes f to the the pair $(p = f(\mathfrak{m}), \phi)$ where ϕ is the induced homomorphism $\mathcal{O}_{X,p} \to \mathcal{O}_{\operatorname{Spec}(A),\mathfrak{m}} = A$ on stalks.

2. Let X be a scheme. Show that the map

$$\begin{aligned} X &\to \{Z \subset X \mid Z \text{ closed and irreducible}\} \\ p &\mapsto \overline{\{p\}} \end{aligned}$$

   is a bijection, i.e. every closed irreducible subset of X has a unique generic point.

3. We say that a property P of rings is a local property if the following conditions hold for every ring A:

   (a) For $f \in A$, we have (A satisfies P) $\implies$ (A_f satisfies P).

   (b) For $f_i \in A$ such that $(f_1, \ldots, f_n) = A$, we have (A_{f_i} satisfies $P, \forall i$) $\implies$ (A satisfies P).

   Given such a property P, we say that a scheme X is *locally* P if there exists a cover $X = \bigcup U_i$ by affine open subschemes $U_i \subset X$ such that $\mathcal{O}_X(U_i)$ satisfies P for all i. (In the above language, we showed in class that the property of being noetherian is local, and defined locally noetherian schemes correspondingly.)

   (a) Show that if P is a local property of rings and X is a scheme which is locally P, then for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ satisfies P.

   (b) Show that the property of being finitely generated as a $\mathbf{Z}$-algebra is a local property of rings.

4. Let X be a scheme. Show that X is reduced if and only if the following condition holds: For any open subset $U \subset X$ and sections $f, g \in \mathcal{O}_X(U)$ such that $f(p) = g(p) \in \kappa(p)$ for all $p \in U$, we have $f = g \in \mathcal{O}_X(U)$. As in class, $f(p)$ denotes the image of f in the residue field $\kappa(p)$ of $p \in X$. In other words, this problem shows that reduced schemes are ones where sections are determined by their values at points.

5. Let X be a scheme. Let $(\mathcal{O}_X)_{\text{red}}$ be the sheafification of the presheaf $U \mapsto \mathcal{O}_X(U)_{\text{red}}$ on X, where for a ring A we denote by A_{red} the reduced ring given by the quotient of A by its nilradical.

(a) Show that $X_{\text{red}} := (X, (\mathcal{O}_X)_{\text{red}})$ is a reduced scheme, and that if $X = \operatorname{Spec}(A)$ is affine then $X_{\text{red}} = \operatorname{Spec}(A_{\text{red}})$. We call X_{red} the *reduction* of X.

(b) Show that there is a closed immersion $i_X \colon X_{\text{red}} \to X$.

(c) Let $f \colon X \to Y$ be a morphism of schemes. Show that there is a unique morphism of schemes $f_{\text{red}} \colon X_{\text{red}} \to Y_{\text{red}}$ such that $f \circ i_X = i_Y \circ f_{\text{red}}$.

6. Let k be a field and let $X = V_+(x_0^2) \subset \mathbf{P}_k^2$ be the scheme defined by $x_0^2 \in k[x_0, x_1, x_2]$ (via glueing as in class).

(a) Show that X is not a reduced scheme, but by explicit computation show that $\mathcal{O}_X(X)$ is reduced.

(b) Identify explicitly the reduction X_{red} of X (which was constructed in general in Problem 5).