

Universal properties of Delannoy categories

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The paper this talk is based on

Universal properties of Delannoy categories

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<https://arxiv.org/abs/2510.10317>

See the paper for references

Slides on my website

<https://public.websites.umich.edu/~asnowden/ams-slides.pdf>

§1. Overview

Background

Finding new symmetric tensor categories is a challenging problem.

Deligne (2006) gave an interesting example $\underline{\text{Rep}}(\mathfrak{S}_t)$.

Knop (2006) gave some additional examples.

Harman–Snowden (2022) gave a general construction using oligomorphic groups. It includes the above, but produces many truly new examples, such as the Delannoy categories.

General research direction

Explore oligomorphic tensor categories.

Universal properties

Deligne's category has a universal property, as does Knop's $\text{Rep}(\mathbf{GL}_t(\mathbf{F}_q))$ by Entova-Aizenbud and Heidersdorf (2022).

Main question for this talk

Do other oligomorphic categories have a universal property?

Our work provides a positive answer:

- There is a universal property in general.
- For the Delannoy categories, we can make it more precise.

This talk will explain these results.

§2. Étale and Frobenius algebras

Tensor categories

Definition

A **tensor category** is an additive k -linear category $\mathcal{T}$ equipped with a k -bilinear symmetric monoidal structure. A **tensor functor** is a k -linear symmetric monoidal functor. (k is a field.)

Definition

A **pre-Tannakian category** is a tensor category $\mathcal{T}$ such that:

- $\mathcal{T}$ is abelian and all objects have finite length.
- All Hom spaces in $\mathcal{T}$ are finite dimensional.
- $\mathcal{T}$ is rigid, i.e., every objects has a dual.
- $\text{End}(\mathbf{1}) = k$, where $\mathbf{1}$ is the tensor unit.

Frobenius algebras

Definition

A (special commutative) Frobenius algebra in a tensor category is an object A with maps

$$\eta: \mathbf{1} \rightarrow A, \quad \epsilon: A \rightarrow \mathbf{1}, \quad m: A \otimes A \rightarrow A, \quad \Delta: A \rightarrow A \otimes A$$

such that:

- (A, m, η) is an associative commutative unital algebra
- (A, Δ, ϵ) is a co-ACU co-algebra
- $(\mu \otimes \text{id}) \circ (\text{id} \otimes \Delta) = \Delta \circ m = (\text{id} \otimes \mu) \circ (\Delta \otimes \text{id})$
- $\mu \circ \Delta = \text{id}_A$ (the special condition).

Warning: this is different from a bi-algebra or Hopf algebra!

A construction of Frobenius algebras

Construction

Let X be a finite set, and put $A = k[X]$ with basis $\{e_x\}$. Put

$$\eta = \sum_{x \in X} e_x, \quad \epsilon(e_x) = 1, \quad m(e_{x,y}) = \delta_{x,y} e_x, \quad \Delta(e_x) = e_{x,x}.$$

This is a Frobenius algebra in the category $\mathbf{Vec}$.

Functorial behavior

If $f: Y \rightarrow X$ is a function, there is an algebra homomorphism

$$f^*: k[X] \rightarrow k[Y], \quad f^*(e_x) = \sum_{f(y)=x} e_y.$$

This is typically not a co-algebra homomorphism.

Another construction of Frobenius algebras

Let G be an algebraic group.

Construction

If X is a finite G -set then $k[X]$ is a Frobenius algebra in $\text{Rep}(G)$.

Fact

Every Frobenius algebra in $\text{Rep}(G)$ has the form $k[X]$.

Remark

The identity component of G acts trivially on a finite G -set.

Étale algebras

Definition

An **étale algebra** in a tensor category is an ACU algebra A s.t.:

- A is rigid, i.e., it has a dual.
- The trace pairing $A \otimes A \rightarrow \mathbf{1}$ is perfect.

Proposition

Étale algebras are the same as Frobenius algebras:

- *If A is a Frobenius algebra then the underlying commutative algebra is étale.*
- *If A is étale, dualizing the algebra structure wrt the trace pairing gives a co-algebra structure, which makes A into a Frobenius algebra.*

The category of étale algebras

Definition

Given a tensor category $\mathcal{T}$, we let $\text{Et}(\mathcal{T})$ be the category of étale algebras in $\mathcal{T}$, where morphisms are algebra homomorphisms.

Example

If $\mathcal{T} = \text{Rep}(G)$ then $\text{Et}(\mathcal{T})^{\text{op}}$ is the category of finite G -sets.

Remark

In general, $\text{Et}(\mathcal{T})^{\text{op}}$ is a “set-like” category: it is extensive, finitely complete, and sub-objects have complements.

§3. Deligne's category

Definition of the category

Construction

$\underline{\text{Rep}}(\mathfrak{S}_t)$ is the additive envelope of the following category:

- Objects are formal symbols $A^{\otimes n}$ with $n \in \mathbf{N}$.
- Morphisms are linear combinations of string diagrams built from η , ϵ , m , and Δ with $\epsilon \circ \eta = t$ and FA axioms.

Theorem (Deligne)

If $\text{char}(k) = 0$ and $t \notin \mathbf{N}$ then the Karoubi envelope of $\underline{\text{Rep}}(\mathfrak{S}_t)$ is a semi-simple pre-Tannakian category.

Remark

There is also a pre-Tannakian category for $\text{char}(k) = 0$ and $t \in \mathbf{N}$ by work of Deligne and Comes–Ostrik.

Universal property of Deligne's category

The object A of $\underline{\text{Rep}}(\mathfrak{S}_t)$ is an étale algebra of dimension t .

Let $\mathcal{T}$ be an arbitrary tensor category.

Theorem

Giving a tensor functor $\Phi: \underline{\text{Rep}}(\mathfrak{S}_t) \rightarrow \mathcal{T}$ is equivalent to giving an étale algebra in $\mathcal{T}$ of dimension t , via $\Phi \leftrightarrow \Phi(A)$.

§4. Oligomorphic tensor categories

Oligomorphic groups

Definition

An **oligomorphic group** is a permutation group (G, Ω) such that G has finitely many orbits on Ω^n for all $n \geq 0$.

Examples

- The infinite symmetric group acting on $\Omega = \{1, 2, \dots\}$.
- $\text{GL}_\infty(\mathbf{F}_q)$ acting on $\Omega = \mathbf{F}_q^\infty$.
- The group $\text{Aut}(\mathbf{R}, <)$ acting on $\Omega = \mathbf{R}$.

Let (G, Ω) be an oligomorphic group.

Definition

Let $\mathbf{S}(G)$ be the category of finitary smooth G -sets:

- **finitary** means finitely many orbits.
- **smooth** means each orbit is a subquotient of some Ω^n .

Remark

The category $\mathbf{S}(G)$ has all the same basic categorical properties of the category of finite Γ -sets, for a finite group Γ . In particular, $\mathbf{S}(G)$ has finite products.

Measures

Definition (Harman–Snowden)

A **measure** for G is a rule μ that assigns to each morphism $f: Y \rightarrow X$ in $\mathbf{S}(G)$, with X transitive, a quantity $\mu(f)$ in k such that certain axioms hold.

Convention

$\mu(X)$ is defined to be $\mu(X \rightarrow \text{pt})$.

Intuition

$\mu(f)$ is like the size of a fiber of f , and $\mu(X)$ is like the size of X .

The main construction

Construction (Harman–Snowden)

Define a tensor category $\underline{\text{Perm}}(G, \mu)$:

- Objects are $\mathcal{C}(X)$ with X finitary smooth G -set.
- Morphisms $\mathcal{C}(X) \rightarrow \mathcal{C}(Y)$ are functions $G \backslash (Y \times X) \rightarrow k$.
- Composition is convolution, defined using μ .
- $\mathcal{C}(X) \oplus \mathcal{C}(Y) = \mathcal{C}(X \amalg Y)$ and $\mathcal{C}(X) \otimes \mathcal{C}(Y) = \mathcal{C}(X \times Y)$.

The object $\mathcal{C}(X)$ is rigid, self-dual, of dimension $\mu(X)$.

Intuition

$\mathcal{C}(X)$ is like a permutation representation with basis indexed by X . Morphisms can be thought of as G -invariant matrices.

Recovering Deligne's category

Let (G, Ω) be the infinite symmetric group, and suppose $\text{char}(k) = 0$.

For $t \in k$, there is a unique measure μ_t such that $\mu_t(\Omega) = t$. The μ_t account for all the measures for G .

We have $\underline{\text{Perm}}(G, \mu_t)^{\text{kar}} = \underline{\text{Rep}}(\mathfrak{S}_t)^{\text{kar}}$.

In this way, the oligomorphic theory recovers Deligne's example.

§5. Universal properties

Étale algebras

Fix an oligomorphic group G with measure μ .

Fact

$\mathcal{C}(X)$ is naturally an étale algebra in $\underline{\text{Perm}}(G, \mu)$.

Consequence

There is a natural functor $\mathbf{S}(G) \rightarrow \text{Et}(\underline{\text{Perm}}(G, \mu))^{\text{op}}$.

This functor is often an equivalence, but not always.

Tensor functors

Suppose $\Phi: \underline{\text{Perm}}(G, \mu) \rightarrow \mathfrak{T}$ is a tensor functor.

Define Ψ to be the composition

$$\mathbf{S}(G) \rightarrow \text{Et}(\underline{\text{Perm}}(G, \mu))^{\text{op}} \rightarrow \text{Et}(\mathfrak{T})^{\text{op}}$$

Observations:

- Ψ is additive, i.e., it commutes with finite co-products.
- Ψ is left-exact, i.e., it commutes with finite limits.
- Ψ is compatible with μ , e.g., the dimension of $\Psi(X)$ is $\mu(X)$.

The main theorem

Let $\mathcal{T}$ be an arbitrary tensor category.

Theorem

Giving a tensor functor $\Phi: \underline{\text{Perm}}(G, \mu) \rightarrow \mathcal{T}$ is equivalent to giving a functor $\Psi: \mathbf{S}(G) \rightarrow \text{Et}(\mathcal{T})^{\text{op}}$ that is additive, left-exact, and compatible with μ .

Remark

We think of Φ as an algebraic object, but Ψ as a combinatorial object. This is why the theorem is useful.

Remark

The theorem recovers the universal property of $\underline{\text{Rep}}(\mathfrak{S}_t)$.

§6. Delannoy categories

The categories

Let $\mathbb{G} = \text{Aut}(\mathbf{R}, <)$, which acts oligomorphically on $\mathbf{R}$.

Fact

$\mathbb{G}$ has exactly four measures μ_1, μ_2, μ_3 and μ_4 .

Definition

The i th Delannoy category is $\mathfrak{C}_i = \underline{\text{Perm}}(\mathbb{G}, \mu_i)$.

Remark

$\mathfrak{C}_1^{\text{kar}}$ is semi-simple pre-Tannakian. It was studied in depth by Harman, Snowden, Snyder (2022), and found to have many remarkable properties. The other $\mathfrak{C}_i$'s have been mysterious.

Let $\mathbf{R}^{(n)} \subset \mathbf{R}^n$ be the set of increasing tuples. The following provides an essentially complete picture of $\mathbf{S}(\mathbb{G})$.

Fact

- The $\mathbf{R}^{(n)}$ are the transitive objects in $\mathbf{S}(\mathbb{G})$.
- Any $\mathbb{G}$ -map $\mathbf{R}^{(n)} \rightarrow \mathbf{R}^{(m)}$ is a projection.

Notation

Write $\mathcal{C}_i(\mathbf{R}^{(n)})$ for $\mathcal{C}(\mathbf{R}^{(n)})$ in the category $\mathfrak{C}_i$.

Universal property of $\mathbf{S}(\mathbb{G})$

Let $\mathcal{S}$ be an extensive category with finite limits, e.g., $\mathbf{Et}(\mathcal{T})^{\text{op}}$.

Definition

An **ordered object** of $\mathcal{S}$ is an object X equipped with a subobject of $X \times X$ satisfying the axioms of a total order.

Example

$\mathbf{R}$ is an ordered object in $\mathbf{S}(\mathbb{G})$.

Theorem

Giving an additive left-exact functor $\Psi: \mathbf{S}(\mathbb{G}) \rightarrow \mathcal{S}$ is equivalent to giving an ordered object of $\mathcal{S}$, via $\Psi \leftrightarrow \Psi(\mathbf{R})$.

Ordered étale algebras

Let $\mathfrak{T}$ be a tensor category.

Definition

An **ordered étale algebra** is an ordered object in $\text{Et}(\mathfrak{T})^{\text{op}}$.

(Non-)example

In $\text{Rep}(G)$ ($G = \text{alg. gp.}$), an ordered étale algebra A corresponds to a finite G -set X equipped with a G -invariant total order. The action of G on such an X is trivial $\implies A \cong \mathbf{1}^{\oplus n}$.

Simplest example

$\mathcal{C}_i(\mathbf{R})$ is an ordered étale algebra in $\mathfrak{C}_i$.

Delannic algebras

Let A be an ordered étale algebra. There is an étale algebra $A^{(n)}$ of “ordered n -tuples,” and n (co-)projection maps $A \rightarrow A^{(n)}$.

Definition

A is **Delannic** of type i , for $1 \leq i \leq 4$, if it satisfies the following three numeric conditions:

- $\dim A = \mu_i(\mathbf{R})$.
- $\dim_A A^{(2)} = \mu_i(\mathbf{R}^{(2)} \rightarrow \mathbf{R})$, for both choices of maps.

Here $\dim_A$ is dimension in the tensor category of A -modules.

Example

$\mathcal{C}_i(\mathbf{R})$ is Delannic of type i .

The main theorem

Let $\mathfrak{T}$ be an arbitrary tensor category.

Theorem

Giving a tensor functor $\Phi: \mathfrak{C}_i \rightarrow \mathfrak{T}$ is equivalent to giving a Delannic algebra of type i in $\mathfrak{T}$, via $\Phi \leftrightarrow \Phi(\mathfrak{C}_i(\mathbf{R}))$.

Proof

By our general theorem, Φ corresponds to $\Psi: \mathbf{S}(\mathbb{G}) \rightarrow \mathrm{Et}(\mathfrak{T})^{\mathrm{op}}$ that is additive, left-exact, and compatible with μ_i .

By the universal property of $\mathbf{S}(\mathbb{G})$, giving Ψ with the first two conditions is equivalent to giving an ordered étale algebra in $\mathfrak{T}$.

Compatibility with μ_i is the Delannic condition; this is non-trivial, since the former is an infinite list of numeric conditions.

Other universal properties

There has been other work on universal properties for $\mathcal{C}_1$:

- S. Kriz (2023) gave a universal property based on the simple decomposition of $\mathcal{C}_1(\mathbf{R})$, and not using the order structure.
- Khovanov and Snyder (see next talk!) give a variant of Kriz's universal property (among other things).

Not clear if there is a Kriz-style universal property for the other $\mathcal{C}_i$.

Examples

If A and B are OEA then so is $A \oplus B$ (lexicographic sum).

- $\mathcal{C}_1(\mathbf{R}) \oplus \mathbf{1}$ is Delannic of type 2 $\implies \mathfrak{C}_2 \rightarrow \mathfrak{C}_1$.
- $\mathbf{1} \oplus \mathcal{C}_1(\mathbf{R})$ is Delannic of type 3 $\implies \mathfrak{C}_3 \rightarrow \mathfrak{C}_1$.
- $\mathbf{1} \oplus \mathcal{C}_1(\mathbf{R}) \oplus \mathbf{1}$ is Delannic of type 4 $\implies \mathfrak{C}_4 \rightarrow \mathfrak{C}_1$.

Significance

Each $\mathfrak{C}_i$ admits a map to a pre-Tannakian category.

More examples

If A and B are OEA then so are $A \otimes B$ and $A^{(n)}$.

- $\mathcal{C}_1(\mathbf{R}^{(2)})$ is Delannic of type 4 $\implies \mathfrak{C}_4 \rightarrow \mathfrak{C}_1$.
- $\mathcal{C}_1(\mathbf{R}) \oplus \mathcal{C}_1(\mathbf{R}^{(2)})$ is Delannic of type 2 $\implies \mathfrak{C}_2 \rightarrow \mathfrak{C}_1$.
- More generally, $\mathcal{C}_1(\mathbf{R}^{(n)})$ is Delannic of type 1 if n is odd, and type 4 if n is even $\implies$ many functors $\mathfrak{C}_1 \rightarrow \mathfrak{C}_1$ and $\mathfrak{C}_4 \rightarrow \mathfrak{C}_1$.
- Can use $\oplus$, $\otimes$, and $(-)^{(n)}$ to obtain many more functors.

Local abelian envelopes

Let $\mathcal{T}$ be a tensor category with finite Hom's and $\text{End}(\mathbf{1}) = k$.

Theorem (Coulembier)

There exists $\{\Phi_i: \mathcal{T} \rightarrow \mathcal{U}_i\}_{i \in I}$ where each $\mathcal{U}_i$ is pre-Tannakian such that any faithful $\Phi: \mathcal{T} \rightarrow \mathcal{U}$ (pre-Tannakian) factors uniquely as $\Psi \circ \Phi_i$ with $\Psi: \mathcal{U}_i \rightarrow \mathcal{U}$ exact and faithful.

Definition

The $\mathcal{U}_i$ are the **local abelian envelopes** of $\mathcal{T}$.

Definition

If $\#I = 1$ the unique $\mathcal{U}_i$ is **the abelian envelope** of $\mathcal{T}$.

Local envelopes for Delannoy

Theorem

$\mathfrak{C}_2$ has at least two local abelian envelopes.

Proof

The functors $\mathfrak{C}_2 \rightarrow \mathfrak{C}_1$ provided by the Delannic algebras

$$\mathfrak{C}_1(\mathbf{R}) \oplus \mathbf{1} \quad \text{and} \quad \mathfrak{C}_1(\mathbf{R}) \oplus \mathfrak{C}_1(\mathbf{R}^{(2)})$$

belong to different local abelian envelopes.

Remark

Forthcoming work of Coulembier and Snowden: $\mathfrak{C}_2$ has exactly two local abelian envelopes. One is equivalent to $\mathfrak{C}_1^{\text{kar}}$, the other is a new pre-Tannakian category.