

Math 614: Commutative Algebra, Fall 2026

Problem Set 6

Due Friday, October 23, 2026 at 10pm

1. Let k be an algebraically closed field. Recall that we have a contravariant functor

$$\mathcal{O}: \text{Aff}_k \rightarrow \text{Alg}_k,$$

where Aff_k is the category of affine varieties over k and Alg_k is the category of k -algebras. Moreover, in Problem Set 1 you proved that this functor is fully faithful.¹

- (a) Let $\text{Alg}_k^{\text{red,ft}}$ be the full subcategory of Alg_k consisting of reduced finite type k -algebras.² Explain why $\mathcal{O}$ can also be regarded as a contravariant functor

$$\mathcal{O}: \text{Aff}_k \rightarrow \text{Alg}_k^{\text{red,ft}}.$$

- (b) Let $(\text{Aff}_k)^{\text{op}}$ denote the opposite category of Aff_k .³ Prove that the functor $\mathcal{O}$ gives an equivalence of categories $(\text{Aff}_k)^{\text{op}} \simeq \text{Alg}_k^{\text{red,ft}}$.

2. Let R be a finite type k -algebra, where k is a field.

- (a) Prove that any radical ideal $I \subset R$ is the intersection of all maximal ideals in R which contain I .
- (b) Prove that for any closed subset $Z \subset \text{Spec}(R)$, the intersection $Z \cap \text{mSpec}(R)$ is dense in Z , where $\text{mSpec}(R)$ is the maximal spectrum of R .

3. Let M be a noetherian module over a ring R . Let $f: M \rightarrow M$ be a surjective R -module homomorphism. Prove that f is an isomorphism.

4. For each of the following, either provide an example or prove that none exists:

- (a) A subring R of a field K such that R is not noetherian.
- (b) A field k , a finite type k -algebra R , and a sub- k -algebra $S \subset R$ such that S is not noetherian.

5. For each of the following rings R , find the irreducible components of $\text{Spec}(R)$.

(a) $R = \mathbf{C}[x, y]/(x^3 + x - x^2y - y)$.

(b) $R = \mathbf{Z}[x, y, z]/(xy, xz, yz)$.

¹Technically, there we assumed that $k = \mathbf{C}$ was the field of complex numbers, but nothing changes for more general algebraically closed fields.

²In other words, the objects of $\text{Alg}_k^{\text{red,ft}}$ are the so-described k -algebras, and the morphisms are morphisms of k -algebras.

³In other words, $(\text{Aff}_k)^{\text{op}}$ has the same objects as Aff_k , but for affine varieties X and Y over k the morphisms in $(\text{Aff}_k)^{\text{op}}$ are defined by $\text{Hom}_{(\text{Aff}_k)^{\text{op}}}(X, Y) = \text{Hom}_{\text{Aff}_k}(Y, X)$.

6. If X is a topological space and $Z \subset X$ is an irreducible closed subset, a *generic point* of Z is a point $\xi \in Z$ whose closure is Z , i.e. $Z = \overline{\{\xi\}}$. Prove that when $X = \operatorname{Spec}(R)$ for a ring R , every irreducible closed subset $Z \subset X$ has a unique generic point, given explicitly by $I(Z)$.
7. For each of the following statements, either provide an example or prove that none exists:
 - (a) A $\mathbf{Z}$ -module M whose support is not closed.
 - (b) A ring R and an R -module M such that the support $\operatorname{Supp}(M) \subset \operatorname{Spec}(R)$ is not stable under specialization. (In general, given a topological space X , a subset $S \subset X$ is said to be stable under specialization if whenever $x \in X$ is a point which is a specialization of a point $s \in S$, then $x \in S$.)