

Math 614: Commutative Algebra, Fall 2026

Problem Set 5

Due Friday, October 9, 2026 at 10pm

1. This problem reviews a general form of the Chinese remainder theorem.

Let R be a ring. Let $I_1, \dots, I_n \subset R$ be ideals. Let $\pi: R \rightarrow \prod_{i=1}^n R/I_i$ be the homomorphism given by $x \mapsto (\pi_1(x), \dots, \pi_n(x))$, where $\pi_i: R \rightarrow R/I_i$ is the quotient map. We will say that the ideals I_i are pairwise coprime if for all $i \neq j$ the ideals I_i and I_j are coprime (i.e. $I_i + I_j = R$).

- (a) Prove that if the ideals I_i are pairwise coprime, then $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n$.
- (b) Give an example to show that the equality $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n$ can fail if the ideals I_i are not pairwise coprime.
- (c) Prove that $\pi: R \rightarrow \prod_{i=1}^n R/I_i$ is surjective if and only if the ideals I_i are pairwise coprime.
- (d) Prove that π is injective if and only if $I_1 \cap I_2 \cap \cdots \cap I_n = 0$.

Combining the above, we see that if the I_i are pairwise coprime and $I_1 I_2 \cdots I_n = 0$, then $\pi: R \rightarrow \prod_{i=1}^n R/I_i$ is an isomorphism.

2. Let R be a domain with fraction field K . Recall that the normalization $\tilde{R}$ of R is the integral closure of R in K , i.e. $\tilde{R} = \{y \in K \mid y \text{ is integral over } R\}$, and we say that R is normal if $R = \tilde{R}$. Prove that any UFD is normal.
3. (a) Show that $\mathbf{C}[x, y]/(y^3 - x^5)$ is a domain which is not normal, and compute its normalization.
- (b) Compute the normalization of $\mathbf{Z}[\sqrt{-3}]$ (the subring of the complex numbers generated by $\mathbf{Z}$ and $\sqrt{-3}$).
4. (a) (Normalization commutes with localization). Let R be a domain and $S \subset R$ a multiplicative system. Via the inclusion $R \hookrightarrow \tilde{R}$, we can also regard S as a multiplicative system in the normalization $\tilde{R}$. Prove that there is an isomorphism of rings

$$S^{-1} \tilde{R} \cong \widetilde{S^{-1} R},$$

where $\widetilde{S^{-1} R}$ is the normalization of $S^{-1} R$.

- (b) (Normality is a local property). Let R be a domain. Prove that the following are equivalent:
- i. R is normal.
 - ii. $R_{\mathfrak{p}}$ is normal for all $\mathfrak{p} \in \text{Spec}(R)$.
 - iii. $R_{\mathfrak{m}}$ is normal for all $\mathfrak{m} \in \text{mSpec}(R)$.
5. Let $f: R \rightarrow S$ be a finite ring homomorphism. Prove that the fibers of the induced map on spectra $\text{Spec}(f): \text{Spec}(S) \rightarrow \text{Spec}(R)$ are finite sets.

6. If $f: R \rightarrow S$ is a ring homomorphism, we say that f *satisfies going up* if for every pair of prime ideals $\mathfrak{p}, \mathfrak{p}' \in \operatorname{Spec}(R)$ such that $\mathfrak{p} \subset \mathfrak{p}'$ and $\mathfrak{q} \in \operatorname{Spec}(S)$ such that $f^{-1}(\mathfrak{q}) = \mathfrak{p}$, there exists $\mathfrak{q}' \in \operatorname{Spec}(S)$ such that $\mathfrak{q} \subset \mathfrak{q}'$ and $f^{-1}(\mathfrak{q}') = \mathfrak{p}'$.

If X is a topological space and $x, x' \in X$ are points, we say that x' is a *specialization* of x (or x *specializes* to x') if x' is contained in the closure of x , i.e. $x' \in \overline{\{x\}}$. If $\varphi: X \rightarrow Y$ is a continuous map of topological spaces, we say that *specializations lift along φ* if for every $y, y' \in Y$ such that y' is a specialization of y and $x \in X$ such that $\varphi(x) = y$, there exists a specialization $x' \in X$ of x such that $\varphi(x') = y'$.

- (a) Check that a ring homomorphism $f: R \rightarrow S$ satisfies going up if and only if specializations lift along the map $\operatorname{Spec}(f): \operatorname{Spec}(S) \rightarrow \operatorname{Spec}(R)$.
- (b) Let $f: R \rightarrow S$ be a ring homomorphism and let $T \subset \operatorname{Spec}(R)$ be the image of the map $\operatorname{Spec}(f): \operatorname{Spec}(S) \rightarrow \operatorname{Spec}(R)$. Assume that T satisfies the following property: if $\mathfrak{p} \in \operatorname{Spec}(R)$ is any point which is a specialization of a point of T , then $\mathfrak{p} \in T$. Prove that T is closed.

Hint: Let $\mathfrak{p} \in \operatorname{Spec}(R)$ be a point in the closure of T . Let $S_{\mathfrak{p}}$ denote the localization of S at the multiplicative system $f(R \setminus \mathfrak{p})$ (or equivalently, $S_{\mathfrak{p}} = S \otimes_R R_{\mathfrak{p}}$). Prove that the ring $S_{\mathfrak{p}}$ is nonzero and that any point in the image of the natural map $\operatorname{Spec}(S_{\mathfrak{p}}) \rightarrow \operatorname{Spec}(R)$ specializes to $\mathfrak{p}$.

- (c) For a ring homomorphism $f: R \rightarrow S$, prove that f satisfies going up if and only if the map $\operatorname{Spec}(f): \operatorname{Spec}(S) \rightarrow \operatorname{Spec}(R)$ is closed.