Algebraic Geometry I, Fall 2021

Problem Set 6

Due Friday, October 22, 2021 at 5 pm

1. (a) Let $f\colon X \to Y$ be a closed immersion of schemes which is a homeomorphism (of underlying topological spaces). Prove that if Y is reduced, then f is an isomorphism.

   (b) Give an example of finitely generated reduced $\mathbf{C}$-algebras A and B and a morphism $f\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ of affine schemes such that f is a homeomorphism, but not a closed immersion.

2. Let X be a topological space, and $i\colon Z \to X$ the inclusion of a closed subset $Z \subset X$.

   (a) Prove that the pushforward functor $i_*\colon \mathrm{Ab}(Z) \to \mathrm{Ab}(X)$ is exact.

   (b) Prove that $i^{-1} \circ i_*\colon \mathrm{Ab}(Z) \to \mathrm{Ab}(Z)$ is isomorphic to the identity functor, and deduce that i_* is fully faithful.

   (c) Prove that the essential image of the functor $i_*\colon \mathrm{Ab}(Z) \to \mathrm{Ab}(X)$ is the subcategory $\mathrm{Ab}_Z(X) \subset \mathrm{Ab}(X)$ of abelian sheaves with support contained in Z, and thus i_* induces an equivalence of categories $\mathrm{Ab}(Z) \simeq \mathrm{Ab}_Z(X)$.

   We used the following terminology: The support of a sheaf $\mathcal{F} \in \mathrm{Ab}(X)$ is defined as the subset $\operatorname{Supp}(\mathcal{F}) := \{p \in X \mid \mathcal{F}_p \neq 0\} \subset X$. The essential image of a functor $F\colon \mathcal{C} \to \mathcal{D}$ is the full subcategory of $\mathcal{D}$ consisting of objects $D \in \mathcal{D}$ such that there exists an object $C \in \mathcal{C}$ and an isomorphism $F(C) \cong D$.

3. Let $U = \mathbf{A}^1_k \setminus \{(x)\}$ be the complement of the origin in the affine line $\mathbf{A}^1_k = \operatorname{Spec}(k[x])$ over a field k. Let $j\colon U \to \mathbf{A}^1_k$ be the inclusion, and let $\mathcal{I} = j_!\mathcal{O}_U$, where $j_!$ is the extension by 0 functor defined in Problem Set 2, Problem 6. Show that $\mathcal{I}$ is a sheaf of ideals on $\mathbf{A}^1_k$, but that the pair $(Z := \operatorname{Supp}(\mathcal{O}_{\mathbf{A}^1_k}/\mathcal{I}), (\mathcal{O}_{\mathbf{A}^1_k}/\mathcal{I})|_Z)$ is *not* a scheme.

4. (a) Show that if $f\colon X \to Y$ is a morphism of affine schemes such that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective, then f is a closed immersion.

   (b) Give an example to show that the conclusion of part (a) fails if the hypothesis that X and Y are affine is dropped.

5. You don't need to submit any work for this problem, but please check the following:

   If $f\colon X \to Y$ is a morphism of schemes whose (set-theoretic) image is contained in an open subset $V \subset Y$, then f factors uniquely as $f = j \circ f'$ where $j\colon V \to Y$ is the open immersion including V into Y and $f'\colon X \to V$ is a morphism of schemes. In particular, if $f\colon X \to Y$ is any morphism of schemes and $V \subset Y$, then there is a natural induced morphism $f^{-1}(V) \to V$.

6. (a) Show that the composition of two closed immersions of schemes is a closed immersion.

   (b) Let $f\colon X \to Y$ be a closed immersion of schemes. Show that if $V \subset Y$ is an open subset, then the morphism $f^{-1}(V) \to V$ induced by f is a closed immersion.

(c) Show that if $f\colon X \to Y$ is a morphism of schemes and $Y = \bigcup V_i$ is an open cover such that for each i the induced morphism $f^{-1}(V_i) \to V_i$ is a closed immersion, then f is a closed immersion.

(d) Let $f \in k[x_0, \ldots, x_n]$ be a homogeneous polynomial, and let $X = V_+(f)$ be the corresponding scheme defined in class. By construction, the underlying topological space of X is a subset of $\mathbf{P}^n_k$. Show that there is a natural closed immersion of schemes $X \to \mathbf{P}^n_k$.