Oligomorphic groups and tensor categories

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Logistical information

Joint work

The core ideas in this talk are joint with Nate Harman.

Other joint work appearing

with Coulembier, Etingof, Nekrasov, Snyder, Zelingher.

Slides on my website

https://websites.umich.edu/~asnowden/icm-slides.pdf

§1. Background

Representation categories

Goal

Understand the representation theory of an algebraic group G.

One approach

Study the category $\mathrm{Rep}(G)$.

Problem

The abelian category $\mathrm{Rep}(G)$ carries little information.

Solution

Treat $\mathrm{Rep}(G)$ as a tensor category.

Tensor categories

Definition

A pre-Tannakian category is a k-linear abelian category $\mathcal{C}$ with a k-bilinear symmetric monoidal structure $\otimes$ such that:

(a) All objects have finite length.

(b) Every object admits a dual.

(c) $\mathrm{End}(\mathbf{1}) = k$, where $\mathbf{1} =$ monoidal unit.

The axioms imply all Hom spaces are finite dimensional.

Remark

- The definition abstracts the key properties of $\mathrm{Rep}(G)$.
- $\exists$ other kinds of tensor categories (e.g., braided).

Motivation

Question

Why pre-Tannakian categories?

Answers

(a) Tannakian reconstruction: recover G from $\mathrm{Rep}(G)$.

(b) Motives: a pre-Tannakian category not of the form $\mathrm{Rep}(G)$.

(c) Rep. thy. of algebraic groups very fertile $\implies$ this natural generalization might contain interesting new math.

The main problem

Main problem

Understand pre-Tannakian categories.

Sub-problems

- Construct new examples, i.e., not of the form $\mathrm{Rep}(G)$.
- Explain new examples: if $\mathcal{C}$ does not come from an algebraic group, does it come from a more general group-like object?
- Classify (subclasses of) pre-Tannakian categories.

Recognition theorems

Question

When is a pre-Tannakian category $\mathcal{C}$ of the form $\mathrm{Rep}(G)$?

Theorem (Tannakian reconstruction, 1972, 1982)

If $\mathcal{C}$ admits a fiber functor, i.e., a faithful exact $\otimes$-functor to Vec, *then $\mathcal{C}$ is equivalent to $\mathrm{Rep}(G)$ for some affine group scheme G.*

Theorem (Deligne, 1990, 2002)

Suppose $k = \overline{k}$ has char. 0 and $\mathcal{C}$ has moderate growth, i.e., for all objects X the function $n \mapsto \mathrm{len}(X^{\otimes n})$ has at most exponential growth. Then $\mathcal{C}$ admits a (super) fiber functor.

Two branches

Due to Deligne's theorem, the field has bifurcated:

First branch

Classify pre-Tannakian categories of moderate growth in char. p. $\exists$ conjecture from Benson–Etingof–Ostrik (2023) and Coulembier (2021) based on higher Verlinde categories.

Second branch

Study pre-Tannakian categories of super-exponential growth.

This talk focuses on the second branch.

Trace and dimension

Definition

Let $f\colon X \to X$ be an endomorphism in a pre-Tannakian category. The categorical trace of f, denoted $\underline{\mathrm{tr}}(f)$, is the composition

$$\mathbf{1} \xrightarrow{\mathrm{cv}} X \otimes X^\vee \xrightarrow{f \otimes \mathrm{id}} X \otimes X^\vee \xrightarrow{\mathrm{ev}} \mathbf{1}.$$

It is an element of $\mathrm{End}(\mathbf{1}) = k$.

Definition

The categorical dimension of X is $\underline{\dim}\, X = \underline{\mathrm{tr}}(\mathrm{id}_X)$.

Proposition

Trace is additive in short exact sequences. In particular, if f is nilpotent then $\underline{\mathrm{tr}}(f) = 0$.

Deligne interpolation

Notation

- S^λ = irreducible $\mathbf{C}[\mathfrak{S}_n]$-module (Specht module) for $\lambda \vdash n$
- $\lambda[m] = (m - |\lambda|, \lambda_1, \lambda_2, \ldots)$

Deligne's category (2007)

Semi-simple pre-Tannakian category $\underline{\mathrm{Rep}}(\mathfrak{S}_t)$ for $t \in \mathbf{C} \setminus \mathbf{N}$.

- Simples are X_λ with λ a partition; intuition: $X_\lambda = S^{\lambda[t]}$
- $X_\lambda \otimes X_\mu$ decomposes like $S^{\lambda[n]} \otimes S^{\mu[n]}$ for $n \gg 0$
- $\underline{\dim}\, X_\lambda = p_\lambda(t)$ where $p_\lambda(n) = \dim S^{\lambda[n]}$ for $n \gg 0$

First* example with super-exponential growth.

Deligne interpolation (continued)

- Deligne: $\underline{\mathrm{Rep}}(\mathfrak{S}_t)$ is an "interpolation" of the $\mathrm{Rep}(\mathfrak{S}_n)$.
- Deligne constructed $\underline{\mathrm{Rep}}(\mathfrak{S}_t)$ using the partition category.
- Harman constructed $\underline{\mathrm{Rep}}(\mathfrak{S}_t)$ as an ultraproduct of $\mathrm{Rep}(\mathfrak{S}_n)$.
- $\underline{\mathrm{Rep}}(\mathfrak{S}_t)$ exists for $t \in \mathbf{N}$, but not s.s. (Comes–Ostrik).
- Interpolation applies to other sequences of finite groups:
  - Finite classical groups $\mathbf{GL}_n(\mathbf{F}_q)$, $\mathbf{O}_n(\mathbf{F}_q)$, etc.
  - Wreath products $\mathfrak{S}_n \wr \Gamma$, with Γ a fixed finite group.
  - $\mathfrak{S}_n \wr \mathfrak{S}_m$ with both indices varying (two parameters).
- Also applies to $\mathbf{GL}_n$ (Deligne–Milne) and other classical groups
- Knop (2007) gave a general construction of pre-Tannakian categories. All known examples it produces can also be obtained from interpolation.

Summary

Objects of interest

Pre-Tannakian categories of super-exponential growth.

Known examples prior to 2022

Interpolations of finite groups or classical (super)groups.

Known classification results prior to 2022

None.

§2. Oligomorphic tensor categories

Oligomorphic groups

Definition

An oligomorphic group is a permutation group (G, Ω) such that G has finitely many orbits on Ω^n for all $n \geq 0$.

Remark

Peter Cameron coined the term and wrote an excellent introductory book *Oligomorphic permutation groups* (1990).

Examples of oligomorphic groups

- Any finite permutation group.
- The infinite symmetric group $\mathfrak{S}$.
- The infinite general linear group over a finite field.
- $\mathrm{Aut}(\mathbf{Q}, <)$ or $\mathrm{Aut}(\mathbf{R}, <)$.
- Choose a 2-coloring $c \colon \mathbf{Q} \to [2]$ such that each color is dense. Then $\mathrm{Aut}(\mathbf{Q}, c, <)$ is oligomorphic (and independent of c).
- Let X be a countable set and let $<_1$ and $<_2$ be two random total orders on X. Then $\mathrm{Aut}(X, <_1, <_2)$ is oligomorphic.
- Let X be the Rado graph (random countable graph). Then $\mathrm{Aut}(X)$ is oligomorphic.

Remark

The random infinite structures appearing above are constructed rigorously using the Fraïssé limit.

G-sets

Definition

For an oligomorphic group (G, Ω) we let $\mathbf{S}(G)$ be the category of G-sets generated by Ω: an object is a subquotient of a finite disjoint union of powers of Ω. The term "G-set" will always mean "object of $\mathbf{S}(G)$."

Important observation

The category $\mathbf{S}(G)$ is closed under finite products.

Remark

$\mathbf{S}(G)$ only depends on the topological group G, which is why we suppress Ω from the notation.

Classical permutation modules

Let $\mathrm{Perm}(\Gamma)$ denote the category of permutation modules of the finite group Γ over the field k.

We can realize $\mathrm{Perm}(\Gamma)$ as a "linearization" of $\mathbf{S}(\Gamma)$:

- An object of $\mathrm{Perm}(\Gamma)$ is a formal symbol $[X]$ with $X \in \mathbf{S}(\Gamma)$.
- A map $[X] \to [Y]$ is a Γ-invariant function $Y \times X \to k$.
- Composition is matrix multiplication.
- $[X] \oplus [Y] = [X \amalg Y]$ and $[X] \otimes [Y] = [X \times Y]$.

Key idea

Do the same with oligomorphic groups. Additional data is needed to define matrix multiplication.

Measures

Definition (Harman–S.)

A measure for the oligomorphic group G valued in a ring k is a rule μ that assigns to each morphism $f\colon Y \to X$ of transitive G-sets a value $\mu(f) \in k$ such that:

(a) $\mu(\mathrm{isom}) = 1$.

(b) $\mu(g \circ f) = \mu(g) \cdot \mu(f)$, when defined.

(c) If $f'\colon Y' \to X'$ is the base change of f along $X' \to X$, with X' transitive, then $\mu(f') = \mu(f)$.
  - Here $\mu(f') = \sum_{i=1}^{n} \mu(f_i')$ where $Y_1', \ldots, Y_n'$ are the orbits on Y' and f_i' is the restriction of f' to Y_i'.

Remark

Intuition: $\mu(f)$ measures the size of the fiber of f.

Oligomorphic permutation modules

Construction (Harman–S.)

Let G be an oligomorphic group with measure μ. Define a symmetric monoidal category $\underline{\mathrm{Perm}}(G, \mu)$ as follows:

- An object is a formal symbol $[X]$ with $X \in \mathbf{S}(G)$.
- A map $[X] \to [Y]$ is a G-invariant function $Y \times X \to k$.
- Composition is matrix multiplication, defined using μ.
- $[X] \oplus [Y] = [X \amalg Y]$ and $[X] \otimes [Y] = [X \times Y]$.

Remark

- There is some work in showing that this is all well-defined.
- The category $\underline{\mathrm{Perm}}(G, \mu)$ is rigid, and every object is self-dual.
- It is not abelian, and thus not pre-Tannakian, in general.

An example of composition

Define $\mu(X)$ to be $\mu(X \to \mathbb{1})$, where $\mathbb{1}$ is the one-point set.

Proposition

The composition $\mathbf{1} \to [X] \to \mathbf{1}$ *is* $\mu(X)$.

Corollary

We have $\underline{\dim}\,[X] = \mu(X)$.

Abelian envelopes

Definition

Let $\mathcal{C}$ be a k-linear symmetric monoidal category. An abelian envelope of $\mathcal{C}$ is a faithful $\otimes$-functor $\Phi\colon \mathcal{C} \to \mathcal{T}$, with $\mathcal{T}$ pre-Tannakian, that is universal:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\Phi"] \arrow[dr, "\Phi'"'] & \mathcal{T} \arrow[d, dotted, "\exists!\Psi"] \\ & \mathcal{T}' \end{tikzcd}$$

Given a faithful $\otimes$-functor Φ', with $\mathcal{T}'$ pre-Tannakian, a faithful $\otimes$-functor Ψ exists such that the diagram commutes.

Remark

Abelian envelopes need not exist, but are unique when they do.

Pre-Tannakian categories from oligomorphic groups

Definition

A field-valued measure μ for an oligomorphic group is regular if $\mu(X) \neq 0$ for all transitive G-sets X. It is quasi-regular if it restricts to a regular measure on some open subgroup.

Theorem (Harman–S.)

If μ is quasi-regular and nilpotent endomorphisms in $\underline{\mathrm{Perm}}(G, \mu)$ have trace 0 then $\underline{\mathrm{Perm}}(G, \mu)$ admits an abelian envelope $\underline{\mathrm{Rep}}(G, \mu)$. If μ is regular then $\underline{\mathrm{Rep}}(G, \mu)$ is semi-simple.

Summary

Main construction

An oligomorphic group G with a measure μ gives a (non-abelian) rigid tensor category $\underline{\mathrm{Perm}}(G, \mu)$.

Abelian envelopes

Under certain assumptions (μ quasi-regular and $\underline{\mathrm{tr}}(\mathrm{nilp}) = 0$), the category $\underline{\mathrm{Perm}}(G, \mu)$ admits an abelian envelope.

§3. Examples

Deligne's category revisited

Let $\mathfrak{S}$ be the infinite symmetric group acting on $\Omega = \{1, 2, \ldots\}$.

Theorem

(a) *For* $t \in \mathbf{C}$, $\exists!$ $\mathbf{C}$*-valued measure* μ_t *with* $\mu_t(\Omega) = t$*.*

(b) *Every* $\mathbf{C}$*-valued measure is one of the* μ_t*.*

(c) μ_t *is regular for* $t \in \mathbf{C} \setminus \mathbf{N}$*, and quasi-regular for* $t \in \mathbf{N}$*.*

(d) *Nilpotent endomorphisms in* $\underline{\mathrm{Perm}}(\mathfrak{S}, \mu_t)$ *have trace 0.*

(e) $\underline{\mathrm{Rep}}(\mathfrak{S}, \mu_t)$ *is equivalent to Deligne's category* $\underline{\mathrm{Rep}}(\mathfrak{S}_t)$*.*

Remark

Similar story for interpolation categories of other sequences of finite groups, such as $\mathbf{GL}_n(\mathbf{F}_q)$.

The Delannoy category

Let $G = \mathrm{Aut}(\mathbf{R}, <)$.

Let $\mathbf{R}^{(n)} \subset \mathbf{R}^n$ be the strictly ordered tuples. This is a transitive G-set, and these account for all the transitive G-sets.

Proposition

There is a measure μ for G given by

$$\mu(\mathbf{R}^{(n)} \to \mathbf{R}^{(m)}) = (-1)^{n+m}$$

This measure is regular over any field.

Remark

We have $\mu(f) = \chi_c(\mathrm{fiber})$, and standard properties of χ_c translate to the measure axioms.

The Delannoy category (continued)

Theorem (Harman–S.)

The abelian envelope $\underline{\mathrm{Rep}}(G, \mu)$ *exists as a semi-simple pre-Tannakian category over any field.*

Definition

This category is called the Delannoy category.

Remark

- Remarkable that Delannoy is semi-simple in all characteristics.
- First example of a semi-simple pre-Tannakian category of super-exponential growth in positive characteristic.

The Delannoy category (continued)

Other properties of the Delannoy category (Harman–S.–Snyder):

- Simple objects indexed by words in $\{\bullet, \circ\}$.
- Categorical dimension of the simple L_λ is $(-1)^{\ell(\lambda)}$.
- Combinatorial rule for $L_\lambda \otimes L_\mu$, closely related to shuffling.
- $\mathrm{End}([\mathbf{R}^{(n)}])$ admits a combinatorial description related to Delannoy paths; hence the name of the category.
- The Adams operations on the Grothendieck group are trivial, i.e., $\psi^p = \mathrm{id}$ for all p. Delannoy is essentially the only known pre-Tannakian category with this property.

Additional examples

- Vector Delannoy category (Kriz, Deligne).
- Arboreal categories (Harman–Nekrasov–S.) coming from Fraïssé classes of trees. First 1-parameter family of pre-Tannakian categories not coming from interpolation.
- Jacobi categories (Harman–S.–Zelingher). Categorical dimensions are related to Gauss and Jacobi sums.
- Second Delannoy category (Coulembier–S.). First pre-Tannakian category from non-quasi-regular measure.

§4. Further topics

Finding all measures

Problem

Given an oligomorphic group G, describe all measures.

Observation

There exists a universal measure valued in a ring $\Theta(G)$. To find all measures, it suffices to compute $\Theta(G)$.

Finding all measures (continued)

Theorem (Harman–S.)

$\Theta(G)$ *is a binomial ring.*

Examples

(a) $\Theta(\mathfrak{S})$ is the ring of integer-valued polynomials (Harman–S.).

(b) For $G = \mathrm{Aut}(\mathbf{R}, <)$, we have $\Theta(G) = \mathbf{Z}^4$ (Harman–S.).

(c) Let X be a countable set with random total orders $<_1$ and $<_2$, and let $G = \mathrm{Aut}(X, <_1, <_2)$. Then $\Theta(G) = \mathbf{Z}^{37}$ (S.).

(d) Let G be as above, but with r random total orders. Then $\Theta(G) = \mathbf{Z}^n$ for some $n \lesssim \exp(\exp(r^2 \log r))$ (Nekrasov–S.).

(e) $\exists G$ with $\Theta(G) = 0$ (Harman–Nekrasov).

Discrete categories

Observation

An algebraic group G is finite if and only if every object of $\mathrm{Rep}(G)$ is a quotient of an étale algebra.

Definition (Harman–S.)

A pre-Tannakian category is discrete if every object is a quotient of an étale algebra.

Remark

- Oligomorphic tensor categories are discrete.
- Pre-Tannakian categories are a generalization of algebraic groups. The discrete ones generalize finite groups.

Discrete categories (continued)

Theorem (Harman–S.)

Let $\mathcal{C}$ be a discrete pre-Tannakian category. Then $\mathcal{C}$ is equivalent to $\underline{\mathrm{Rep}}(G, \mu)$ for some oligomorphic group G and measure μ.

Remark

This is the first general classification result for pre-Tannakian categories of super-exponential growth.

Linearly oligomorphic groups

- The interpolation category $\underline{\mathrm{Rep}}(\mathbf{GL}_t)$ is not discrete $\implies$ it does not come from the oligomorphic theory.
- Idea: $\underline{\mathrm{Rep}}(\mathfrak{S}_t)$ comes from infinite symmetric group $\mathfrak{S}$ $\implies$ $\underline{\mathrm{Rep}}(\mathbf{GL}_t)$ should come from the infinite $\mathbf{GL}$.
- Idea: $\mathfrak{S}$ is the prototype for oligomorphic groups $\implies$ $\mathbf{GL}$ should be the prototype for "linearly oligomorphic" groups.
- Hope: these will lead to more tensor categories.

Linearly oligomorphic groups (continued)

Definition (Harman–S.)

A linear group (G, V) is linearly oligomorphic if for every d there is a finite dimensional $E \subset V$ such that any d-dimensional subspace of V can be moved into E using G.

- We constructed many interesting examples of these groups using a variant of Fraïssé theory.
- Open problem: find a construction

$$\text{linearly oligomorphic group} + ?? \implies \text{tensor category}$$

- Potentially interesting connections to model theory, e.g., linear analog of Ryll-Nardzewski theorem.