Algebraic Geometry I, Fall 2021

Problem Set 7

Due Friday, October 22, 2021 at 5 pm

1. Let $f\colon X \to Y$ be a morphism of schemes. Prove that the following are equivalent:

   (a) $f\colon X \to Y$ is a closed immersion (as defined in class).

   (b) For every affine open $V \subset Y$, the preimage $U = f^{-1}(V)$ is affine and the induced ring map $\mathcal{O}_V(V) \to \mathcal{O}_U(U)$ is surjective.

   (c) There exists an affine open cover $Y = \bigcup V_i$ such that each $U_i = f^{-1}(V_i)$ is affine and the induced ring map $\mathcal{O}_{V_i}(V_i) \to \mathcal{O}_{U_i}(U_i)$ is surjective.

   Use this to prove that the property of being a closed immersion is stable under base change. More precisely, show that if $f\colon X \to Y$ is a closed immersion and $Y' \to Y$ is any morphism of schemes, then the morphism $X \times_Y Y' \to Y'$ is a closed immersion.

2. Let k be a field, and let $\pi\colon \mathbf{A}^1_k \to \mathbf{A}^1_k$ be the morphism corresponding to the ring map $k[x] \to k[x]$ given by $x \mapsto x^2$. Define X to be the fiber product

$$\begin{array}{ccc} X & \longrightarrow & \mathbf{A}^1_k \\ \downarrow & & \downarrow \pi \\ \mathbf{A}^1_k & \xrightarrow{\pi} & \mathbf{A}^1_k \end{array}$$

   (a) If $\operatorname{char}(k) \neq 2$, show that X is reduced and has two irreducible components.

   (b) If $\operatorname{char}(k) = 2$, determine whether X is reduced and the number of irreducible components.

3. This exercise is about the underlying topological spaces of fiber products of schemes. For a scheme X, we denote by $|X|$ the underlying topological space of X.

   (a) Let $f\colon X \to S$ be a morphism of schemes, let $i\colon Z \to S$ be a closed immersion, and consider the fiber product

$$\begin{array}{ccc} X_Z & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Z & \xrightarrow{i} & S \end{array}$$

   Show that the morphism $X_Z \to X$ induces a homeomorphism $|X_Z| \to f^{-1}(i(|Z|))$.

   (b) Let $f\colon X \to S$ and $g\colon Y \to S$ be morphisms of schemes. There is a natural map of topological spaces $|X \times_S Y| \to |X| \times_{|S|} |Y|$ (why?). Let $(x, y) \in |X| \times_{|S|} |Y|$ and set $s = f(x) = g(y)$; note that the pullback maps $f^\#$ and $g^\#$ induce local ring homomorphisms $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ and $\mathcal{O}_{S,s} \to \mathcal{O}_{Y,y}$, and hence maps on residue fields $\kappa(s) \to \kappa(x)$ and $\kappa(s) \to \kappa(y)$. Show that the fiber of $|X \times_S Y| \to |X| \times_{|S|} |Y|$ over (x, y) is homeomorphic to $\operatorname{Spec}(\kappa(x) \otimes_{\kappa(s)} \kappa(y))$.

(c) Show that surjectivity is stable under base change: if $f\colon X \to S$ is a surjective morphism of schemes and $S' \to S$ is any morphism of schemes, then $X \times_S S' \to S'$ is surjective. However, show by example that the same is not true for injectivity.