Serre functors of semiorthogonal components

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(joint work with Alexander Kuznetsov)

Given an algebraic variety X, its bounded derived category of coherent sheaves $\mathrm{D^b}(X)$ can studied by breaking it into smaller pieces, via the notion of a semiorthogonal decomposition. The components appearing in such decompositions can fruitfully be thought of as noncommutative algebraic varieties, although they have no underlying space of points. A motivating example is a Fano complete intersection $X \subset \mathbf{P}^n$ of multidegree $(d_1, d_2, \dots, d_c)$, for which there is a semiorthogonal decomposition

$$\mathrm{D^b}(X) = \langle \mathcal{R}_X, \mathcal{O}_X, \mathcal{O}_X(1), \dots, \mathcal{O}_X(\mathrm{ind}(X)-1) \rangle,$$

where $\mathrm{ind}(X) = n+1-\sum d_i$ is the index of X and

$$\mathcal{R}_X = \{ E \in \mathrm{D^b}(X) \mid \mathrm{Ext}^\bullet(\mathcal{O}_X(i), E) = 0 \text{ for } 0 \leq i \leq \mathrm{ind}(X)-1 \}$$

is the *residual category* or *Kuznetsov component*. Recently, residual categories of Fano varieties have attracted attention due to their connections and applications to birational geometry, moduli spaces of sheaves, hyperkähler varieties, and Hodge theory. For instance, in the most famous case of a cubic fourfold $X \subset \mathbf{P}^5$, the residual category $\mathcal{R}_X$ conjecturally controls the rationality of X [Kuz10], and its moduli spaces of stable objects give rise to unirational locally complete families of polarized hyperkähler varieties of K3 type, as well as to a new proof of the integral Hodge conjecture for X [BLM+21, Per22].

Given a semiorthogonal component $\mathcal{C}$ of a smooth proper variety, its Serre functor $\mathrm{S}_{\mathcal{C}}$ is an autoequivalence characterized by the existence of functorial isomorphisms

$$\mathrm{Hom}(E, F) \cong \mathrm{Hom}(F, \mathrm{S}_{\mathcal{C}}(E))^\vee$$

for $E, F \in \mathcal{C}$. The Serre functor of $\mathcal{C}$ is one of its most important invariants, playing the role of the dualizing complex in noncommutative algebraic geometry. For instance, the applications mentioned above for a cubic fourfold X depend crucially on Kuznetsov's result that $\mathrm{S}_{\mathcal{R}_X} \cong [2]$, i.e. that $\mathcal{R}_X$ is a 2-dimensional Calabi–Yau category. The main theorem of the talk was a vast generalization of this result, which in particular applies to the previously mysterious case of Fano complete intersections of higher codimension.

To state the theorem, we recall that if $\Psi \colon \mathcal{C} \to \mathcal{D}$ is a functor between (suitably enhanced) triangulated categories which admits a right adjoint $\Psi^!$, then there are associated *twist endofunctors* $\mathrm{T}_{\Psi^!,\Psi}$ of $\mathcal{C}$ and $\mathrm{T}_{\Psi,\Psi^!}$ of $\mathcal{D}$ defined by exact triangles

$$\mathrm{T}_{\Psi^!,\Psi} \to \mathrm{id}_{\mathcal{C}} \xrightarrow{\text{unit}} \Psi^! \circ \Psi \qquad \text{and} \qquad \Psi \circ \Psi^! \xrightarrow{\text{counit}} \mathrm{id}_{\mathcal{D}} \to \mathrm{T}_{\Psi,\Psi^!}.$$

The functor Ψ is called *spherical* if it also admits a left adjoint and its twists $\mathrm{T}_{\Psi^!,\Psi}$ and $\mathrm{T}_{\Psi,\Psi^!}$ are autoequivalences [AL17].

Theorem 1 ([KP21])**.** *Let M be an n-dimensional smooth proper variety equipped with a line bundle $\mathcal{O}_M(1)$ such that $\omega_M \cong \mathcal{O}_M(-m)$ for a positive integer m, and assume given a semiorthogonal decomposition of the form*

$$\mathrm{D^b}(M) = \langle \mathcal{R}_M, \mathcal{B}_M, \mathcal{B}_M(1), \dots, \mathcal{B}_M(m-1) \rangle,$$

where $\mathcal{B}_M(k) = \mathcal{B}_M \otimes \mathcal{O}_M(k)$ for $k \in \mathbf{Z}$. Let $i \colon X \to M$ be the inclusion of a smooth divisor in the linear system $|\mathcal{O}_M(d)|$ for some $1 \leq d < m$.

(1) *$i^* \colon \mathrm{D^b}(M) \to \mathrm{D^b}(X)$ is fully faithful when restricted to $\mathcal{B}_M$, and there is a semiorthogonal decomposition*

$$\mathrm{D^b}(X) = \langle \mathcal{R}_X, \mathcal{B}_X, \mathcal{B}_X(1), \dots, \mathcal{B}_X(m-d-1) \rangle,$$

where $\mathcal{B}_X(k) = i^*(\mathcal{B}_M(k))$ for $k \in \mathbf{Z}$.

(2) *i^* restricts to a spherical functor $\Psi \colon \mathcal{R}_M \to \mathcal{R}_X$, and there are isomorphisms of functors*

$$\mathrm{S}_{\mathcal{R}_M}^{d/c} \cong \mathrm{T}_{\Psi^!,\Psi}^{m/c} \circ \left[\frac{dn}{c}\right],$$

$$\mathrm{S}_{\mathcal{R}_X}^{d/c} \cong \mathrm{T}_{\Psi,\Psi^!}^{(m-d)/c} \circ \left[\frac{d(n+1)-2m}{c}\right],$$

where $c = \gcd(d, m)$.

Remark 2. Let us note some special cases and extensions of this result:

(1) There are many examples of varieties M to which the theorem applies, including Fano complete intersections in a (stacky weighted) projective space or in a Grassmannian.

(2) If $\mathcal{R}_M = 0$ then $\mathrm{T}_{\Psi,\Psi^!} = \mathrm{id}_{\mathcal{R}_X}$ and the theorem shows that $\mathcal{R}_X$ is *fractional Calabi–Yau* in the sense that a power of its Serre functor is a shift. This recovers the main result of [Kuz19].

(3) A more general version of the theorem is proved in [KP21], where the categories $\mathrm{D^b}(M)$ and $\mathrm{D^b}(X)$ are replaced by suitably enhanced triangulated categories $\mathcal{C}$ and $\mathcal{D}$, the line bundles $\mathcal{O}_M(1)$ and $\mathcal{O}_X(1)$ by autoequivalences of $\mathcal{C}$ and $\mathcal{D}$, and i^* by a spherical functor $\mathcal{C} \to \mathcal{D}$ satisfying some compatibilities. This general formulation applies to other geometric situations as well, like double covers $X \to M$.

Theorem 1 has an interesting application to the dimension of residual categories. For a triangulated category $\mathcal{C}$, we consider the *upper* $\overline{\mathrm{Sdim}}(\mathcal{C})$ and *lower* $\underline{\mathrm{Sdim}}(\mathcal{C})$ *Serre dimensions* introduced by Elagin and Lunts [EL21], which are defined when $\mathcal{C}$ is proper and admits a Serre functor and a generator, e.g. when $\mathcal{C}$ is the residual category of a smooth proper Fano variety; roughly, these dimensions measure how fast the maximal and minimal cohomological amplitudes of the functor $\mathrm{S}_{\mathcal{C}}^{-k}$ grow as $k \to \infty$. For a smooth proper variety X we have $\underline{\mathrm{Sdim}}(\mathrm{D^b}(X)) = \overline{\mathrm{Sdim}}(\mathrm{D^b}(X)) = \dim(X)$, but in general only the inequality $\underline{\mathrm{Sdim}}(\mathcal{C}) \leq \overline{\mathrm{Sdim}}(\mathcal{C})$ holds. This is illustrated by the following result, which verifies a corrected version of a conjecture of Katzarkov and Kontsevich.

Theorem 3 ([KP21])**.** *Let $X \subset \mathbf{P}^n$ be a complex smooth Fano complete intersection of multidegree $(d_1, d_2, \dots, d_c)$ with all $d_i > 1$. Let $d_{\max}$ and $d_{\min}$ be the maximum and minimum of the degrees d_i. Then*

$$\overline{\mathrm{Sdim}}(\mathcal{R}_X) = \dim(X) - 2\frac{\mathrm{ind}(X)}{d_{\max}},$$
$$\underline{\mathrm{Sdim}}(\mathcal{R}_X) = \dim(X) - 2\frac{\mathrm{ind}(X)}{d_{\min}}.$$

The idea of the proof is to use Theorem 1 to inductively control the Serre functor $\mathrm{S}_{\mathcal{R}_X}$ in terms of those of residual categories of Fano complete intersections of smaller codimension.

One simple consequence of Theorem 3 is that in most cases $\mathcal{R}_X$ is not equivalent to the derived category of a variety, since that would require the upper and lower Serre dimensions to be equal and integral.

There is also an interesting consequence for stability conditions. For many Fano threefolds, there exist stability conditions on the residual category which are *Serre invariant*, i.e. fixed by the Serre functor modulo the natural $\widetilde{\mathrm{GL}}_2^+(\mathbf{R})$-action (see [PS23, §5] for a survey of the known results). This property has played a key role in the analysis of moduli spaces of stable objects in these examples, raising the hope that Serre invariant stability conditions might always exist on the residual categories of Fano varieties. However, Theorem 3 implies this is far from true, since one can show that the existence of such a stability condition forces the equality of the upper and lower Serre dimensions.

References

[AL17] Rina Anno and Timothy Logvinenko, *Spherical DG-functors*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 9, 2577–2656.

[BLM+21] Arend Bayer, Martí Lahoz, Emanuele Macrì, Howard Nuer, Alexander Perry, and Paolo Stellari, *Stability conditions in families*, Publ. Math. Inst. Hautes Études Sci. **133** (2021), 157–325.

[BLMS23] Arend Bayer, Martí Lahoz, Emanuele Macrì, and Paolo Stellari, *Stability conditions on Kuznetsov components*, Ann. Sci. Éc. Norm. Supér. (4) **56** (2023), no. 2, 517–570, With an appendix by Bayer, Lahoz, Macrì, Stellari and X. Zhao.

[EL21] Alexey Elagin and Valery Lunts, *Three notions of dimension for triangulated categories*, J. Algebra **569** (2021), 334–376.

[KP21] Alexander Kuznetsov and Alexander Perry, *Serre functors and dimensions of residual categories*, arXiv:2109.02026 (2021).

[Kuz10] Alexander Kuznetsov, *Derived categories of cubic fourfolds*, Cohomological and geometric approaches to rationality problems, Progr. Math., vol. 282, Birkhäuser Boston, Boston, MA, 2010, pp. 219–243.

[Kuz19] ______, *Calabi-Yau and fractional Calabi-Yau categories*, J. Reine Angew. Math. **753** (2019), 239–267.

[Per22] Alexander Perry, *The integral Hodge conjecture for two-dimensional Calabi-Yau categories*, Compos. Math. **158** (2022), no. 2, 287–333.

[PS23] Laura Pertusi and Paolo Stellari, *Categorical Torelli theorems: results and open problems*, Rend. Circ. Mat. Palermo (2) **72** (2023), no. 5, 2949–3011.