Algebraic Geometry I, Fall 2021

Problem Set 9

Due Friday, November 5, 2021 at 11:59 pm

1. Let $f\colon X \to S$ be a morphism of schemes, and consider the fiber product

$$\begin{array}{ccc} X \times_S X & \xrightarrow{\pi_1} & X \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & S \end{array}$$

   Show that the image of the diagonal $\Delta_f\colon X \to X \times_S X$ is contained in the subset

$$Z = \{p \in X \times_S X \mid \pi_1(p) = \pi_2(p)\},$$

   but show by example that in general the image is not equal to Z.

2. Recall that we defined a morphism of schemes to be a *locally closed immersion* if it factors as a closed immersion followed by an open immersion.

   (a) Show that $f\colon X \to Y$ is a locally closed immersion if and only if it gives a homeomorphism from X onto a locally closed subset of Y and for all $p \in X$ the homomorphism $f_p^\sharp\colon \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is surjective.

   (b) Show that the property of being a locally closed immersion is stable under composition, stable under base change, and local on the target.

   (c) Show that a locally closed immersion $f\colon X \to Y$ is a closed immersion if and only if $f(X)$ is closed.

   (d) Show that the previous statement is not true for open immersions, i.e. give an example of a locally closed immersion $f\colon X \to Y$ such that $f(X)$ is open but f is not an open immersion.

3. This problem gives some examples of separated morphisms.

   (a) Show that if $f\colon X \to Y$ is a monomorphism in the category of schemes, then f is separated.

   (b) Show that if $f\colon X \to Y$ is a locally closed immersion of schemes, then f is a monomorphism, and hence separated.

   (c) Give an example of a monomorphism $f\colon X \to Y$ which is not a locally closed immersion.

4. In class we will see that if $f, g\colon X \to Y$ are morphisms of S-schemes where X is reduced and $Y \to S$ is separated, and if there exists an open dense subscheme $U \subset X$ such that $f|_U = g|_U$, then $f = g$.

   (a) Show by example that the above result fails if X is not assumed to be reduced.

(b) Show by example that the above result fails if $Y \to S$ is not assumed to be separated.

5. Let k be a field. A *k-variety* is a scheme X over k (i.e. a scheme equipped with a morphism $X \to \operatorname{Spec}(k)$) such that X is reduced and $X \to \operatorname{Spec}(k)$ is separated and finite type. A morphism of k-varieties is a morphism of schemes over $\operatorname{Spec}(k)$.

   (a) Show that if $k = \overline{k}$ is algebraically closed, then morphisms of k-varieties are determined on k-valued points. More precisely, show that if $f, g \colon X \to Y$ are morphisms of varieties over k which induce the same map

   $$X(k) := \operatorname{Hom}_{\operatorname{Spec}(k)}(\operatorname{Spec}(k), X) \to Y(k) := \operatorname{Hom}_{\operatorname{Spec}(k)}(\operatorname{Spec}(k), Y),$$

   then $f = g$. (Here, $\operatorname{Hom}_{\operatorname{Spec}(k)}(\operatorname{Spec}(k), X)$ means morphisms of $\operatorname{Spec}(k)$-schemes.)

   (b) Show by example that if k is not algebraically closed, then morphisms of varieties are not determined on k-valued points.

   (c) Show that if $f \colon X \to Y$ is a morphism of k-varieties, then f is separated and finite type.