Universal properties of Delannoy categories

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Logistical information

The paper this talk is based on

Universal properties of Delannoy categories

by Kevin Coulembier, Nate Harman, and Andrew Snowden

https://arxiv.org/abs/2510.10317

See the paper for references

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Slides on my website

https://public.websites.umich.edu/~asnowden/ams-slides.pdf

§1. Overview

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Harman–Snowden (2022) gave a general construction using oligomorphic groups. It includes the above, but produces many truly new examples, such as the Delannoy categories.

General research direction

Explore oligomorphic tensor categories.

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- There is a universal property in general.

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- For the Delannoy categories, we can make it more precise.

This talk will explain these results.

§2. Étale and Frobenius algebras

Tensor categories

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- $\mathfrak{T}$ is rigid, i.e., every objects has a dual.
- $\mathrm{End}(\mathbf{1}) = k$, where $\mathbf{1}$ is the tensor unit.

Frobenius algebras

Definition

A (special commutative) Frobenius algebra in a tensor category is an object A with maps

$$\eta\colon \mathbf{1} \to A, \quad \epsilon\colon A \to \mathbf{1}, \quad m\colon A \otimes A \to A, \quad \Delta\colon A \to A \otimes A$$

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Warning: this is different from a bi-algebra or Hopf algebra!

A construction of Frobenius algebras

Construction

Let X be a finite set, and put $A = k[X]$ with basis $\{e_x\}$. Put

$$\eta = \sum_{x \in X} e_x, \quad \epsilon(e_x) = 1, \quad m(e_{x,y}) = \delta_{x,y} e_x, \quad \Delta(e_x) = e_{x,x}.$$

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Functorial behavior

If $f\colon Y \to X$ is a function, there is an algebra homomorphism

$$f^*\colon k[X] \to k[Y], \qquad f^*(e_x) = \sum_{f(y)=x} e_y.$$

This is typically not a co-algebra homomorphism.

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Let G be an algebraic group.

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Remark

The identity component of G acts trivially on a finite G-set.

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Proposition

Étale algebras are the same as Frobenius algebras:

- *If A is a Frobenius algebra then the underlying commutative algebra is étale.*
- *If A is étale, dualizing the algebra structure wrt the trace pairing gives a co-algebra structure, which makes A into a Frobenius algebra.*

The category of étale algebras

Definition

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Remark

In general, $\mathrm{Et}(\mathfrak{T})^{\mathrm{op}}$ is a "set-like" category: it is extensive, finitely complete, and sub-objects have complements.

§3. Deligne's category

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Theorem (Deligne)

If $\mathrm{char}(k) = 0$ *and* $t \notin \mathbf{N}$ *then the Karoubi envelope of* $\underline{\mathrm{Rep}}(\mathfrak{S}_t)$ *is a semi-simple pre-Tannakian category.*

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Remark

There is also a pre-Tannakian category for $\mathrm{char}(k) = 0$ and $t \in \mathbf{N}$ by work of Deligne and Comes–Ostrik.

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Giving a tensor functor $\Phi\colon \underline{\mathrm{Rep}}(\mathfrak{S}_t) \to \mathfrak{T}$ is equivalent to giving an étale algebra in $\mathfrak{T}$ of dimension t, via $\Phi \leftrightarrow \Phi(A)$.

§4. Oligomorphic tensor categories

Oligomorphic groups

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- $\mathbf{GL}_\infty(\mathbf{F}_q)$ acting on $\Omega = \mathbf{F}_q^\infty$.
- The group $\mathrm{Aut}(\mathbf{R}, <)$ acting on $\Omega = \mathbf{R}$.

G-sets

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Definition

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Remark

The category $\mathbf{S}(G)$ has all the same basic categorical properties of the category of finite Γ-sets, for a finite group Γ. In particular, $\mathbf{S}(G)$ has finite products.

Measures

Definition (Harman–Snowden)

A measure for G is a rule μ that assigns to each morphism $f\colon Y \to X$ in $\mathbf{S}(G)$, with X transitive, a quantity $\mu(f)$ in k such that certain axioms hold.

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Intuition

$\mu(f)$ is like the size of a fiber of f, and $\mu(X)$ is like the size of X.

The main construction

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The object $\mathcal{C}(X)$ is rigid, self-dual, of dimension $\mu(X)$.

Intuition

$\mathcal{C}(X)$ is like a permutation representation with basis indexed by X. Morphisms can be thought of as G-invariant matrices.

Recovering Deligne's category

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In this way, the oligomorphic theory recovers Deligne's example.

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This functor is often an equivalence, but not always.

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- Ψ is left-exact, i.e., it commutes with finite limits.
- Ψ is compatible with μ, e.g., the dimension of $\Psi(X)$ is $\mu(X)$.

The main theorem

Let $\mathfrak{T}$ be an arbitrary tensor category.

Theorem

Giving a tensor functor $\Phi\colon \underline{\mathrm{Perm}}(G,\mu) \to \mathfrak{T}$ *is equivalent to giving a functor* $\Psi\colon \mathbf{S}(G) \to \mathrm{Et}(\mathfrak{T})^{\mathrm{op}}$ *that is additive, left-exact, and compatible with* μ.

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We think of Φ as an algebraic object, but Ψ as a combinatorial object. This is why the theorem is useful.

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Remark

The theorem recovers the universal property of $\underline{\mathrm{Rep}}(\mathfrak{S}_t)$.

§6. Delannoy categories

The categories

Let $\mathbb{G} = \mathrm{Aut}(\mathbf{R}, <)$, which acts oligomorphically on $\mathbf{R}$.

Fact

$\mathbb{G}$ has exactly four measures μ_1, μ_2, μ_3 and μ_4.

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Remark

$\mathfrak{C}_1^{\mathrm{kar}}$ is semi-simple pre-Tannakian. It was studied in depth by Harman, Snowden, Snyder (2022), and found to have many remarkable properties. The other $\mathfrak{C}_i$'s have been mysterious.

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$\mathbb{G}$-sets

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Notation

Write $\mathcal{C}_i(\mathbf{R}^{(n)})$ for $\mathcal{C}(\mathbf{R}^{(n)})$ in the category $\mathfrak{C}_i$.

Universal property of $\mathbf{S}(\mathbb{G})$

Let $\mathcal{S}$ be an extensive category with finite limits, e.g., $\mathsf{Et}(\mathfrak{T})^{\mathsf{op}}$.

Definition

An ordered object of $\mathcal{S}$ is an object X equipped with a subobject of $X \times X$ satisfying the axioms of a total order.

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Theorem

Giving an additive left-exact functor $\Psi\colon \mathbf{S}(\mathbb{G}) \to \mathcal{S}$ is equivalent to giving an ordered object of $\mathcal{S}$, via $\Psi \leftrightarrow \Psi(\mathbf{R})$.

Ordered étale algebras

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Simplest example

$\mathcal{C}_i(\mathbf{R})$ is an ordered étale algebra in $\mathfrak{C}_i$.

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The main theorem

Let $\mathfrak{T}$ be an arbitrary tensor category.

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Giving a tensor functor $\Phi\colon \mathfrak{C}_i \to \mathfrak{T}$ is equivalent to giving a Delannic algebra of type i in $\mathfrak{T}$, via $\Phi \leftrightarrow \Phi(\mathcal{C}_i(\mathbf{R}))$.

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Compatibility with μ_i is the Delannic condition; this is non-trivial, since the former is an infinite list of numeric conditions.

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Not clear if there is a Kriz-style universal property for the other $\mathfrak{C}_i$.

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Significance

Each $\mathfrak{C}_i$ admits a map to a pre-Tannakian category.

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- More generally, $\mathcal{C}_1(\mathbf{R}^{(n)})$ is Delannic of type 1 if n is odd, and type 4 if n is even $\implies$ many functors $\mathfrak{C}_1 \to \mathfrak{C}_1$ and $\mathfrak{C}_4 \to \mathfrak{C}_1$.

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- Can use $\oplus$, $\otimes$, and $(-)^{(n)}$ to obtain many more functors.

Local abelian envelopes

Let $\mathfrak{T}$ be a tensor category with finite Hom's and $\mathrm{End}(\mathbf{1}) = k$.

Theorem (Coulembier)

There exists $\{\Phi_i \colon \mathfrak{T} \to \mathfrak{U}_i\}_{i \in I}$ where each $\mathfrak{U}_i$ is pre-Tannakian such that any faithful $\Phi \colon \mathfrak{T} \to \mathfrak{U}$ (pre-Tannakian) factors uniquely as $\Psi \circ \Phi_i$ with $\Psi \colon \mathfrak{U}_i \to \mathfrak{U}$ exact and faithful.

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If $\# I = 1$ the unique $\mathfrak{U}_i$ is **the** abelian envelope of $\mathfrak{T}$.

Local envelopes for Delannoy

Theorem

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$\mathfrak{C}_2$ *has at least two local abelian envelopes.*

Proof

The functors $\mathfrak{C}_2 \to \mathfrak{C}_1$ provided by the Delannic algebras

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belong to different local abelian envelopes.

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belong to different local abelian envelopes.

Remark

Forthcoming work of Coulembier and Snowden: $\mathfrak{C}_2$ has exactly two local abelian envelopes. One is equivalent to $\mathfrak{C}_1^{\mathrm{kar}}$, the other is a new pre-Tannakian category.