Algebraic Geometry I, Fall 2021

Problem Set 8

Due Friday, October 29, 2021 at 5 pm

1. Let k be a field, let $\mathbf{A}_k^\infty = \operatorname{Spec}(k[x_1, x_2, x_3, \dots])$, and let $U = \mathbf{A}_k^\infty \setminus V(x_1, x_2, x_3, \dots)$ be the complement of the origin.
   (a) Show that U is not quasi-compact, and therefore the inclusion $j\colon U \to \mathbf{A}_k^\infty$ is an example of an open immersion that is not quasi-compact.
   (b) Construct an example of a morphism $f\colon X \to Y$ of schemes such that X and Y are quasi-compact, but f is not quasi-compact.
   Hint: Glue together two copies of $\mathbf{A}_k^\infty$ along U.

2. Let Y be a noetherian scheme and $f\colon X \to Y$ a finite type morphism. Prove that X is noetherian.

3. Read Section 7.3.3 on affine morphisms from *Foundations of Algebraic Geometry* by Vakil. In particular, make sure you understand the definition of an affine morphism and the statement of Proposition 7.3.4.

4. Let $f\colon X \to Y$ be a morphism of schemes.
   (a) Show that the following two conditions are equivalent:
      i. For every affine open $V \subset Y$, the preimage $U = f^{-1}(V)$ is affine and the corresponding ring map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is finite. (Recall a ring map $A \to B$ is called finite if B is finitely generated as an A-module.)
      ii. There exists an affine open cover $Y = \bigcup V_i$ such that each $U_i = f^{-1}(V_i)$ is affine and the corresponding ring map $\mathcal{O}_Y(V_i) \to \mathcal{O}_X(U_i)$ is finite.

      In this case, we say that f is a *finite* morphism.
   (b) Prove that the property of a morphism being finite is stable under composition, stable under base change, and local on the target.

5. This exercise deals with some further properties of finite morphisms:
   (a) Prove that a finite morphism is closed.
   (b) Prove that a finite morphism is quasi-finite, where by definition a morphism $f\colon X \to Y$ is *quasi-finite* if it is of finite type and for every $p \in Y$ the fiber $f^{-1}(p)$ is a finite set.
   (c) Give an example of a morphism $f\colon X \to Y$ which is surjective, finite type, and quasi-finite, but *not* finite.

6. A morphism $f\colon X \to S$ of schemes is called *closed* if the map on underlying topological spaces is closed. Show that the property of a morphism being closed is not stable under base change.
   Hint: Consider $\mathbf{A}_k^1 \to \operatorname{Spec}(k)$.

7. The following exercise about constructible sets will be used in our proof of Chevalley's Theorem in class:

   Let X be a noetherian topological space. A subset $S \subset X$ is *locally closed* if it can be written as $S = U \cap Z$ where $U \subset X$ is open and $Z \subset X$ is closed. A subset $E \subset X$ is called *constructible* if it is a finite union of locally closed subsets of X.

   Assume that $E \subset X$ is a subset such that the following condition holds: for every irreducible closed subset $Z \subset X$, the intersection $E \cap Z$ either contains a nonempty open subset of Z or is not dense in Z. Prove that E is constructible.