

GRÖBNER METHODS FOR REPRESENTATIONS OF COMBINATORIAL CATEGORIES

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ABSTRACT. Given a category \mathcal{C} of a combinatorial nature, we study the following fundamental question: how does the combinatorial behavior of \mathcal{C} affect the algebraic behavior of representations of \mathcal{C} ? We prove two general results. The first gives a combinatorial criterion for representations of \mathcal{C} to admit a theory of Gröbner bases. From this, we obtain a criterion for noetherianity of representations. The second gives a combinatorial criterion for a general “rationality” result for Hilbert series of representations of \mathcal{C} . This criterion connects to the theory of formal languages, and makes essential use of results on the generating functions of languages, such as the transfer-matrix method and the Chomsky–Schützenberger theorem.

Our work is motivated by recent work in the literature on representations of various specific categories. Our general criteria recover many of the results on these categories that had been proved by ad hoc means, and often yield cleaner proofs and stronger statements. For example: we give a new, more robust, proof that FI-modules (originally introduced by Church–Ellenberg–Farb), and a family of natural generalizations, are noetherian; we give an easy proof of a generalization of the Lannes–Schwartz artinian conjecture from the study of generic representation theory of finite fields; we significantly improve the theory of Δ -modules, introduced by Snowden in connection to syzygies of Segre embeddings; and we establish fundamental properties of twisted commutative algebras in positive characteristic.

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1. INTRODUCTION

Informally, a **combinatorial category** is a category whose objects are finite sets, possibly with extra structure, and whose morphisms are functions, possibly with extra structure. A **representation** of a category over a ring \mathbf{k} is a functor to the category of \mathbf{k} -modules. For simplicity of exposition, we assume that \mathbf{k} is a field in the introduction. Typically, a representation can be thought of as a sequence of representations of finite groups together with transition maps. Some examples of interest include:

- The category **FI** of finite sets with injections. A representation of this category can be thought of as a sequence $(M_n)_{n \geq 0}$, where M_n is a representation of the symmetric group S_n , together with transition maps $M_n \rightarrow M_{n+1}$ satisfying certain compatibilities. This category was studied in [CEF, CEFN], where many examples of representations occurring in algebra and topology are discussed, and from a different point of view in [SS1]. See [Fa] for a survey of its uses and additional references.
- Variants of **FI**. In [Sn], modules over twisted commutative algebras are studied; these can be viewed (in certain cases) as representations of a category \mathbf{FI}_d generalizing **FI**. In [Wi] analogs of **FI** for other classical Weyl groups are studied. In [WG] the category **FA** of finite sets with all maps is studied.
- The category \mathbf{FS}_G of finite sets with G -surjections, G being a finite group (see §10.1 for the definition). Really, it is the opposite category that is of interest. A representation of $\mathbf{FS}_G^{\text{op}}$ can be thought of as a sequence $(M_n)_{n \geq 0}$, where M_n is a representation of the wreath product $S_n \wr G$, together with transition maps $M_n \rightarrow M_{n+1}$ satisfying certain compatibilities (quite different from those in the **FI** case). The theory of $\mathbf{FS}_G^{\text{op}}$ representations, with G a symmetric group, is closely tied to the theory of Δ -modules studied in [Sn].
- The Brauer categories of [SS3], in which the objects are finite sets and the morphisms are certain Brauer diagrams. Representations of this category are equivalent to algebraic representations of the infinite orthogonal group.
- The category $\mathbf{VA}_{\mathbf{F}_q}$ of finite-dimensional vector spaces over a finite field \mathbf{F}_q with all linear maps. Representations of this category (in particular when $\mathbf{k} = \mathbf{F}_q$) have been studied in relation to algebraic K-theory, rational cohomology, and the Steenrod algebra, see [K4] for a survey and additional references.

The referenced works use a variety of methods, often ad hoc, to study representations. However, one is struck by the fact that many of the results appear to be quite similar. For instance, each proves (or conjectures) a noetherianity result. This suggests that there are general principles at play, and leads to the subject of our paper:

Main Problem. Find practical combinatorial criteria for categories that imply interesting algebraic properties of their representations.

We give solutions to this problem for the algebraic properties of noetherianity and rationality of Hilbert series. Our criteria easily recover and strengthen most known results, and allow us to resolve some open questions. Without a doubt, they will be applicable to many categories not yet considered.

In the remainder of the introduction, we summarize our results and applications in more detail, and indicate some interesting open problems. See §1.6 for a guide for the paper.

1.1. Noetherianity. The category $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ of representations of \mathcal{C} is called **noetherian** if any subrepresentation of a finitely generated representation is again finitely generated;

see §4.1 for the definition of “finitely generated.” This is a fundamental property, and has played a crucial role in applications. The first main theoretical result of this paper is a combinatorial criterion on \mathcal{C} that ensures $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian, for any left-noetherian ring \mathbf{k} . We now explain this criterion, and the motivation behind it.

We start by recalling a combinatorial proof of the Hilbert basis theorem (i.e., $\mathbf{k}[x_1, \dots, x_n]$ is a noetherian ring). Pick an admissible order on the monomials, i.e., a well-order compatible with multiplication. Using the order, we can define initial ideals, and reduce the study of the ascending chain condition to monomial ideals. Now, the set of monomial ideals is naturally in bijection with the set of ideals in the poset \mathbf{N}^r . Thus noetherianity of $\mathbf{k}[x_1, \dots, x_n]$ follows from noetherianity of the poset \mathbf{N}^r (Dixon’s lemma), which is a simple combinatorial exercise: given infinitely many vectors v_1, v_2, \dots in \mathbf{N}^r , there exists $i \neq j$ with $v_i \leq v_j$, where \leq means coordinate-by-coordinate comparison.

To apply this method to $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ we must first make sense of what “monomials” are. Given an object x of \mathcal{C} , define a representation P_x of \mathcal{C} by $P_x(y) = \mathbf{k}[\text{Hom}(x, y)]$, i.e., $P_x(y)$ is the free \mathbf{k} -module with basis $\text{Hom}(x, y)$. We call P_x the **principal projective** at x . These representations take the place of free modules; in fact, one should think of P_x as the free representation with one generator of degree x . The **monomials** are the basis elements $e_f \in P_x(y)$ for $f: x \rightarrow y$. A subrepresentation M of P_x is **monomial** if $M(y)$ is spanned by the monomials it contains, for all $y \in \mathcal{C}$.

We now carry over the combinatorial proof of the Hilbert basis theorem. For simplicity of exposition, we assume that \mathcal{C} is directed, i.e., if $f: x \rightarrow x$ then $f = \text{id}_x$. For an object x of \mathcal{C} , we write $|\mathcal{C}_x|$ for the set of isomorphism classes of morphisms $x \rightarrow y$. This can be thought of as the set of monomials in P_x . Suppose that \mathcal{C} satisfies the following condition:

- (G1) For each $x \in \mathcal{C}$, the set $|\mathcal{C}_x|$ admits an admissible order \prec , that is, a well-order compatible with post-composition, i.e., $f \prec f'$ implies $gf \prec gf'$ for all g .

Given a subrepresentation M of a principal projective P_x , we can use \prec to define the **initial subrepresentation** $\text{init}(M)$. This allows us to reduce the study of the ascending chain condition for subrepresentations of P_x to monomial subrepresentations. Monomial subrepresentations are naturally in bijection with ideals in the poset $|\mathcal{C}_x|$, where the order is defined by $f \leq g$ if $g = hf$ for some h . (Note: \prec and \leq are two different orders on $|\mathcal{C}_x|$; the former is chosen, while the latter is canonical.) We now assume that \mathcal{C} satisfies one further condition:

- (G2) For each $x \in \mathcal{C}$, the poset $|\mathcal{C}_x|$ is noetherian.

Given this, ascending chains of monomial subrepresentations stabilize, and so P_x is noetherian. One easily deduces from this that $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is a noetherian category.

The above discussion motivates one of the main definitions in this paper:

Definition 1.1.1. A directed category \mathcal{C} is **Gröbner** if (G1) and (G2) hold. \square

We can summarize the previous paragraph as: if \mathcal{C} is a directed Gröbner category then $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian. In the body of the paper, we give a definition of Gröbner that does not require directed. However, it still precludes non-trivial automorphisms, and hence many of the categories of primary interest. This motivates a weakening of the above definition:

Definition 1.1.2. A category \mathcal{C} is **quasi-Gröbner** if there is a Gröbner category \mathcal{C}' and an essentially surjective functor $\mathcal{C}' \rightarrow \mathcal{C}$ satisfying property (F) (see Definition 4.2.1). \square

Property (F) is a finiteness condition that intuitively means \mathcal{C}' locally has finite index in \mathcal{C} . Our main combinatorial criterion for noetherianity is the following theorem:

Theorem 1.1.3. *If \mathcal{C} is quasi-Gröbner and \mathbf{k} is left-noetherian, then $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian.*

This is a solution of an instance of the Main Problem: “(quasi-)Gröbner” is a purely combinatorial condition on \mathcal{C} , which can be checked easily in practice, and the above theorem connects it to an important algebraic property of representations.

Example 1.1.4. Recall that **FI** is the category whose objects are finite sets and whose morphisms are injections. The automorphism groups in this category are symmetric groups, so **FI** is not Gröbner. Define **OI** to be the category whose objects are totally ordered finite sets and whose morphisms are order-preserving injections. We show that **OI** is a Gröbner category (Theorem 7.1.1) and the forgetful functor $\mathbf{OI} \rightarrow \mathbf{FI}$ shows that **FI** is quasi-Gröbner. In particular, we see that $\text{Rep}_{\mathbf{k}}(\mathbf{FI})$ is noetherian for any left-noetherian ring \mathbf{k} . See Remark 7.1.4 for the history of this result and its generalizations. \square

The above example is a typical application of the theory of Gröbner categories: The main category of interest (in this case **FI**) has automorphisms, and is therefore not Gröbner. One therefore adds extra structure (e.g., a total order) to obtain a more rigid category. One then shows that this rigidified category is Gröbner, which usually comes down to an explicit combinatorial problem (in this case, Dixon’s lemma). Finally, one deduces that the original category is quasi-Gröbner, which is usually quite easy.

1.2. Hilbert series. A **norm** on a category \mathcal{C} is a function $\nu: |\mathcal{C}| \rightarrow \mathbf{N}$, where $|\mathcal{C}|$ is the set of isomorphism classes of \mathcal{C} . Suppose that \mathcal{C} is equipped with a norm and M is a representation of \mathcal{C} over a field \mathbf{k} . We then define the **Hilbert series** of M by

$$H_M(t) = \sum_{x \in |\mathcal{C}|} \dim_{\mathbf{k}} M(x) \cdot t^{\nu(x)},$$

when this makes sense, i.e., when the coefficient of t^n is finite for all n . The second main theoretical result of this paper is a combinatorial condition on (\mathcal{C}, ν) that ensures $H_M(t)$ has a particular form, for any finitely generated representation M .

The key idea is to connect to the theory of formal languages. Let Σ be a finite set, and let Σ^* denote the set of all finite words in Σ (i.e., the free monoid generated by Σ). A **language** on Σ is a subset of Σ^* . Given a language \mathcal{L} , we define its **Hilbert series** by

$$H_{\mathcal{L}}(t) = \sum_{w \in \mathcal{L}} t^{\ell(w)},$$

where $\ell(w)$ is the length of the word w . There are many known results on Hilbert series of languages: for example, regular languages have rational Hilbert series and unambiguous context-free languages have algebraic Hilbert series (Chomsky–Schützenberger theorem). We review these results, and establish some new ones, in §3.

Let \mathcal{P} be a class of languages. A **\mathcal{P} -lingual structure** on \mathcal{C} at x consists of a finite alphabet Σ and an injection $i: |\mathcal{C}_x| \rightarrow \Sigma^*$ such that the following two conditions hold: (1) for any $f: x \rightarrow y$ in $|\mathcal{C}_x|$ we have $\nu(y) = \ell(i(f))$; and (2) if S is a poset ideal of $|\mathcal{C}_x|$ then the language $i(S)$ is of class \mathcal{P} . The first condition essentially means that we can interpret monomials of P_x as words in some alphabet in such a way that their norm agrees with the length of the word. We say that \mathcal{C} is **\mathcal{P} -lingual** if it admits a \mathcal{P} -lingual structure at every object. A special case of our main result on Hilbert series is then:

Theorem 1.2.1. *Suppose that \mathcal{C} is a \mathcal{P} -lingual Gröbner category and let M be a finitely generated representation of \mathcal{C} . Then H_M is a \mathbf{Z} -linear combination of series of the form $H_{\mathcal{L}}$, where each \mathcal{L} is a language of class \mathcal{P} .*

This too is a solution of an instance of the Main Problem: “ \mathcal{P} -lingual” is a purely combinatorial condition on (\mathcal{C}, ν) , which can be easily checked in practice, and the above theorem connects it to an important algebraic property of representations.

Example 1.2.2. Define a norm on \mathbf{OI} by $\nu(x) = \#x$. We show that (\mathbf{OI}, ν) is \mathcal{P} -lingual, where \mathcal{P} is the class of “ordered languages” introduced in §3.3; we refer to Theorem 7.1.1 for the details. As a consequence, if M is a finitely generated \mathbf{FI} -module then $H_M(t) = \frac{p(t)}{(1-t)^k}$ where $p(t)$ is a polynomial and $k \geq 0$, and so the function $n \mapsto \dim_{\mathbf{k}} M([n])$ is eventually polynomial. See Remark 7.1.6 for the history of this result and its generalizations. \square

1.3. Applications. We apply our theory to prove a large number of results about categories of interest. We mention three of these results here.

1.3.1. Lannes–Schwartz artinian conjecture. This conjecture, posed by Jean Lannes and Lionel Schwartz, first appears in print as [K2, Conjecture 3.12], and is equivalent to the statement that $\text{Rep}_{\mathbf{F}_q}(\mathbf{VA}_{\mathbf{F}_q})$ is noetherian. (The conjecture asserts the dual statement that the principal injectives are artinian.) Some previous work on this conjecture appears in [Dj1, Dj2, Dj3, K3, Po1, Po2, Po3]. See [K4, Sc] for the connection to algebraic topology.

The conjecture is a special case of our result (Corollary 8.3.3) that $\text{Rep}_{\mathbf{k}}(\mathbf{VA}_R)$ is noetherian for any left-noetherian ring \mathbf{k} and finite commutative ring R . One of the original motivations for this conjecture is that it implies that Ext modules between finitely generated functors are finite-dimensional (equivalently every finitely generated functor has a resolution by finitely generated projectives — this was previously only known for the restricted class of “finite” functors). See [FFSS] for some general results and calculations for these Ext groups. A similar (but distinct) proof of this conjecture appears in [PS].

1.3.2. Syzygies of Segre embeddings. In [Sn], the p -syzygies of all Segre embeddings (with any number of factors of any dimensions, but with p fixed) are assembled into a single algebraic structure called a Δ -module. The two main results of [Sn] state that, in characteristic 0, this structure is finitely generated and has a rational Hilbert series. Informally, these results mean that the p -syzygies of Segre embeddings admit a finite description.

We improve the results of [Sn] in three ways. First, we show that the main theorems continue to hold in positive characteristic. It is not clear if one would expect this a priori, since the syzygies of the Segre embeddings are known to behave differently in positive characteristic (see [Ha] for determinantal ideals, which for 2×2 determinants are special cases of Segre embeddings). While we only work with Δ -modules over fields, one can work over \mathbf{Z} and show that for any given syzygy module, the type of torsion that appears is bounded.

Second, we greatly improve the rationality result for the Hilbert series and give an affirmative answer to [Sn, Question 5]. This yields strong constraints on how p -syzygies of the Segre vary with the number of factors.

Finally, we remove a technical assumption from [Sn]: that paper only dealt with “small” Δ -modules, whereas our methods handle all finitely generated Δ -modules. This is useful for technical reasons: for instance, it shows that a finitely generated Δ -module admits a resolution by finitely generated projective Δ -modules. In particular, we find that Δ -modules of syzygies are finitely presented, and so can be completely described in a finite manner.

It is worth emphasizing that this paper greatly simplifies the approach to Δ -modules. The treatment in [Sn] is abstract and complicated. Ours is much more elementary and direct.

1.3.3. Twisted commutative algebras in positive characteristic. A twisted commutative algebra (tca) is a graded algebra on which the symmetric group S_n acts on the n th graded piece so that the multiplication is commutative up to a “twist” by the symmetric group. In characteristic 0, one can use Schur–Weyl duality to describe tca’s in terms of commutative algebras with an action of $\mathbf{GL}(\infty)$, and we have fruitfully exploited this to obtain many results [Sn, SS1, SS2, SS3]. This method is inapplicable in positive characteristic, so we know much less about tca’s there. In [CEFN], the univariate tca $\mathbf{k}\langle x \rangle$ was analyzed, for any ring \mathbf{k} , and certain fundamental results (such as noetherianity) were established. We establish many of the same results for the multivariate tca $\mathbf{k}\langle x_1, \dots, x_d \rangle$; see §7.3 for details.

1.4. Relation to previous work.

- The idea of reducing noetherianity of an algebraic structure to that of a poset has been used before in different contexts. We highlight [Co] for an example in universal algebra and [Hi] for examples in abstract algebra.
- The notion of quasi-Gröbner formalizes the intuitive idea of “breaking symmetry.” A similar idea is used in [DK1] to define Gröbner bases for symmetric operads by passing to the weaker structure of shuffle operads. This idea is used in [KP] to study Hilbert series of operads with well-behaved Gröbner bases.
- A related topic (and one that serves as motivation for us) is the notion of “noetherianity up to symmetry” in multilinear algebra and algebraic statistics. We point to [DE, DrK, HM, HS] for some applications of noetherianity and to [Dr] for a survey and further references.
- While preparing this article, we discovered that an essentially equivalent version of Proposition 8.2.1 is proven in the proof of [DrK, Proposition 7.5].
- Another source of motivation is the topic of representation stability [CF, Fa] and its relation to FI-modules [CEF, CEFN]. The main advantage of our approach over the ones in [CEF, CEFN] is its flexibility. For example, it is remarked after [CEFN, Proposition 2.12] that their techniques cannot handle linear analogues of **FI**; we handle these in Theorem 8.3.1.

1.5. Open problems.

1.5.1. Krull dimension. There is a notion of Krull dimension for abelian categories [Ga, Ch. IV] generalizing that for commutative rings. If \mathbf{k} is a field of characteristic 0, then $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FA})$ has Krull dimension 0 [WG], $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI})$ has Krull dimension 1 [SS1, Corollary 2.2.6], and $\mathrm{Rep}_{\mathbf{k}}(\mathbf{OI})$ has infinite Krull dimension (easy). These results hold in positive characteristic as well: $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FA})$ is handled in Theorem 8.4.4, $\mathrm{Rep}_{\mathbf{k}}(\mathbf{OI})$ remains easy, and $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI})$ is an unpublished result of ours.

One would like a combinatorial method to compute the Krull dimension of $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$. In Proposition 5.3.5 we give a criterion for dimension 0, but it is probably far from optimal.

1.5.2. Enhanced Hilbert series. Our definition of the Hilbert series of $M \in \mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ only records the dimension of $M(x)$, for each object x . One could attempt to improve this by recording the representation of $\mathrm{Aut}(x)$ on $M(x)$. Suppose R is a ring and for each $x \in |\mathcal{C}|$ we have an additive function $\mu_x: \mathcal{R}_{\mathbf{k}}(\mathrm{Aut}(x)) \rightarrow R$ ($\mathcal{R}_{\mathbf{k}}$ denotes the Grothendieck group).

We define the **enhanced Hilbert series** of M by

$$\tilde{H}_M = \sum_{x \in |\mathcal{C}|} \mu_x([M(x)]),$$

where $[M(x)]$ is the class of $M(x)$ in $\mathcal{R}_{\mathbf{k}}(\text{Aut}(x))$, when this sum makes sense. For example, if $\mathcal{C} = \mathbf{FI}$ and \mathbf{k} has characteristic 0, we define maps μ_x to $R = \mathbf{Q}[[t_1, t_2, \dots]]$ in [SS1, §5.1], and prove a sort of rationality result there. We can now prove the analogous result for \mathbf{FI}_d -modules as well. A rationality result for enhanced Hilbert series for $\mathbf{FS}_G^{\text{op}}$ -modules is given in §10.1. Is it possible to prove a general rationality result?

1.5.3. Minimal resolutions and Poincaré series. Suppose \mathcal{C} is a weakly directed (i.e., any self-map is invertible) normed category. Suppose furthermore that the automorphism groups in \mathcal{C} are finite and have order invertible in \mathbf{k} . For a \mathcal{C} -module M , let $\Psi(M)$ be the \mathcal{C} -module defined as follows: $\Psi(M)(x)$ is the quotient of $M(x)$ by the images of all maps $M(y) \rightarrow M(x)$ induced by non-isomorphisms $y \rightarrow x$ in \mathcal{C} . One can think of $\Psi(M)$ as analogous to tensoring M with the residue field in the case of modules over an augmented algebra. The left-derived functors of Ψ exist, and one can read off from $L^i \Psi(M)$ the i th projective in the minimal resolution of M . Define the **Poincaré series** of M by

$$P_M(\mathbf{t}, q) = \sum_{i \geq 0} H_{L^i \Psi(M)}(\mathbf{t})(-q)^i.$$

This contains strictly more information than the Hilbert series, but is much more subtle since it does not factor through the Grothendieck group. What can one say about the form of this series? We proved a rationality theorem for Poincaré series of \mathbf{FI} -modules in [SS1, §6.7] and can generalize this result to \mathbf{FI}_d -modules. Preliminary computations with $\mathbf{VI}_{\mathbf{F}_q}$ -modules (defined in §8.3) have seen theta series come into play; it is not clear yet if there is a deeper meaning to this.

1.5.4. Noetherianity results for other categories. There are combinatorial categories which are expected to be noetherian, but do not fall into our framework. For example: Let \mathcal{C} be the category whose objects are finite sets, and where a morphism $S \rightarrow T$ is an injection $f: S \rightarrow T$ together with a perfect matching on $T \setminus f(S)$. When \mathbf{k} has characteristic 0, $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is equivalent to the category of $\text{Sym}(\text{Sym}^2(\mathbf{k}^\infty))$ -modules with a compatible polynomial action of $\mathbf{GL}_\infty(\mathbf{k})$. See also [SS3, §4.2] for a connection between $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ (where \mathcal{C} is called (db)) and the stable representation theory of the orthogonal group.

We expect $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian, but cannot prove it. There is an ordered version \mathcal{D} where the objects are ordered finite sets and the maps $f: S \rightarrow T$ are order-preserving, but the principal projective P_\emptyset is not noetherian: if $M_n: \emptyset \rightarrow [2n]$ is the perfect matching on $\{1, \dots, 2n\}$ consisting of the edges $(i, i+3)$ where $i = 1, 3, \dots, 2n-3$ and the edge $(2n-1, 2)$, then M_3, M_4, M_5, \dots are pairwise incomparable. Is there a way to extend the scope of the methods of this paper to include these categories?

1.5.5. Coherence. The category $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is **coherent** if the kernel of any map of finitely generated projective representations is finitely generated. This is a weaker property than noetherianity, and should therefore be easier to prove. We have some partial combinatorial results on coherence, but none that apply in cases of interest. We would be especially interested in a criterion that applies to the category mentioned in §1.5.4.

1.5.6. *Hilbert series of ideals in permutation posets.* There are many algebraic structures not mentioned in this paper which lead to interesting combinatorial problems. As an example, let \mathfrak{S} be the disjoint union of all finite symmetric groups. Represent a permutation σ in one-line notation: $\sigma(1)\sigma(2)\cdots\sigma(n)$. Say that $\tau \leq \sigma$ if there is a consecutive subword $\sigma(i)\sigma(i+1)\cdots\sigma(i+r-1)\sigma(i+r)$ which gives the same permutation as τ , i.e., $\sigma(j) > \sigma(j')$ if and only if $\tau_j > \tau_{j'}$ for all $j \neq j'$. (In the literature, τ is a **consecutive pattern** in σ .) If we drop the condition “consecutive,” then this is the poset of pattern containment (see [B, §7.2.3]) which has infinite anti-chains ([B, Theorem 7.35]), so the same is true for consecutive pattern containment.

What is the behavior of Hilbert series of finitely generated ideals in this poset? For example, are they always D-finite (see [St2, §6.4] for the definition)? This is asked for the pattern containment poset in [B, Conjecture 5.8]. The relevant algebra is monomial ideals in the free shuffle algebra (see [Ro, Example 2.2(b)] or [DK2, §2.2, Example 2]).

1.6. **Guide to the paper.** The paper is divided into two parts: theory and applications.

The first part on theory begins with background results on noetherian posets (§2) and formal languages (§3). The material on posets is standard and we have included proofs to make it self-contained; we use it for applications to noetherian conditions on representations of categories. The section on formal languages is a mix of review and new material on ordered languages. This material is essential for our applications to Hilbert series.

In §4 we introduce basic terminology and properties of representations of categories and functors between them. We state criteria for noetherian conditions on representations which will be further developed in later sections. §5 introduces and develops the main topic of this paper: Gröbner bases for representations of categories and (quasi-)Gröbner categories. We give a criterion for categories to admit a Gröbner basis theory and relate noetherian properties of representations to those of posets. §6 is concerned with Hilbert series of representations of categories. Here we introduce the notion of lingual structures on categories and connect properties of Hilbert series with formal languages.

The second part of the paper contains applications of the theory developed in the first part. §7 is about categories of finite sets and injective functions of different kinds. This has two sources of motivation: the theory of **FI**-modules [CEF] and the theory of twisted commutative algebras [SS2]. We recover and strengthen known results on noetherianity and Hilbert series for these categories and related ones.

§8 concerns categories of finite sets and surjective functions. These categories are much more complicated than their injective counterparts. A significant application of the results here is the proof of the Lannes–Schwartz artinian conjecture (discussed in §1.3.1).

§9 is about applications to Δ -modules (introduced by the second author in [Sn]). We prove that finitely generated Δ -modules are noetherian over any field, thus significantly improving the results of [Sn] where it is only shown in characteristic 0 under a “smallness” assumption. For Hilbert series of Δ -modules, we affirmatively resolve and strengthen [Sn, §6, Question 5] by proving a stronger rationality result.

Finally, in the last section, we mention some additional examples, some of which will appear in a different article for reasons of space.

We end with some notation:

- If Σ is a set, let Σ^* denote the set of words in Σ , i.e., the free monoid generated by Σ . For $w \in \Sigma^*$, let $\ell(w)$ denote the length of the word.
- For a non-negative integer $n \geq 0$, set $[n] = \{1, \dots, n\}$, with the convention $[0] = \emptyset$.

- Let \mathbf{n} be an element of \mathbf{N}^r . We write $|\mathbf{n}|$ for the sum of the coordinate of \mathbf{n} and $\mathbf{n}!$ for $n_1! \cdots n_r!$. We let $[\mathbf{n}]$ be the tuple $([n_1], \dots, [n_r])$ of finite sets. Given variables t_1, \dots, t_r , we let $\mathbf{t}^{\mathbf{n}}$ be the monomial $t_1^{n_1} \cdots t_r^{n_r}$.
- $\widehat{\text{Sym}}$ denotes the completion of Sym with respect to the homogeneous maximal ideal, i.e., $\widehat{\text{Sym}}(V) = \prod_{n \geq 0} \text{Sym}^n(V)$.

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Part 1. Theory

2. PARTIALLY ORDERED SETS

In this section, we state some basic definitions and properties of noetherian posets. This section can be skipped and referred back to as necessary since it serves a technical role only.

Let X be a poset. Then X satisfies the **ascending chain condition** (ACC) if every ascending chain in X stabilizes, i.e., given $x_1 \leq x_2 \leq \cdots$ in X we have $x_i = x_{i+1}$ for $i \gg 0$. The **descending chain condition** (DCC) is defined similarly. An **anti-chain** in X is a sequence x_1, x_2, \dots such that $x_i \not\leq x_j$ for all $i \neq j$. An **ideal** in X is a subset I of X such that $x \in I$ and $x \leq y$ implies $y \in I$. We write $\mathcal{I}(X)$ for the poset of ideals of X , ordered by inclusion. For $x \in X$, the **principal ideal** generated by x is $\{y \mid y \geq x\}$. An ideal is **finitely generated** if it is a finite union of principal ideals. The following result is standard.

Proposition 2.1. *The following conditions on X are equivalent:*

- The poset X satisfies DCC and has no infinite anti-chains.*
- Given a sequence x_1, x_2, \dots in X , there exists $i < j$ such that $x_i \leq x_j$.*
- The poset $\mathcal{I}(X)$ satisfies ACC.*
- Every ideal of X is finitely generated.*

The poset X is **noetherian** if the above conditions are satisfied. Where we say “ X is noetherian,” one often sees “ \leq is a well-quasi-order” in the literature. Similarly, where we say “ X satisfies DCC” one sees “ \leq is well-founded.”

Proposition 2.2. *Let X be a noetherian poset and let x_1, x_2, \dots be a sequence in X . Then there exists an infinite sequence of indices $i_1 < i_2 < \cdots$ such that $x_{i_1} \leq x_{i_2} \leq \cdots$.*

Proof. Let I be the set of indices such that $i \in I$ and $j > i$ implies that $x_i \not\leq x_j$. If I is infinite, then there is $i < i'$ with $i, i' \in I$ such that $x_i \leq x_{i'}$ by definition of noetherian and hence contradicts the definition of I . So I is finite; let i_1 be any number larger than all elements of I . Then by definition of I , we can find $x_{i_1} \leq x_{i_2} \leq \cdots$. \square

Proposition 2.3. *Let X and Y be noetherian posets. Then $X \times Y$ is noetherian.*

Proof. Let $(x_1, y_1), (x_2, y_2), \dots$ be an infinite sequence in $X \times Y$. Since X is noetherian, there exists $i_1 < i_2 < \cdots$ such that $x_{i_1} \leq x_{i_2} \leq \cdots$ (Proposition 2.2). Since Y is noetherian, there exists $j < j'$ such that $y_j \leq y_{j'}$, and hence $(x_{i_j}, y_{i_j}) \leq (x_{i_{j'}}, y_{i_{j'}})$. \square

Let X and Y be posets and let $f: X \rightarrow Y$ be a function. We say that f is **order-preserving** if $x \leq x'$ implies $f(x) \leq f(x')$, and f is **strictly order-preserving** if $x \leq x'$ is equivalent to $f(x) \leq f(x')$. If f is strictly order-preserving, then it is injective; we can thus regard X as a subposet of Y . In particular, if Y is noetherian then so is X .

Let $\mathcal{F} = \mathcal{F}(X, Y)$ be the set of all order-preserving functions $f: X \rightarrow Y$. We partially order \mathcal{F} by $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$.

Proposition 2.4. *We have the following.*

- (a) *If X is noetherian and Y satisfies ACC then \mathcal{F} satisfies ACC.*
- (b) *If \mathcal{F} satisfies ACC and X is non-empty then Y satisfies ACC.*
- (c) *If \mathcal{F} satisfies ACC and Y has two distinct comparable elements then X is noetherian.*

Proof. (a) Suppose X is noetherian and \mathcal{F} does not satisfy ACC. Let $f_1 < f_2 < \dots$ be an ascending chain in \mathcal{F} . For each i , choose $x_i \in X$ such that $f_i(x_i) < f_{i+1}(x_i)$. By passing to a subsequence we can assume $x_1 \leq x_2 \leq \dots$ (Proposition 2.2). Let $y_i = f_i(x_i)$. Then $f_i(x_i) < f_{i+1}(x_i) \leq f_{i+1}(x_{i+1})$ implies that $y_1 < y_2 < \dots$, so Y does not satisfy ACC.

(b) Now suppose \mathcal{F} satisfies ACC and X is non-empty. Then Y embeds into \mathcal{F} as the set of constant functions, and so Y satisfies ACC.

(c) Finally, suppose \mathcal{F} satisfies ACC and Y contains elements $y_1 < y_2$. Given an ideal I of X , define $\chi_I \in \mathcal{F}$ by

$$\chi_I(x) = \begin{cases} y_2 & x \in I \\ y_1 & x \notin I \end{cases}.$$

Then $I \mapsto \chi_I$ embeds $\mathcal{I}(X)$ into \mathcal{F} , and so $\mathcal{I}(X)$ satisfies ACC, and so X is noetherian. \square

Given a poset X , let X^* be the set of finite words $x_1 \cdots x_n$ with $x_i \in X$. We define $x_1 \cdots x_n \leq x'_1 \cdots x'_m$ if there exist $1 \leq i_1 < \dots < i_n \leq m$ such that $x_j \leq x'_{i_j}$ for $j = 1, \dots, n$.

Theorem 2.5 (Higman's lemma [Hi]). *If X is a noetherian poset, then so is X^* .*

Proof. Suppose that X^* is not noetherian. We use Nash-Williams' theory of minimal bad sequences [NW] to get a contradiction. A sequence w_1, w_2, \dots of elements in X^* is bad if $w_i \not\leq w_j$ for all $i < j$. We pick a bad sequence minimal in the following sense: for all $i \geq 1$, among all bad sequences beginning with w_1, \dots, w_{i-1} (this is the empty sequence for $i = 1$), $\ell(w_i)$ is as small as possible. Let $x_i \in X$ be the first element of w_i and let v_i be the subword of w_i obtained by removing x_i . By Proposition 2.2, there is an infinite sequence $i_1 < i_2 < \dots$ such that $x_{i_1} \leq x_{i_2} \leq \dots$. Then $w_1, w_2, \dots, w_{i_1-1}, v_{i_1}, v_{i_2}, \dots$ is a bad sequence because $v_{i_j} \leq w_{i_j}$ for all j , and $v_{i_j} \leq v_{i_{j'}}$ would imply that $w_{i_j} \leq w_{i_{j'}}$. It is smaller than our minimal bad sequence, so we have reached a contradiction. Hence X^* is noetherian. \square

3. FORMAL LANGUAGES

In this section we give basic definitions and results on formal languages. In particular, we define a few classes of formal languages (regular, ordered, unambiguous context-free) which will be used in this paper along with results on their generating functions. We believe that the material in §3.3 on ordered languages is new. The rest of the material is standard.

3.1. Generalities. Fix a finite set Σ (which we also call an **alphabet**). A **language** on Σ is a subset of Σ^* . Let \mathcal{L} and \mathcal{L}' be two languages. The **union** of \mathcal{L} and \mathcal{L}' , denoted $\mathcal{L} \cup \mathcal{L}'$, is their union as subsets of Σ^* . The **concatenation** of \mathcal{L} and \mathcal{L}' is $\mathcal{L}\mathcal{L}' = \{ww' \mid w \in \mathcal{L}, w' \in \mathcal{L}'\}$. The **Kleene star** of \mathcal{L} is $\mathcal{L}^* = \{w_1 \cdots w_n \mid w_1, \dots, w_n \in \mathcal{L}\}$, i.e., the submonoid (under concatenation) of Σ^* generated by \mathcal{L} .

Let Σ be an alphabet. A **norm** on Σ is a monoid homomorphism $\nu: \Sigma^* \rightarrow \mathbf{N}^I$, for some finite set I which is induced by a function $\Sigma \rightarrow I$. A norm on a language \mathcal{L} over Σ is a

function $\mathcal{L} \rightarrow \mathbf{N}^I$ which is the restriction of a norm on Σ . Let $\mathbf{t} = (t_i)_{i \in I}$ be indeterminates. The **Hilbert series** of \mathcal{L} (with respect to ν) is

$$H_{\mathcal{L}, \nu}(\mathbf{t}) = \sum_{w \in \mathcal{L}} \mathbf{t}^{\nu(w)},$$

when this makes sense (i.e., the coefficient of $\mathbf{t}^{\mathbf{n}}$ is finite for all \mathbf{n}). The coefficient of $\mathbf{t}^{\mathbf{n}}$ in $H_{\mathcal{L}, \nu}(\mathbf{t})$ is the number of words of norm \mathbf{n} . We omit ν from the notation when possible.

Every language admits a canonical norm over \mathbf{N} , namely, the length function $\ell: \mathcal{L} \rightarrow \mathbf{N}$. The coefficient of t^n in $H_{\mathcal{L}, \ell}$ counts the number of words of length n in \mathcal{L} . This series is the **generating function** of \mathcal{L} in the literature.

We say that a norm ν is **universal** if the map $\Sigma \rightarrow I$ is injective. We say that the corresponding Hilbert series is **universal**. Any Hilbert series of \mathcal{L} (under any norm) is a homomorphic image of a universal Hilbert series of \mathcal{L} .

3.2. Regular languages. The set of **regular languages** on Σ is the smallest set of languages on Σ containing the empty language and the singleton languages $\{c\}$ (for $c \in \Sigma$), and closed under finite union, concatenation, and Kleene star. A **deterministic finite-state automata** (DFA) for Σ is a tuple $(Q, T, \sigma, \mathcal{F})$ consisting of:

- A finite set Q , the set of **states**.
- A function $T: \Sigma \times Q \rightarrow Q$, the **transition table**.
- A state $\sigma \in Q$, the **initial state**.
- A subset $\mathcal{F} \subset Q$, the set of **final states**.

Fix a DFA. We write $\alpha \xrightarrow{c} \beta$ to indicate $T(c, \alpha) = \beta$. Given a word $w = w_1 \cdots w_n$, we write $\alpha \xrightarrow{w} \beta$ if there are states $\alpha = \alpha_0, \dots, \alpha_n = \beta$ such that

$$\alpha_0 \xrightarrow{w_1} \alpha_1 \xrightarrow{w_2} \cdots \xrightarrow{w_{n-1}} \alpha_{n-1} \xrightarrow{w_n} \alpha_n.$$

Intuitively, we think of $\alpha \xrightarrow{w} \beta$ as saying that the DFA starts in state α , reads the word w , and ends in the state β . We say that the DFA **accepts** the word w if $\sigma \xrightarrow{w} \tau$ with τ a final state. The set of accepted words is called the language **recognized** by the DFA. The following results are standard (see [HU, Ch. 2] and [St1, Theorem 4.7.2], for example).

Theorem 3.2.1. *A language is regular if and only if it is recognized by some DFA.*

Theorem 3.2.2. *If \mathcal{L} is regular then $H_{\mathcal{L}, \nu}(\mathbf{t})$ is a rational function of \mathbf{t} , for any norm ν .*

3.3. Ordered languages. The set of **ordered languages** on Σ is the smallest set of languages on Σ that contains the singleton languages and the languages Π^* , for $\Pi \subseteq \Sigma$, and that is closed under finite union and concatenation. We have not found this class of languages considered in the literature.

A DFA is **ordered** if the following condition holds: if α and β are states and there exists words u and w with $\alpha \xrightarrow{u} \beta$ and $\beta \xrightarrow{w} \alpha$, then $\alpha = \beta$. The states of an ordered DFA admit a natural partial order by $\alpha \leq \beta$ if there exists a word w with $\alpha \xrightarrow{w} \beta$.

We prove two theorems about ordered languages. The first is the following.

Theorem 3.3.1. *A language is ordered if and only if it is recognized by some ordered DFA.*

Proof. If \mathcal{L} is an ordered language then we can write $\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n$, where each \mathcal{L}_i is a concatenation of singleton languages and languages of the form Π^* with $\Pi \subset \Sigma$. By Lemmas 3.3.4 and 3.3.5 below, each \mathcal{L}_i is recognized by an ordered DFA. Thus by Lemma 3.3.2 below, \mathcal{L} is recognized by an ordered DFA. The converse is proved in Lemma 3.3.6. \square

Lemma 3.3.2. *Suppose that \mathcal{L} and \mathcal{L}' are languages recognized by ordered DFA's. Then $\mathcal{L} \cup \mathcal{L}'$ is recognized by an ordered DFA.*

Proof. Suppose $(Q, T, \sigma, \mathcal{F})$ is an ordered DFA recognizing \mathcal{L} and $(Q', T', \sigma', \mathcal{F}')$ is an ordered DFA recognizing \mathcal{L}' . Let $Q'' = Q \times Q'$. Define $T'' : \Sigma \times Q'' \rightarrow Q''$ by

$$T''(c, (\alpha, \alpha')) = (T(c, \alpha), T'(c, \alpha')).$$

Let $\sigma'' = (\sigma, \sigma')$, and $\mathcal{F}'' = (\mathcal{F} \times Q') \cup (Q \times \mathcal{F}')$. Then $(Q'', T'', \sigma'', \mathcal{F}'')$ is a DFA recognizing $\mathcal{L} \cup \mathcal{L}'$. Since $(\alpha, \alpha') \xrightarrow{w} (\beta, \beta')$ if and only if $\alpha \xrightarrow{w} \beta$ and $\alpha' \xrightarrow{w} \beta'$, this DFA is ordered. \square

Lemma 3.3.3. *If \mathcal{L} is a non-empty language recognized by an ordered DFA then we can write \mathcal{L} as a finite union of languages recognized by an ordered DFA with a single final state.*

Proof. Suppose \mathcal{L} is recognized by the ordered DFA $(Q, T, \sigma, \mathcal{F})$. Enumerate \mathcal{F} as $\{\tau_1, \dots, \tau_n\}$. Let \mathcal{L}_i be the set of words $w \in \mathcal{L}$ for which $\sigma \xrightarrow{w} \tau_i$. Then \mathcal{L} is clearly the union of the \mathcal{L}_i . Moreover, \mathcal{L}_i is recognized by the ordered DFA $(Q, T, \sigma, \{\tau_i\})$. \square

Lemma 3.3.4. *Let \mathcal{L} be a language recognized by an ordered DFA and let $\mathcal{L}' = \{c\}$ be a singleton language. Then the concatenation $\mathcal{L}\mathcal{L}'$ is recognized by an ordered DFA.*

Proof. Suppose first that \mathcal{L} is recognized by the ordered DFA $(Q, T, \sigma, \mathcal{F})$, where $\mathcal{F} = \{\tau\}$ has a single element. If $T(c, \tau) = \tau$, then nothing needs to be changed since $\mathcal{L} = \mathcal{L}\mathcal{L}'$. Suppose then that $T(c, \tau) = \rho \neq \tau$. Let $Q' = Q \amalg \{\tau'\}$, and define a transition function T' as follows. First, $T'(b, \alpha) = T(b, \alpha)$ if $\alpha \in Q$, unless $\alpha = \tau$ and $b = c$. We define $T'(c, \tau) = \tau'$. Finally, we define $T'(b, \tau') = \rho$, for any b . Then $(Q', T', \sigma, \{\tau'\})$ is an ordered DFA recognizing $\mathcal{L}\mathcal{L}'$. For the general case, use Lemmas 3.3.2 and 3.3.3. \square

Lemma 3.3.5. *Let \mathcal{L} be a language recognized by an ordered DFA and let $\mathcal{L}' = \Pi^*$ for some subset Π of Σ . Then the concatenation $\mathcal{L}\mathcal{L}'$ is recognized by an ordered DFA.*

Proof. Suppose first that \mathcal{L} is accepted by the ordered DFA $(Q, T, \sigma, \mathcal{F})$, where $\mathcal{F} = \{\tau\}$. Let $Q' = Q \amalg \{\tau', \rho\}$, and define a transition function T' as follows. First, $T'(c, \alpha) = T(c, \alpha)$ for $\alpha \in Q \setminus \{\tau\}$. Let Δ be the set of elements $c \in \Sigma$ such that $T(c, \tau) = \tau$. We define

$$T'(c, \tau) = \begin{cases} \tau & \text{if } c \in \Delta \\ \tau' & \text{if } c \in \Pi \setminus \Delta \\ \rho & \text{if } c \notin \Pi \cup \Delta \end{cases}, \quad T'(c, \tau') = \begin{cases} \tau' & \text{if } c \in \Pi \\ \rho & \text{if } c \notin \Pi \end{cases}.$$

Finally, we define $T'(c, \rho) = \rho$ for all $c \in \Sigma$. One easily verifies that $(Q', T', \sigma, \{\tau, \tau'\})$ is an ordered DFA accepting $\mathcal{L}\mathcal{L}'$. The deduction of the general case from this special case proceeds exactly as the corresponding argument in the proof of Lemma 3.3.4. \square

Lemma 3.3.6. *The language recognized by an ordered automata is ordered.*

Proof. Let $(Q, T, \sigma, \mathcal{F})$ be an ordered automata, and let \mathcal{L} be the language it recognizes. We show that \mathcal{L} is ordered, following the proof of [HU, Thm. 2.4]. Enumerate the states Q as $\{\alpha_1, \dots, \alpha_s\}$ in such a way that if $\alpha_i \leq \alpha_j$ then $i \leq j$. Let $\mathcal{L}_{i,j}^k$ be the set of words $w = w_1 \cdots w_n$ such that

$$\alpha_i = \beta_0 \xrightarrow{w_1} \beta_1 \xrightarrow{w_2} \cdots \xrightarrow{w_n} \beta_n = \alpha_j \quad \text{with} \quad \beta_1, \dots, \beta_{n-1} \in \{\alpha_1, \dots, \alpha_k\}.$$

In other words, $w \in \mathcal{L}_{i,j}^k$ if it induces a transition from α_i to α_j via intermediate states of the form α_ℓ with $\ell \leq k$. We prove by induction on k that each $\mathcal{L}_{i,j}^k$ is an ordered language.

To begin with, when $k = 0$ no intermediate states are allowed (i.e., $n = 1$), and so $\mathcal{L}_{i,j}^k$ is a subset of Σ , and therefore an ordered language. For $k \geq 1$, we have

$$\mathcal{L}_{i,j}^k = \mathcal{L}_{i,k}^{k-1} (\mathcal{L}_{k,k}^{k-1})^* \mathcal{L}_{k,j}^{k-1} \cup \mathcal{L}_{i,j}^{k-1}.$$

Since there is no way to transition from α_k to α_i with $i < k$, any word in $\mathcal{L}_{k,k}^{k-1}$ must have length 1. Thus $\mathcal{L}_{k,k}^{k-1}$ is a subset of Σ , and so $(\mathcal{L}_{k,k}^{k-1})^*$ is an ordered language. The above formula then establishes inductively that $\mathcal{L}_{i,j}^k$ is ordered for all k .

Let $\mathcal{F} = \{\alpha_j\}_{j \in J}$ be the set of final states, and let $\sigma = \alpha_i$ be the initial state. Then $\mathcal{L} = \bigcup_{j \in J} \mathcal{L}_{i,j}^s$, and is therefore an ordered language. \square

The following is our second main result about ordered languages. To state it, we need one piece of terminology: we say that a subset Π of Σ is **repeatable** with respect to some language \mathcal{L} if there exist w and w' in Σ^* such that $w\Pi^*w' \subset \mathcal{L}$.

Theorem 3.3.7. *Suppose \mathcal{L} is an ordered language with a norm ν . Let Π_1, \dots, Π_r be the repeatable subsets of Σ , and let $\lambda_i = \sum_{c \in \Pi_i} \mathbf{t}^{\nu(c)}$. Then $H_{\mathcal{L}}(\mathbf{t}) = f(\mathbf{t})/g(\mathbf{t})$ where $f(\mathbf{t})$ and $g(\mathbf{t})$ are polynomials, and $g(\mathbf{t})$ factors as $\prod_{i=1}^r (1 - \lambda_i)^{e_i}$ where $e_i \geq 0$.*

Proof. Choose an ordered DFA recognizing \mathcal{L} . We can assume that every state is \geq the initial state. Call a state α **prefinal** if there exists a final state τ such that $\alpha \leq \tau$. Enumerate the states as $\alpha_1, \dots, \alpha_s$ such that the following conditions hold: (1) α_1 is the initial state; (2) if $\alpha_i \leq \alpha_j$ then $i \leq j$; and (3) there exists n such that $\{\alpha_1, \dots, \alpha_n\}$ is the set of prefinal states. Let A be the $s \times s$ matrix $A_{i,j} = \sum_{\substack{\alpha_i \xrightarrow{w} \alpha_j \\ \ell(w)=1}} \mathbf{t}^{\nu(w)}$; then $(A^k)_{i,j} = \sum_{\substack{\alpha_i \xrightarrow{w} \alpha_j \\ \ell(w)=k}} \mathbf{t}^{\nu(w)}$. Then A

is upper-triangular and the i th diagonal entry is $\sum_{c \in \Pi} \mathbf{t}^{\nu(c)}$, where Π is the set of letters c which induce a transition from α_i to itself. If α_i is prefinal then Π is one of the Π_j 's, and so the diagonal entry at (i, i) is one of the λ_j 's. If $\alpha_{t_1}, \dots, \alpha_{t_s}$ are the final states then

$$H_{\mathcal{L}}(\mathbf{t}) = \sum_{i=0}^s \sum_{k \geq 0} (A^k)_{1,t_i} = \sum_{i=1}^s \frac{(-1)^{1+t_i} \det(1 - A : t_i, 1)}{\det(1 - A)},$$

where the notation $(1 - A : t_i, 1)$ means that we remove the t_i th row and first column from $1 - A$. It is clear that this is of the stated form. \square

Corollary 3.3.8. *Suppose \mathcal{L} is an ordered language. Then $H_{\mathcal{L},\ell}(t)$ can be written in the form $f(t)/g(t)$ where $f(t)$ and $g(t)$ are polynomials, and $g(t)$ factors as $\prod_{a=1}^r (1 - at)^{e(a)}$ where $e(a) \geq 0$ and r is the cardinality of the largest repeatable subset of Σ with respect to \mathcal{L} .*

3.4. Unambiguous context-free languages. Let Σ be a finite alphabet, which we also call **terminal** symbols. Let N be another finite set, disjoint from Σ , which we call the **non-terminal** symbols. A **production rule** is $n \rightarrow P(n)$ where $n \in N$ and $P(n)$ is a word in $\Sigma \cup N$. A **context-free grammar** is a tuple $(\Sigma, N, \mathcal{P}, n_0)$, where Σ and N are as above, \mathcal{P} is a finite set of production rules, and n_0 is a distinguished element of N . Given such a grammar, a word w in Σ is a **valid n -expression** if there is a production rule $n \rightarrow P(n)$, where $P(n) = c_1 \cdots c_r$, and a decomposition $w = w_1 \cdots w_r$ (with $r > 1$), such that w_i is a valid c_i -expression if c_i is non-terminal, and $w_i = c_i$ if c_i is terminal. The language **recognized** by a grammar is the set of valid n_0 -expressions, and a language of this form is called a **context-free language**. A grammar is **unambiguous** if for each valid n -expression w the production rule $n \rightarrow P(n)$ and decomposition of w above is unique. An **unambiguous**

context-free language is one defined by such a grammar. See [St2, Definition 6.6.4] for an alternative description.

Theorem 3.4.1 (Chomsky–Schützenberger [CS]). *Let \mathcal{L} be an unambiguous context-free language equipped with a norm ν . Then $H_{\mathcal{L},\nu}(\mathbf{t})$ is an algebraic function.*

Proof. Let \mathcal{L} be a context-free language and pick a grammar that recognizes \mathcal{L} . Consider the modified Hilbert series $\sum_{w \in \mathcal{L}} a_w \mathbf{t}^{\nu(w)}$ where ν is the universal norm and a_w is the number of ways that w can be built using the production rules. This is an algebraic function [St2, Theorem 6.6.10]. If \mathcal{L} is unambiguous, then we can pick a grammar so that $a_w = 1$ for all $w \in \mathcal{L}$, and we obtain the usual Hilbert series. \square

4. REPRESENTATIONS OF CATEGORIES

This section introduces the main topic of this paper: representations of categories. Our goal is to lay out the main definitions and basic properties of representations and functors between categories of representations and to state some criteria for representations to be noetherian. More specifically, definitions are given in §4.1, properties of functors between categories are in §4.2, and tensor products of representations are discussed in §4.3.

4.1. Basic definitions and results. Let \mathcal{C} be a category. If \mathcal{C} is essentially small, we denote by $|\mathcal{C}|$ the set of isomorphism classes in \mathcal{C} . For an object x of \mathcal{C} , we let \mathcal{C}_x be the category of morphisms from x ; thus the objects of \mathcal{C}_x are morphisms $x \rightarrow y$ (with y variable), and the morphisms in \mathcal{C}_x are the obvious commutative triangles. We say that \mathcal{C} is **directed** if every self-map in \mathcal{C} is the identity. If \mathcal{C} is directed then so is \mathcal{C}_x , for any x . If \mathcal{C} is essentially small and directed, then $|\mathcal{C}|$ is naturally a poset by defining $x \leq y$ if there exists a morphism $x \rightarrow y$. We say that \mathcal{C} is Hom-finite if all Hom sets are finite.

Fix a nonzero ring \mathbf{k} (not necessarily commutative) and let $\text{Mod}_{\mathbf{k}}$ denote the category of left \mathbf{k} -modules. A **representation** of \mathcal{C} (or a **\mathcal{C} -module**) over \mathbf{k} is a functor $\mathcal{C} \rightarrow \text{Mod}_{\mathbf{k}}$. A map of \mathcal{C} -modules is a natural transformation. We write $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ for the category of representations, which is abelian. Let M be a representation of \mathcal{C} . By an **element** of M we mean an element of $M(x)$ for some object x of \mathcal{C} . Given any set S of elements of M , there is a smallest subrepresentation of M containing S ; we call this the subrepresentation **generated by** S . We say that M is **finitely generated** if it is generated by a finite set of elements. For a morphism $f: x \rightarrow y$ in \mathcal{C} , we typically write f_* for the given map of \mathbf{k} -modules $M(x) \rightarrow M(y)$.

Let x be an object of \mathcal{C} . Define a representation P_x of \mathcal{C} by $P_x(y) = \mathbf{k}[\text{Hom}(x, y)]$, i.e., $P_x(y)$ is the free left \mathbf{k} -module with basis $\text{Hom}(x, y)$. For a morphism $f: x \rightarrow y$, we write e_f for the corresponding element of $P_x(y)$. If M is another representation then $\text{Hom}(P_x, M) = M(x)$. This shows that $\text{Hom}(P_x, -)$ is an exact functor, and so P_x is a projective object of $\text{Rep}_{\mathbf{k}}(\mathcal{C})$. We call it the **principal projective** at x . A \mathcal{C} -module is finitely generated if and only if it is a quotient of a finite direct sum of principal projective objects.

An object of $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is **noetherian** if every ascending chain of subobjects stabilizes; this is equivalent to every subrepresentation being finitely generated. The category $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is **noetherian** if every finitely generated object in it is.

Proposition 4.1.1. *The category $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian if and only if every principal projective is noetherian.*

Proof. Obviously, if $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian then so is every principal projective. Conversely, suppose every principal projective is noetherian. Let M be a finitely generated object. Then M is a quotient of a finite direct sum P of principal projectives. Since noetherianity is preserved under finite direct sums, P is noetherian. And since noetherianity descends to quotients, M is noetherian. This completes the proof. \square

4.2. Pullback functors. Let $\Phi: \mathcal{C} \rightarrow \mathcal{C}'$ be a functor. There is then a pullback functor on representations $\Phi^*: \text{Rep}_{\mathbf{k}}(\mathcal{C}') \rightarrow \text{Rep}_{\mathbf{k}}(\mathcal{C})$. In this section, we study how Φ^* interacts with finiteness conditions. The following definition is of central importance:

Definition 4.2.1. We say that Φ satisfies **property (F)** if the following condition holds: given any object x of \mathcal{C}' there exist finitely many objects y_1, \dots, y_n of \mathcal{C} and morphisms $f_i: x \rightarrow \Phi(y_i)$ in \mathcal{C}' such that for any object y of \mathcal{C} and any morphism $f: x \rightarrow \Phi(y)$ in \mathcal{C}' , there exists a morphism $g: y_i \rightarrow y$ in \mathcal{C} such that $f = \Phi(g) \circ f_i$. \square

The following proposition is the motivation for introducing property (F).

Proposition 4.2.2. *Suppose $\Phi: \mathcal{C} \rightarrow \mathcal{C}'$ satisfies property (F). Then Φ^* takes finitely generated objects of $\text{Rep}_{\mathbf{k}}(\mathcal{C}')$ to finitely generated objects of $\text{Rep}_{\mathbf{k}}(\mathcal{C})$.*

Proof. It suffices to show that Φ^* takes principal projectives to finitely generated representations. Thus let P_x be the principal projective of $\text{Rep}_{\mathbf{k}}(\mathcal{C}')$ at an object x . Note that $\Phi^*(P_x)(y)$ has for a basis the elements e_f for $f \in \text{Hom}_{\mathcal{C}'}(x, \Phi(y))$. Let $f_i: x \rightarrow \Phi(y_i)$ be as in the definition of property (F). Then the e_{f_i} generate $\Phi^*(P_x)$. \square

Proposition 4.2.3. *Suppose that $\Phi: \mathcal{C} \rightarrow \mathcal{C}'$ is an essentially surjective functor. Let M be an object of $\text{Rep}_{\mathbf{k}}(\mathcal{C}')$ such that $\Phi^*(M)$ is finitely generated (resp. noetherian). Then M is finitely generated (resp. noetherian).*

Proof. Let S be a set of elements of $\Phi^*(M)$. Let S' be the corresponding set of elements of M . (Thus if S contains $m \in \Phi^*(M)(y)$ then S' contains $m \in M(\Phi(y))$.) If N is a subrepresentation of M containing S' then $\Phi^*(N)$ is a subrepresentation of $\Phi^*(M)$ containing S . It follows that if N (resp. N') is the subrepresentation of M (resp. $\Phi^*(M)$) generated by S' (resp. S), then $N' \subset \Phi^*(N)$. Thus if S generates $\Phi^*(M)$ then $\Phi^*(N) = \Phi^*(M)$, which implies $N = M$ since Φ is essentially surjective, i.e., S generates M . In particular, if $\Phi^*(M)$ is finitely generated then so is M .

Now suppose that $\Phi^*(M)$ is noetherian. Given a subrepresentation N of M , we obtain a subrepresentation $\Phi^*(N)$ of $\Phi^*(M)$. Since $\Phi^*(M)$ is noetherian, it follows that $\Phi^*(N)$ is finitely generated. Thus N is finitely generated, and so M is noetherian. \square

Proposition 4.2.4. *Let $\Phi: \mathcal{C} \rightarrow \mathcal{C}'$ be an essentially surjective functor satisfying property (F) and suppose $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian. Then $\text{Rep}_{\mathbf{k}}(\mathcal{C}')$ is noetherian.*

Proof. Let M be a finitely generated \mathcal{C}' -module. Then $\Phi^*(M)$ is finitely generated by Proposition 4.2.2, and therefore noetherian, and so M is noetherian by Proposition 4.2.3. \square

The next two results follow from the definitions, so we omit their proofs.

Proposition 4.2.5. *Suppose $\Phi: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ and $\Psi: \mathcal{C}_2 \rightarrow \mathcal{C}_3$ satisfy property (F). Then the composition $\Psi \circ \Phi$ satisfies property (F).*

Proposition 4.2.6. *Let $\Phi: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ and $\Psi: \mathcal{C}_2 \rightarrow \mathcal{C}_3$ be functors. Suppose that Φ is essentially surjective and $\Psi \circ \Phi$ satisfies property (F). Then Ψ satisfies property (F).*

4.3. Tensor products. Suppose \mathbf{k} is a commutative ring. Given representations M and N of categories \mathcal{C} and \mathcal{D} over \mathbf{k} , we define their **external tensor product**, denoted $M \boxtimes N$, to be the representation of $\mathcal{C} \times \mathcal{D}$ given by $(x, y) \mapsto M(x) \otimes_{\mathbf{k}} N(y)$. One easily sees that if M and N are finitely generated then so is $M \boxtimes N$. If $\mathcal{C} = \mathcal{D}$ then we define the **pointwise tensor product** of M and N , denoted $M \odot N$, to be the representation of \mathcal{C} given by $x \mapsto M(x) \otimes_{\mathbf{k}} N(x)$. The two tensor products are related by the identity $M \odot N = \Delta^*(M \boxtimes N)$, where $\Delta: \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ is the diagonal functor.

A category \mathcal{C} satisfies **property (F)** if the diagonal functor $\Delta: \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ satisfies property (F). The following result follows from the above discussion and Proposition 4.2.2.

Proposition 4.3.1. *Suppose \mathcal{C} satisfies property (F). Then for any commutative ring \mathbf{k} , the pointwise tensor product of finitely generated representations is finitely generated,*

5. NOETHERIANITY AND GRÖBNER CATEGORIES

This section introduces another main topic of this paper: Gröbner bases for representations of categories. Definitions and basic properties of Gröbner bases and initial submodules are given in §5.1 and we state a Gröbner-theoretic approach to proving the noetherian property. In §5.2, we introduce the notions of Gröbner and quasi-Gröbner categories, which are those categories for which the formalism of the first section can be applied.

5.1. Gröbner bases. Let \mathcal{C} be an essentially small category and let **Set** be the category of sets. Fix a functor $S: \mathcal{C} \rightarrow \mathbf{Set}$, and let $P = \mathbf{k}[S]$, i.e., $P(x)$ is the free \mathbf{k} -module on the set $S(x)$. The purpose of this section is to develop a theory of Gröbner bases for P , and use this theory to give a combinatorial criterion for P to be noetherian.

We begin by defining a partially ordered set $|S|$ associated to S , which is one of the main combinatorial objects of interest. A subfunctor of S is **principal** if it is generated by a single element. (Here we use “element” and “generated” as with representations of \mathcal{C} .) We define $|S|$ to be the set of principal subfunctors of S , partially ordered by reverse inclusion. We can describe this poset more concretely as follows, at least when \mathcal{C} is small. Set $\tilde{S} = \bigcup_{x \in \mathcal{C}} S(x)$. Given $f \in S(x)$ and $g \in S(y)$, define $f \leq g$ if there exists $h: x \rightarrow y$ with $h_*(f) = g$. Define an equivalence relation on \tilde{S} by $f \sim g$ if $f \leq g$ and $g \leq f$. The poset $|S|$ is the quotient of \tilde{S} by \sim , with the induced partial order.

Given $f \in S(x)$, we write e_f for the corresponding element of $P(x)$. An element of P is a **monomial** if it is of the form λe_f for some $\lambda \in \mathbf{k}$ and $f \in S(x)$. A submodule M of P is **monomial** if $M(x)$ is spanned by the monomials it contains, for all objects x .

We now classify the monomial subrepresentations of P in terms of $|S|$. Given $f \in \tilde{S}$, let $I_M(f) = \{\lambda \in \mathbf{k} \mid \lambda e_f \in M\}$. Then $I_M(f)$ is an ideal of \mathbf{k} . If $f \leq g$ then $I_M(f) \subset I_M(g)$. In particular, $I_M(f) = I_M(g)$ if $f \sim g$. Let $\mathcal{I}(\mathbf{k})$ be the poset of left-ideals in \mathbf{k} and let $\mathcal{M}(P)$ be the poset of monomial subrepresentations of P . Given $M \in \mathcal{M}(P)$, we have constructed an order-preserving function $I_M: |S| \rightarrow \mathcal{I}(\mathbf{k})$, i.e., an element of $\mathcal{F}(|S|, \mathcal{I}(\mathbf{k}))$ (see §2).

Proposition 5.1.1. *The map $I: \mathcal{M}(P) \rightarrow \mathcal{F}(|S|, \mathcal{I}(\mathbf{k}))$ is an isomorphism of posets.*

Proof. Suppose that for every $f \in |S|$ we have a left-ideal $I(f)$ of \mathbf{k} such that for $f \leq g$ we have $I(f) \subseteq I(g)$. We then define a monomial submodule $M \subseteq P$ by $M(x) = \sum_{f \in S(x)} I(f) e_f$. This defines a function $\mathcal{F}(|S|, \mathcal{I}(\mathbf{k})) \rightarrow \mathcal{M}(P)$ inverse to I . It is clear from the constructions that I and its inverse are order-preserving, and so I is an isomorphism of posets. \square

Corollary 5.1.2. *The following are equivalent (assuming P is non-zero).*

- (a) *Every monomial submodule of P is finitely generated.*
- (b) *The poset $\mathcal{M}(P)$ satisfies ACC.*
- (c) *The poset $|S|$ is noetherian and \mathbf{k} is left-noetherian.*

Proof. The equivalence of (a) and (b) is standard, while the equivalence of (b) and (c) follows from Propositions 5.1.1 and 2.4. (Note: $|S|$ is non-empty if $P \neq 0$, and $\mathcal{I}(\mathbf{k})$ contains two distinct comparable elements, namely the zero and unit ideals.) \square

To connect arbitrary submodules of P to monomial submodules, we need a theory of monomial orders. Let \mathbf{WO} be the category of well-ordered sets and strictly order-preserving functions. There is a forgetful functor $\mathbf{WO} \rightarrow \mathbf{Set}$. An **ordering** on S is a lifting of S to \mathbf{WO} . More concretely, an ordering on S is a choice of well-order on $S(x)$, for each $x \in \mathcal{C}$, such that for every morphism $x \rightarrow y$ in \mathcal{C} the induced map $S(x) \rightarrow S(y)$ is strictly order-preserving. We write \preceq for an ordering; S is **orderable** if it admits an ordering.

Suppose \preceq is an ordering on S . Given non-zero $\alpha = \sum_{f \in S(x)} \lambda_f e_f$ in $P(x)$, we define the **initial term** of α , denoted $\text{init}(\alpha)$, to be $\lambda_g e_g$, where $g = \max_{\preceq} \{f \mid \lambda_f \neq 0\}$. The **initial variable** of α , denoted $\text{init}_0(\alpha)$, is g . Now let M be a subrepresentation of P . We define the **initial subrepresentation** of M , denoted $\text{init}(M)$, as follows: $\text{init}(M)(x)$ is the \mathbf{k} -span of the elements $\text{init}(\alpha)$ for non-zero $\alpha \in M(x)$. The name is justified by the following result.

Proposition 5.1.3. *Notation as above, $\text{init}(M)$ is a subrepresentation of M .*

Proof. Let $\alpha = \sum_{i=1}^n \lambda_i e_{f_i}$ be an element of $M(x)$, ordered so that $f_i \prec f_1$ for all $i > 1$. Thus $\text{init}(\alpha) = \lambda_1 e_{f_1}$. Let $g: x \rightarrow y$ be a morphism. Then $g_*(\alpha) = \sum_{i=1}^n \lambda_i e_{g_*(f_i)}$. Since $g_*: S(x) \rightarrow S(y)$ is strictly order-preserving, we have $g_*(f_i) \prec g_*(f_1)$ for all $i > 1$. Thus $\text{init}(g_*(\alpha)) = \lambda_1 e_{g_*(f_1)}$, or, in other words, $\text{init}(g_*(\alpha)) = g_*(\text{init}(\alpha))$. This shows that g_* maps $\text{init}(M)(x)$ into $\text{init}(M)(y)$, which completes the proof. \square

Proposition 5.1.4. *If $N \subseteq M$ are submodules of P and $\text{init}(N) = \text{init}(M)$, then $M = N$.*

Proof. Assume that $M(x) \neq N(x)$ for $x \in \mathcal{C}$. Let $K \subset S(x)$ be the set of all elements which appear as the initial variable of some element of $M(x) \setminus N(x)$. Then $K \neq \emptyset$, so has a minimal element f with respect to \preceq . Pick $\alpha \in M(x) \setminus N(x)$ with $\text{init}_0(\alpha) = f$. By assumption, there exists $\beta \in N(x)$ with $\text{init}(\alpha) = \text{init}(\beta)$. But then $\alpha - \beta \in M(x) \setminus N(x)$, and $\text{init}_0(\alpha - \beta) \prec \text{init}_0(\alpha)$, a contradiction. Thus $M = N$. \square

Let M be a subrepresentation of P . A set \mathfrak{G} of elements of M is a **Gröbner basis** of M if $\{\text{init}(\alpha) \mid \alpha \in \mathfrak{G}\}$ generates $\text{init}(M)$. Note that M has a finite Gröbner basis if and only if $\text{init}(M)$ is finitely generated. As usual, we have:

Proposition 5.1.5. *Let \mathfrak{G} be a Gröbner basis of M . Then \mathfrak{G} generates M .*

Proof. Let $N \subseteq M$ be the subrepresentation generated by \mathfrak{G} . Then $\text{init}(N)$ contains $\text{init}(\alpha)$ for all $\alpha \in \mathfrak{G}$, and so $\text{init}(N) = \text{init}(M)$. Thus $M = N$ by Proposition 5.1.4. \square

We now come to our main result, which follows from the above discussion:

Theorem 5.1.6. *Suppose \mathbf{k} is left-noetherian, S is orderable, and $|S|$ is noetherian. Then every subrepresentation of P has a finite Gröbner basis. In particular, P is a noetherian object of $\text{Rep}_{\mathbf{k}}(\mathcal{C})$.*

Remark 5.1.7. We have not discussed the important topic of algorithms for Gröbner bases. See [Ei, Chapter 15] for an exposition of this theory for modules over polynomial rings. Two important results are Buchberger’s criterion using S-pairs for determining if a set of elements is a Gröbner basis, and Schreyer’s extension of this idea to calculate free resolutions. These ideas can be extended to our settings and will be developed in future work. \square

5.2. Gröbner categories. Let \mathcal{C} be an essentially small category. For an object x , let $S_x: \mathcal{C} \rightarrow \mathbf{Set}$ be the functor given by $S_x(y) = \text{Hom}(x, y)$. Note that $P_x = \mathbf{k}[S_x]$.

Definition 5.2.1. We say that \mathcal{C} is **Gröbner** if, for all objects x , the functor S_x is orderable and the poset $|S_x|$ is noetherian. We say that \mathcal{C} is **quasi-Gröbner** if there exists a Gröbner category \mathcal{C}' and an essentially surjective functor $\mathcal{C}' \rightarrow \mathcal{C}$ satisfying property (F). \square

The following theorem is one of the two main theoretical results of this paper. It connects the purely combinatorial condition “(quasi-)Gröbner” with the algebraic condition “noetherian” for representations.

Theorem 5.2.2. *Let \mathcal{C} be a quasi-Gröbner category. Then for any left-noetherian ring \mathbf{k} , the category $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian.*

Proof. First suppose that \mathcal{C} is a Gröbner category. Then every principal projective of $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian, by Theorem 5.1.6, and so $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian by Proposition 4.1.1.

Now suppose that \mathcal{C} is quasi-Gröbner, and let $\Phi: \mathcal{C}' \rightarrow \mathcal{C}$ be an essentially surjective functor satisfying property (F), with \mathcal{C}' Gröbner. Then $\text{Rep}_{\mathbf{k}}(\mathcal{C}')$ is noetherian, by the previous paragraph, and so $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian by Proposition 4.2.4. \square

Remark 5.2.3. If the functor S_x is orderable, then the group $\text{Aut}(x) = S_x(x)$ admits a well-order compatible with the group operation, and is therefore trivial. Thus, in a Gröbner category, there are no non-trivial automorphisms. \square

The definition of Gröbner is rather abstract. We now give a more concrete reformulation when \mathcal{C} is a directed category, which is the version we will apply in practice. Let x be an object of \mathcal{C} . An **admissible order** on $|\mathcal{C}_x|$ is a well-order \preceq satisfying the following additional condition: given $f, f' \in \text{Hom}(x, y)$ with $f \prec f'$ and $g \in \text{Hom}(y, z)$, we have $gf \prec gf'$.

Proposition 5.2.4. *Let \mathcal{C} be a directed category. Then \mathcal{C} is Gröbner if and only if for all x the poset $|\mathcal{C}_x|$ is noetherian and admits an admissible order.*

Proof. It suffices to treat the case where \mathcal{C} is small. Let x be an object of \mathcal{C} . The sets $\text{Ob}(\mathcal{C}_x)$ and \tilde{S}_x are equal: both are the set of all morphisms $x \rightarrow y$. In $|\mathcal{C}_x|$, two morphisms f and g are identified if $g = hf$ for some isomorphism h . In $|S_x|$, two morphisms f and g are identified if there are morphisms h and h' such that $g = hf$ and $f = h'g$. Since \mathcal{C} is directed, h and h' are necessarily isomorphisms. So $|\mathcal{C}_x|$ and $|S_x|$ are the same quotient of $\text{Ob}(\mathcal{C}_x) = \tilde{S}_x$. The orders on each are defined in the same way, and thus the two are isomorphic posets. Thus $|\mathcal{C}_x|$ is noetherian if and only if $|S_x|$ is.

Now let \preceq be an admissible order on $|\mathcal{C}_x|$. Since \mathcal{C} is directed, the natural map $S_x(y) \rightarrow |\mathcal{C}_x|$ is an injection. We define a well-ordering on $S_x(y)$ by restricting \preceq to it. One readily verifies that this defines an ordering of S_x .

Finally, suppose that \preceq is an ordering on S_x . Let \mathcal{C}_0 be a set of isomorphism class representatives for \mathcal{C} . Since \mathcal{C} is directed, the natural map $\coprod_{y \in \mathcal{C}_0} S_x(y) \rightarrow |\mathcal{C}_x|$ is a bijection. Choose an arbitrary well-ordering \preceq on \mathcal{C}_0 . Define a well-order \preceq on $\coprod_{y \in \mathcal{C}_0} S_x(y)$ as follows.

If $f: x \rightarrow y$ and $g: x \rightarrow z$ then $f \preceq g$ if $y \prec z$, or $y = z$ and $f \prec g$ as elements of $S_x(y)$. One easily verifies that this induces an admissible order on $|\mathcal{C}_x|$. \square

The following two results follow easily from the definitions, so we omit their proofs.

Proposition 5.2.5. *The cartesian product of finitely many (quasi-)Gröbner categories is (quasi-)Gröbner.*

Proposition 5.2.6. *Suppose that $\Phi: \mathcal{C}' \rightarrow \mathcal{C}$ is an essentially surjective functor satisfying property (F) and \mathcal{C}' is quasi-Gröbner. Then \mathcal{C} is quasi-Gröbner.*

5.3. Other properties. We end this section with some useful combinatorial properties.

Definition 5.3.1. A functor $\Phi: \mathcal{C}' \rightarrow \mathcal{C}$ satisfies **property (S)** if the following condition holds: if $f: x \rightarrow y$ and $g: x \rightarrow z$ are morphisms in \mathcal{C}' and there exists $\tilde{h}: \Phi(y) \rightarrow \Phi(z)$ such that $\Phi(g) = \tilde{h}\Phi(f)$ then there exists $h: y \rightarrow z$ such that $g = hf$. A subcategory $\mathcal{C}' \subset \mathcal{C}$ satisfies property (S) if the inclusion functor does. \square

Proposition 5.3.2. *Let $\Phi: \mathcal{C}' \rightarrow \mathcal{C}$ be a faithful functor satisfying property (S) and suppose \mathcal{C} is Gröbner. Then \mathcal{C}' is Gröbner.*

We end with a combinatorial criterion for $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ to have Krull dimension 0. Assume that \mathbf{k} is a field. Given $M \in \text{Rep}_{\mathbf{k}}(\mathcal{C})$, define $M^\vee \in \text{Rep}_{\mathbf{k}}(\mathcal{C}^{\text{op}})$ by $M^\vee(x) = \text{Hom}_{\mathbf{k}}(M(x), \mathbf{k})$.

Definition 5.3.3. We say that \mathcal{C} satisfies **property (D)** if the following condition holds: for every object x of \mathcal{C} there exist finitely many objects $\{y_i\}_{i \in I}$ and finite subsets S_i of $\text{Hom}(x, y_i)$ such that if $f: x \rightarrow z$ is any morphism then there exists $i \in I$ and a morphism $g: z \rightarrow y_i$ such that $g_*^{-1}(S_i) = \{f\}$, where g_* is the map $\text{Hom}(x, z) \rightarrow \text{Hom}(x, y_i)$. \square

Proposition 5.3.4. *Suppose that \mathcal{C} is Hom-finite and satisfies property (D), and \mathbf{k} is a field. Then for any object x , the representation P_x^\vee of \mathcal{C}^{op} is finitely generated.*

Proof. The space $P_x(y)$ has a basis $\{e_f\}$. We let e_f^* be the dual basis of $P_x^\vee(y)$. We note that the e_f^* generate P_x^\vee as a representation. Let y_i and S_i be as in the definition of property (D). For $i \in I$, define $\varphi_i \in \mathbf{k}[\text{Hom}(x, y_i)]^*$ by $\varphi_i = \sum_{f \in S_i} e_f^*$. Then the φ_i generate P_x^\vee . \square

Proposition 5.3.5. *Assume that \mathbf{k} is a field. Suppose that \mathcal{C} is Hom-finite and satisfies property (D), and that \mathcal{C} and \mathcal{C}^{op} are both quasi-Gröbner. Then $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ has Krull dimension 0, that is, every finitely generated representation of \mathcal{C} has finite length.*

Proof. Let M be a finitely generated \mathcal{C} -module and pick a chain $M \supseteq M_1 \supseteq M_2 \supseteq \dots$. Let K_i be the kernel of the surjection $M^\vee \rightarrow M_i^\vee$ dual to the inclusion $M_i \subseteq M$; we get a chain $K_1 \subseteq K_2 \subseteq \dots \subseteq M^\vee$. Choose a surjection $P \rightarrow M$, where P is a finite direct sum of principal projectives. Then $M^\vee \subseteq P^\vee$, so is noetherian since P^\vee is finitely generated (Proposition 5.3.4) and $\text{Rep}_{\mathbf{k}}(\mathcal{C}^{\text{op}})$ is noetherian. So the chain K_\bullet stabilizes, and hence M is artinian. Since \mathcal{C} is quasi-Gröbner, M is also noetherian, so M has finite length. \square

6. HILBERT SERIES AND LINGUAL CATEGORIES

In this section we use the theory of formal languages (discussed in §3) together with the Gröbner techniques in §5 to study Hilbert series of representations of quasi-Gröbner categories. Our goal is to introduce norms, Hilbert series, lingual structures, and lingual categories and to prove a few basic properties about lingual categories.

6.1. Normed categories and Hilbert series. Let \mathcal{C} be an essentially small category. A **norm** on \mathcal{C} is a function $\nu: |\mathcal{C}| \rightarrow \mathbf{N}^I$, where I is a finite set. A **normed category** is a category equipped with a norm. Fix a category \mathcal{C} with a norm ν with values in \mathbf{N}^I . As in §3.1, we let $\{t_i\}_{i \in I}$ be indeterminates. Let M be a representation of \mathcal{C} over a field \mathbf{k} . We define the **Hilbert series** of M as

$$H_{M,\nu}(\mathbf{t}) = \sum_{x \in |\mathcal{C}|} \dim_{\mathbf{k}} M(x) \cdot \mathbf{t}^{\nu(x)},$$

when this makes sense. We omit the norm ν from the notation when possible.

6.2. Lingual structures. We return to the set-up of §5.1: let $S: \mathcal{C} \rightarrow \mathbf{Set}$ be a functor, and let $P = \mathbf{k}[S]$. However, we now also assume that \mathcal{C} is directed and normed over \mathbf{N}^I . We define a norm on $|S|$ as follows: given $f \in |S|$, let $\tilde{f} \in S(x)$ be a lift, and put $\nu(f) = \nu(x)$. This is well-defined because \mathcal{C} is ordered: if $\tilde{f}' \in S(y)$ is a second lift then x and y are necessarily isomorphic. Let \mathcal{P} be a class of languages (e.g., regular languages).

Definition 6.2.1. A **lingual structure** on $|S|$ is a pair (Σ, i) consisting of a finite alphabet Σ normed over \mathbf{N}^I and an injection $i: |S| \rightarrow \Sigma^*$ compatible with the norms, i.e., such that $\nu(i(f)) = \nu(f)$. A **\mathcal{P} -lingual structure** is a lingual structure satisfying the following additional condition: for every poset ideal J of $|S|$, the language $i(J)$ is of class \mathcal{P} . \square

Theorem 6.2.2. Suppose \mathcal{C} is directed, S is orderable, $|S|$ is noetherian, and $|S|$ admits a \mathcal{P} -lingual structure. Let M be a subrepresentation of P . Then $H_M(\mathbf{t})$ is of the form $H_{\mathcal{L}}(\mathbf{t})$, where \mathcal{L} is a language of class \mathcal{P} .

Proof. Choose an ordering on S and let N be the initial subrepresentation of M . Then $N(x)$ is an associated graded of $M(x)$, for any object x , and so the two have the same dimension. Thus M and N have the same Hilbert series. Let $J \subset |S|$ be the set of elements f for which e_f belongs to N . Then N and the language $i(J)$ have the same Hilbert series. \square

Remark 6.2.3. We abbreviate “ordered,” “regular,” and “unambiguous context-free” to O, R, and UCF. So an R-lingual structure is one for which $i(I)$ is always regular. \square

Remark 6.2.4. If $|S|$ admits a lingual structure then $S(x)$ is finite for all x . Indeed, given $\mathbf{n} \in \mathbf{N}^I$, let $\Sigma_{\mathbf{n}}^*$ denote the set of words of norm \mathbf{n} ; this is a finite set since all such words have length $|\mathbf{n}|$. Since i maps $S(x)$ injectively into $\Sigma_{\nu(x)}^*$, we see that $S(x)$ is finite. \square

6.3. Lingual categories. Recall that S_x denotes the functor $\text{Hom}_{\mathcal{C}}(x, -)$.

Definition 6.3.1. A directed normed category \mathcal{C} is **\mathcal{P} -lingual** if $|S_x|$ admits a \mathcal{P} -lingual structure for all objects x . \square

The following theorem is the second main theoretical result of this paper. It connects the purely combinatorial condition “ \mathcal{P} -lingual” with the algebraic invariant “Hilbert series.”

Theorem 6.3.2. Let \mathcal{C} be a \mathcal{P} -lingual Gröbner category and let M be a finitely generated representation of \mathcal{C} . Then $H_M(\mathbf{t})$ is a \mathbf{Z} -linear combination of series of the form $H_{\mathcal{L}}(\mathbf{t})$ where \mathcal{L} is a language of class \mathcal{P} . In particular,

- If $\mathcal{P} = \text{R}$, then $H_M(\mathbf{t})$ is a rational function of the t_i .
- If $\mathcal{P} = \text{O}$, then $H_M(\mathbf{t}) = f(\mathbf{t})/g(\mathbf{t})$, where $f(\mathbf{t})$ is a polynomial in the t_i and $g(\mathbf{t}) = \prod_{j=1}^n (1 - \lambda_j)$ and each λ_j is an \mathbf{N} -linear combination of the t_i .

- If $\mathcal{P} = \text{UCF}$, then $H_M(\mathbf{t})$ is an algebraic function of the t_i .

Proof. If M is a subrepresentation of a principal projective then the first statement follows from Theorem 6.2.2. As such representations span the Grothendieck group of finitely generated representations, the first statement holds.

The remaining statements follow from the various results about Hilbert series of \mathcal{P} -languages: in the regular case, Theorem 3.2.2; in the ordered case, Theorem 3.3.7; and in the unambiguous context-free case, the relevant result is Theorem 3.4.1. \square

Suppose \mathcal{C}_1 is normed over \mathbf{N}^{I_1} and \mathcal{C}_2 is normed over \mathbf{N}^{I_2} . We give $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ the structure of a normed category over \mathbf{N}^I , where $I = I_1 \amalg I_2$, by $\nu(x, y) = \nu(x) + \nu(y)$. Here we have identified $|\mathcal{C}|$ with $|\mathcal{C}_1| \times |\mathcal{C}_2|$. We call \mathcal{C} , with this norm, the **product** of \mathcal{C}_1 and \mathcal{C}_2 .

Proposition 6.3.3. *Let \mathcal{C}_1 and \mathcal{C}_2 be \mathcal{P} -lingual normed categories. Suppose that the posets $|\mathcal{C}_{1,x}|$ and $|\mathcal{C}_{2,y}|$ are noetherian for all x and y and that \mathcal{P} is stable under finite unions and concatenations of languages on disjoint alphabets. Then $\mathcal{C}_1 \times \mathcal{C}_2$ is also \mathcal{P} -lingual.*

Proof. Keep the notation from the previous paragraph. Let x be an object of \mathcal{C}_1 , and let $i_1: |\mathcal{C}_{1,x}| \rightarrow \Sigma_1^*$ be a \mathcal{P} -lingual structure at x . Similarly, let y be an object of \mathcal{C}_2 , and let $i_2: |\mathcal{C}_{2,y}| \rightarrow \Sigma_2^*$ be a \mathcal{P} -lingual structure at y . Let $\Sigma = \Sigma_1 \amalg \Sigma_2$, normed over \mathbf{N}^I in the obvious manner. Then the following diagram commutes:

$$\begin{array}{ccccc} |\mathcal{C}_{(x,y)}| & = & |\mathcal{C}_{1,x}| \times |\mathcal{C}_{2,y}| & \xrightarrow{i_1 \times i_2} & \Sigma_1^* \times \Sigma_2^* & \longrightarrow & \Sigma^* \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \mathbf{N}^{I_1} \oplus \mathbf{N}^{I_2} & = & \mathbf{N}^{I_1} \oplus \mathbf{N}^{I_2} & = & \mathbf{N}^I \end{array}$$

The top right map is concatenation of words. We let $i: |\mathcal{C}_{(x,y)}| \rightarrow \Sigma^*$ be the composition along the first line, which is clearly injective. We claim that this is a \mathcal{P} -lingual structure on $|\mathcal{C}_{(x,y)}|$. The commutativity of the above diagram shows that it is a lingual structure. Now suppose S is an ideal of $|\mathcal{C}_{(x,y)}|$. Since this poset is noetherian (Proposition 2.3), it is a finite union of principal ideals S_1, \dots, S_n . Each S_j is of the form $T_j \times T'_j$, where T_j is an ideal of $\mathcal{C}_{1,x}$ and T'_j is an ideal of $\mathcal{C}_{2,y}$. By assumption, the languages $i_1(T_j)$ and $i_2(T'_j)$ satisfy property \mathcal{P} . Regarding $i_1(T_j)$ and $i_2(T'_j)$ as languages over Σ , the language $i(S_j)$ is their concatenation, and therefore satisfies \mathcal{P} . Finally, $i(S)$ satisfies \mathcal{P} , as it is the union of the $i(S_j)$. \square

Part 2. Applications

7. CATEGORIES OF INJECTIONS

In this first applications section, we study the category **FI** of finite sets and injective functions along with generalizations and variations. The main results are listed in §7.1, proofs are in §7.2, applications to twisted commutative algebras are in §7.3, and complementary results are listed in §7.4.

7.1. The categories \mathbf{OI}_d and \mathbf{FI}_d . Let d be a positive integer. Define \mathbf{FI}_d to be the following category. The objects are finite sets. Given two finite sets S and T , a morphism $S \rightarrow T$ is a pair (f, g) where $f: S \rightarrow T$ is an injection and $g: T \setminus f(S) \rightarrow \{1, \dots, d\}$ is a function. Define \mathbf{OI}_d to be the ordered version of \mathbf{FI}_d : its objects are totally ordered finite sets and its morphisms are pairs (f, g) with f an order-preserving injection (no condition is placed on g). We norm \mathbf{FI}_d and \mathbf{OI}_d over \mathbf{N} by $\nu(x) = \#x$.

Our main result about \mathbf{OI}_d is the following theorem, which is proven in §7.2.

Theorem 7.1.1. *The category \mathbf{OI}_d is O-lingual and Gröbner.*

Theorem 7.1.2. *The category \mathbf{FI}_d is quasi-Gröbner.*

Proof. We claim that the forgetful functor $\Phi: \mathbf{OI}_d \rightarrow \mathbf{FI}_d$ satisfies property (F). Let $x = [n]$ be a given object of \mathbf{FI}_d . If y is any totally ordered set, then any morphism $f: x \rightarrow y$ can be factored as $x \xrightarrow{\sigma} x \xrightarrow{f'} y$, where σ is a permutation and f' is order-preserving. It follows that we can take $y_1, \dots, y_{n!}$ to all be $[n]$, and $f_i: x \rightarrow \Phi(y_i)$ to be the i th element of the symmetric group S_n (under any enumeration). This establishes the claim. Since \mathbf{OI}_d is Gröbner, this shows that \mathbf{FI}_d is quasi-Gröbner. \square

Corollary 7.1.3. *If \mathbf{k} is left-noetherian then $\text{Rep}_{\mathbf{k}}(\mathbf{FI}_d)$ is noetherian.*

Remark 7.1.4. This result was first proved for \mathbf{k} a field of characteristic 0 in [Sn, Thm. 2.3], though the essential idea goes back to Weyl; see also [SS2, Proposition 9.2.1]. It was reproved independently for $d = 1$ (and \mathbf{k} a field of characteristic 0) in [CEF]. It was then proved for $d = 1$ and \mathbf{k} an arbitrary ring in [CEFN]. The result is new if $d > 1$ and \mathbf{k} is not a field of characteristic 0. \square

Corollary 7.1.5. *Let M be a finitely generated \mathbf{FI}_d -module over a field \mathbf{k} . The Hilbert series*

$$H_M(t) = \sum_{n \geq 0} \dim_{\mathbf{k}} M([n]) \cdot t^n$$

is of the form $f(t)/g(t)$, where $f(t)$ and $g(t)$ are polynomials and $g(t)$ factors as $\prod_{j=1}^d (1-jt)^{e_j}$ where $e_j \geq 0$. In particular, if $d = 1$ then $n \mapsto \dim_{\mathbf{k}} M([n])$ is eventually polynomial.

Proof. Let $\Phi: \mathbf{OI}_d \rightarrow \mathbf{FI}_d$ be the forgetful functor. Then $H_M(t) = H_{\Phi^*(M)}(t)$, so the statement follows from Theorems 7.1.1 and 6.3.2 except that it only guarantees that $g(t) = \prod_{j=1}^r (1-jt)^{e_j}$ for some r . But, as we will see in §7.2, for each set $x = [n]$, the lingual structure on $|S_x|$ (in the notation of §6.3) is built on the set $\Sigma = \{0, \dots, d\}$ and $\{1, \dots, d\}$ is the largest repeatable subset, so we can refine this statement using Corollary 3.3.8. \square

Remark 7.1.6. This result has a history similar to Corollary 7.1.3. It was first proved for $\text{char}(\mathbf{k}) = 0$ in [Sn, Thm. 3.1]. It was then independently reproved for $d = 1$ and $\text{char}(\mathbf{k}) = 0$ in [CEF]. Refinements and deeper properties were discovered for these Hilbert series in [SS1] (see for example [SS1, §6.8]). It was then proved for $d = 1$ and any \mathbf{k} in [CEFN]. The result is new if $d > 1$ and \mathbf{k} has positive characteristic. \square

7.2. Proof of Theorem 7.1.1. It is clear that $\mathcal{C} = \mathbf{OI}_d$ is directed. Let n be a non-negative integer, and regard $x = [n]$ as an object of \mathcal{C} . Let $\Sigma = \{0, \dots, d\}$, and let \mathcal{L} be the language on Σ in which 0 appears exactly n times. Partially order \mathcal{L} using the subsequence order, i.e., if $s: [i] \rightarrow \Sigma$ and $t: [j] \rightarrow \Sigma$ are words then $s \leq t$ if there exists $I \subseteq [j]$ such that $s = t|_I$.

Lemma 7.2.1. *The poset \mathcal{L} is noetherian and every ideal is an ordered language.*

Proof. Noetherianity is an immediate consequence of Higman's lemma (Theorem 2.5). Let $w = w_1 \cdots w_n$ be a word in \mathcal{L} . Then the ideal generated by w is the language

$$\Pi^* w_1 \Pi^* w_2 \cdots \Pi^* w_n \Pi^*,$$

where $\Pi = \Sigma \setminus \{0\}$, and is therefore ordered. As every ideal is a finite union of principal ideals, and a finite union of ordered languages is ordered, the result follows. \square

Pick $(f, g) \in \text{Hom}_{\mathcal{C}}([n], [m])$. Define $h: [m] \rightarrow \Sigma$ to be the function which is 0 on the image of f , and equal to g on the complement of the image of f . One can recover (f, g) from h since f is required to be order-preserving and injective. This construction therefore defines an isomorphism of posets $i: |\mathcal{C}_x| \rightarrow \mathcal{L}$. It follows that $|\mathcal{C}_x|$ is noetherian. Furthermore, the lexicographic order on \mathcal{L} (using the standard order on Σ) is easily verified to restrict to an admissible order on $|\mathcal{C}_x|$. Thus \mathcal{C} is Gröbner. For $f \in |\mathcal{C}_x|$ we have $|f| = \ell(i(f))$, and so (Σ, i) is a lingual structure on $|\mathcal{C}_x|$ if we norm words in Σ^* by their length. Finally, an ideal of $|\mathcal{C}_x|$ gives an ordered language over Σ , and so this is an O-lingual structure.

Remark 7.2.2. The results about \mathbf{OI}_1 can be made more transparent with the following observation: the set of order-increasing injections $f: [n] \rightarrow [m]$ is naturally in bijection with monomials in x_0, \dots, x_n of degree $m - n$ by assigning the monomial $\mathbf{m}_f = \prod_{i=0}^n x_i^{f(i+1)-f(i)-1}$ using the convention $f(0) = 0$ and $f(n+1) = m+1$. Given $g: [n] \rightarrow [m']$, there is a morphism $h: [m] \rightarrow [m']$ with $g = hf$ if and only if \mathbf{m}_f divides \mathbf{m}_g . Thus the monomial submodules of P_n are in bijection with monomial ideals in the polynomial ring $\mathbf{k}[x_0, \dots, x_n]$. \square

7.3. Twisted commutative algebras. We now discuss the relationship between \mathbf{FI}_d and twisted commutative algebras. For this part, we assume \mathbf{k} is commutative.

Definition 7.3.1. A **twisted commutative algebra** (tca) over \mathbf{k} is an associative and unital graded \mathbf{k} -algebra $A = \bigoplus_{n \geq 0} A_n$ equipped with an action of S_n on A_n such that:

- (a) the multiplication map $A_n \otimes A_m \rightarrow A_{n+m}$ is $(S_n \times S_m)$ -equivariant (where we use the standard embedding $S_n \times S_m \subset S_{n+m}$ for the action on A_{n+m}); and
- (b) given $x \in A_n$ and $y \in A_m$ we have $xy = (yx)^\tau$, where $\tau = \tau_{m,n} \in S_{n+m}$ is defined by

$$\tau(i) = \begin{cases} i + n & \text{if } 1 \leq i \leq m, \\ i - m & \text{if } m + 1 \leq i \leq n + m. \end{cases}$$

This is the “twisted commutativity” condition.

An **A-module** is a graded A -module $M = \bigoplus_{n \geq 0} M_n$ (in the usual sense) equipped with an action of S_n on M_n such that multiplication $A_n \otimes M_m \rightarrow M_{n+m}$ is $(S_n \times S_m)$ -equivariant. \square

Example 7.3.2. Let x_1, \dots, x_d be indeterminates, each regarded as having degree 1. We define a tca $A = \mathbf{k}\langle x_1, \dots, x_d \rangle$, the **polynomial tca** in the x_i , as follows. As a graded \mathbf{k} -algebra, A is just the non-commutative polynomial ring in the x_i . The S_n -action on A_n is the obvious one: on monomials it is given by $\sigma(x_{i_1} \cdots x_{i_n}) = x_{i_{\sigma(1)}} \cdots x_{i_{\sigma(n)}}$. \square

We now give a more abstract way to define tca’s, which clarifies some constructions. Define a representation of S_* over \mathbf{k} to be a sequence $M = (M_n)_{n \geq 0}$, where M_n is a representation of S_n over \mathbf{k} . Given S_* -representations M and N , we define an S_* -representation $M \otimes N$ by

$$(M \otimes N)_n = \bigoplus_{i+j=n} \text{Ind}_{S_i \times S_j}^{S_n} (M_i \otimes_{\mathbf{k}} N_j).$$

There is a symmetry of the tensor product obtained by switching the order of M and N and conjugating $S_i \times S_j$ to $S_j \times S_i$ in S_n via $\tau_{i,j}$. This gives the category $\text{Rep}_{\mathbf{k}}(S_*)$ of S_* -representations the structure of a symmetric abelian tensor category. The general notions of commutative algebra and module in such a category specialize to tca’s and their modules.

Example 7.3.3. For an integer $n \geq 0$ let $\mathbf{k}\langle n \rangle$ be the S_* -representation which is the regular representation in degree n and 0 in all other degrees. The tca $\mathbf{k}\langle x_1, \dots, x_d \rangle$ can then be

identified with the symmetric algebra on the object $\mathbf{k}\langle 1 \rangle^{\oplus d}$. The symmetric algebra on $\mathbf{k}\langle n \rangle$ is poorly understood for $n > 1$ (although some results are known for $n = 2$). \square

If \mathbf{k} is a field of characteristic 0 then $\text{Rep}_{\mathbf{k}}(S_*)$ is equivalent to the category of polynomial representations of $\mathbf{GL}(\infty)$ over \mathbf{k} , as symmetric abelian tensor categories. Under this equivalence, tca's correspond to commutative associative unital \mathbf{k} -algebras equipped with a polynomial action of $\mathbf{GL}(\infty)$. For example, the polynomial tca $\mathbf{k}\langle x \rangle$ corresponds to the polynomial ring $\mathbf{k}[x_1, x_2, \dots]$ in infinitely many variables, equipped with its usual action of $\mathbf{GL}(\infty)$. This point of view has been exploited by us [Sn, SS2, SS1, SS3] to establish properties of tca's in characteristic 0.

There is an obvious notion of generation for a set of elements in a tca or a module over a tca. We say that a tca is **noetherian** if every submodule of a finitely generated module is again finitely generated.

Proposition 7.3.4. *Let $A = \mathbf{k}\langle x_1, \dots, x_d \rangle$. Then Mod_A is equivalent to $\text{Rep}_{\mathbf{k}}(\mathbf{FI}_d)$.*

Proof. Pick $M \in \text{Rep}_{\mathbf{k}}(\mathbf{FI}_d)$ and let $M_m = M([m])$. Then M_m has an action of S_m . To define $A_n \otimes M_m \rightarrow M_{n+m}$, it is enough to define how $x_{i_1} \cdots x_{i_n}$ acts by multiplication for each (i_1, \dots, i_n) . Let $(f, g): [m] \rightarrow [n+m]$ be the morphism in \mathbf{FI}_d where $f: [m] \rightarrow [n+m]$ is the injection $i \mapsto n+i$ and $g: [n] \rightarrow [d]$ is the function $g(j) = i_j$ and let the multiplication map be the induced map $M_{(f,g)}: M_m \rightarrow M_{n+m}$. By definition this is $(S_n \times S_m)$ -equivariant and associativity follows from associativity of composition of morphisms in \mathbf{FI}_d .

It is easy to reverse this process: given an A -module M , we get a functor defined on the full subcategory of \mathbf{FI}_d on objects of the form $[n]$ (note that every morphism $[m] \rightarrow [n+m]$ is a composition of the injections we defined above with an automorphism of $[n+m]$). To extend this to all of \mathbf{FI}_d , pick a total ordering on each finite set to identify it with $[n]$. The two functors we have defined are quasi-inverse to each other. \square

Remark 7.3.5. In particular, the category of \mathbf{FI} -modules studied in [CEF, CEFN] is equivalent to the category of modules over the univariate tca $\mathbf{k}\langle x \rangle$. \square

Corollary 7.3.6. *A tca over a noetherian ring \mathbf{k} finitely generated in degree 1 is noetherian.*

Proof. A tca finitely generated in degree 1 is a quotient of $\mathbf{k}\langle x_1, \dots, x_d \rangle$ for some d . \square

Remark 7.3.7. The same techniques also prove that a twisted graded-commutative algebra finitely generated in degree 1 is noetherian. \square

Suppose \mathbf{k} is a vector space. We define the **Hilbert series** of a module M over a tca by

$$H'_M(t) = \sum_{n \geq 0} \dim_{\mathbf{k}} M([n]) \frac{t^n}{n!}.$$

As the equivalence in Proposition 7.3.4 is clearly compatible with Hilbert series, we obtain:

Corollary 7.3.8. *Let M be a finitely generated $\mathbf{k}\langle x_1, \dots, x_d \rangle$ -module. Then $H'_M(t) \in \mathbf{Q}[t, e^t]$.*

7.4. Additional results.

Proposition 7.4.1. *The category \mathbf{FI}_1 satisfies property (F) (see Definition 4.2.1)*

Proof. Let x and x' be sets, and consider maps $f: x \rightarrow y$ and $f': x' \rightarrow y$. Let \bar{y} be the union of the images of f and f' . Let $g: \bar{y} \rightarrow y$ be the inclusion, and write \bar{f} for the map $x \rightarrow \bar{y}$ induced by f , and similarly for \bar{f}' . Then $f = g \circ \bar{f}$ and $f' = g \circ \bar{f}'$. Since $\#\bar{y} \leq \#x + \#x'$,

it follows that there are only finitely many choices for $(\bar{y}, \bar{f}, \bar{f}')$, up to isomorphism. This completes the proof. \square

The next result follows from Proposition 4.3.1 and reproves [CEF, Proposition 2.61].

Corollary 7.4.2. *For any commutative ring \mathbf{k} , the pointwise tensor product of finitely generated representations of \mathbf{FI}_1 is finitely generated.*

Remark 7.4.3. The above corollary is false for \mathbf{FI}_d for $d > 1$. It follows that these categories do not satisfy property (F). To see this, consider $M = P_1 \odot P_1$. Then $\dim_{\mathbf{k}} M([n]) = (d^2)^n$, so the Hilbert series is $H_M(t) = (1 - d^2n)^{-1}$. By Corollary 7.1.5, the Hilbert series of a finitely generated \mathbf{FI}_d -module cannot have this form if $d > 1$. \square

The next results concern the category \mathbf{FA} of finite sets, with all functions as morphisms.

Theorem 7.4.4. *The category \mathbf{FA} is quasi-Gröbner.*

Proof. We claim that the inclusion functor $\Phi: \mathbf{FI}_1 \rightarrow \mathbf{FA}$ satisfies property (F). Let x be a set and let $f_i: x \rightarrow y_i$ be representatives for the isomorphism classes of surjective maps from x ; these are finite in number. Any morphism $f: x \rightarrow y$ factors as $f = g \circ f_i$ for some i and some injective map $g: y_i \rightarrow y$ which proves the claim. Since \mathbf{FI}_1 is quasi-Gröbner, the result now follows from Proposition 5.2.6. \square

Corollary 7.4.5. *If \mathbf{k} is left-noetherian then $\text{Rep}_{\mathbf{k}}(\mathbf{FA})$ is noetherian.*

We greatly improve this result in Theorem 8.4.4, using our results on \mathbf{FS}^{op} .

Suppose \mathbf{k} is a field. By what we have shown, the Hilbert series of a finitely generated \mathbf{FA} -module is of the form $f(t)/(1-t)^d$ for some polynomial f and $d \geq 0$. Equivalently, the function $n \mapsto \dim_{\mathbf{k}} M([n])$ is eventually polynomial. In fact, one can do better:

Theorem 7.4.6. *If \mathbf{k} is a field and M is a finitely generated \mathbf{FA} -module, then the function $n \mapsto \dim_{\mathbf{k}} M([n])$ agrees with a polynomial for all $n > 0$. Equivalently, the Hilbert series of M is of the form $f(t)/(1-t)^d$ where $\deg f \leq d$.*

Proof. Let \mathbf{FA}° be the category of nonempty finite sets, so that every \mathbf{FA} -module gives an \mathbf{FA}° -module by restriction. Introduce an operator Σ on \mathbf{FA}° -modules by $(\Sigma M)(S) = M(S \amalg \{*\})$. It is easy to see that ΣM is a finitely generated \mathbf{FA}° -module if the same is true for M . For every nonempty set S , the inclusion $S \rightarrow S \amalg \{*\}$ can be split, and so the map $M(S) \rightarrow (\Sigma M)(S)$ is an inclusion for all S . Define $\Delta M = \Sigma M/M$. Since Σ is exact, it follows that $\Delta M \rightarrow \Delta M'$ is a surjection if $M \rightarrow M'$ is a surjection.

Define $h_M(n) = \dim_{\mathbf{k}} M([n])$. Recall that a function $f: \mathbf{Z}_{>0} \rightarrow \mathbf{Z}$ is a polynomial of degree $\leq d$ if and only if the function $g(n) = f(n+1) - f(n)$ is a polynomial of degree $\leq d-1$. Since $h_{\Delta M}(n) = h_M(n+1) - h_M(n)$, we get that h_M is a polynomial function of degree $\leq d$ if and only if $\Delta^{d+1} M = 0$. Pick a surjection $P \rightarrow M \rightarrow 0$ with P a direct sum of principal projectives P_S . Since $h_{P_S}(n) = n^{|S|}$ is a polynomial, P is annihilated by some power of Δ . Since Δ preserves surjections, the same is true for M , and so $h_M(n)$ is a polynomial. \square

Remark 7.4.7. (a) The category of representations $\text{Rep}_{\mathbf{k}}(\mathbf{FA})$ is studied in [WG] in the case that \mathbf{k} is a field of characteristic 0, and Theorems 7.4.6 and 8.4.4 are proved.

(b) In [Do], the functions $n \mapsto |F([n])|$ coming from functors $F: \mathbf{FA} \rightarrow \mathbf{FA}$ are classified.

(c) The analogue of [K1, Proposition 4.10] for \mathbf{FA} holds with the same proof, i.e., if $\deg h_M(n) = r$, then the lattice of \mathbf{FA} -submodules of M is isomorphic to the lattice of $\mathbf{k}[\text{End}([r])]$ -submodules of $M([r])$. So one can prove Theorem 8.4.4 from Theorem 7.4.6. We will give a different proof using Gröbner methods. \square

7.5. Examples of \mathbf{FI}_d -modules. The paper [CEF] gives many examples of \mathbf{FI}_1 -modules appearing “in nature.” We now give some examples of \mathbf{FI}_d -modules. First a general construction. Let M_1, \dots, M_d be \mathbf{FI}_1 -modules over a commutative ring \mathbf{k} . We define $N = M_1 \otimes \dots \otimes M_d$ to be the following \mathbf{FI}_d -module: on sets S , it is defined by

$$N(S) = \bigoplus_{S=S_1 \amalg \dots \amalg S_d} M_1(S_1) \otimes \dots \otimes M_d(S_d).$$

Given a morphism $(f: S \rightarrow T, g: T \setminus f(S) \rightarrow [d])$ in \mathbf{FI}_d , let $T_i = g^{-1}(i)$ and define $N(S) \rightarrow N(T)$ to be the sum of the maps

$$M_1(S_1) \otimes \dots \otimes M_d(S_d) \rightarrow M_1(S_1 \amalg T_1) \otimes \dots \otimes M_d(S_d \amalg T_d).$$

If M_1, \dots, M_d are finitely generated then the same is true for N ; there is an obvious generalization where we allow each M_i to be a \mathbf{FI}_{n_i} -module and N is a $\mathbf{FI}_{n_1+\dots+n_d}$ -module.

Suppose now, for simplicity, that \mathbf{k} is a field. For a topological space Y and finite set S , let $\text{Conf}_S(Y)$ be the space of injective functions $S \rightarrow Y$. For any $r \geq 0$, we get an \mathbf{FI}_1 -module $S \mapsto H^r(\text{Conf}_S(Y); \mathbf{k})$, which we denote by $H^r(\text{Conf}(Y))$. Let X_1, \dots, X_d be connected topological spaces, and let X be their disjoint union. The Künneth theorem gives

$$H^r(\text{Conf}_S(X); \mathbf{k}) = \bigoplus_{p_1+\dots+p_d=r} \bigoplus_{S=S_1 \amalg \dots \amalg S_d} H^{p_1}(\text{Conf}_{S_1}(X_1); \mathbf{k}) \otimes \dots \otimes H^{p_d}(\text{Conf}_{S_d}(X_d); \mathbf{k}).$$

In particular, we can write

$$H^r(\text{Conf}(X)) = \bigoplus_{p_1+\dots+p_d=r} H^{p_1}(\text{Conf}(X_1)) \otimes \dots \otimes H^{p_d}(\text{Conf}(X_d)),$$

so that we can endow $H^r(\text{Conf}(X))$ with the structure of an \mathbf{FI}_d -module. Under reasonable hypotheses on the X_i , each $H^r(\text{Conf}(X_i))$ is a finitely generated \mathbf{FI}_1 -module (see [CEFN, Theorem E]), and so $H^r(\text{Conf}(X))$ will be finitely generated as an \mathbf{FI}_d -module.

8. CATEGORIES OF SURJECTIONS

In this section, we study \mathbf{FS}^{op} , the (opposite of the) category of finite sets with surjective functions and variations, which behaves quite differently from the category of injective functions. The main results are stated in §8.1 and the proofs are in §8.2. The results on \mathbf{FS}^{op} are powerful enough to prove the Lannes–Schwartz artinian conjecture; this is done in §8.3. We give some complementary results in §8.4.

8.1. The categories \mathbf{OS} and \mathbf{FS} . Define \mathbf{FS} to be the category whose objects are non-empty finite sets and whose morphisms are surjective functions. We define an ordered version \mathbf{OS} of this category as follows. The objects are totally ordered finite sets. A morphism $S \rightarrow T$ in \mathbf{OS} is an **ordered surjection**: a surjective map $f: S \rightarrow T$ such that for all $i < j$ in T we have $\min f^{-1}(i) < \min f^{-1}(j)$. We norm \mathbf{FS} and \mathbf{OS} over \mathbf{N} by $\nu(x) = \#x$.

Our main result about \mathbf{OS} is the following theorem, which is proved in the next section.

Theorem 8.1.1. *The category \mathbf{OS}^{op} is O-lingual and Gröbner.*

Theorem 8.1.2. *The category \mathbf{FS}^{op} is quasi-Gröbner.*

Proof. As in the proof of Theorem 7.1.2, one can show that the forgetful functor $\Phi: \mathbf{OS}^{\text{op}} \rightarrow \mathbf{FS}^{\text{op}}$ satisfies property (F). \square

Corollary 8.1.3. *If \mathbf{k} is left-noetherian then $\text{Rep}_{\mathbf{k}}(\mathbf{FS}^{\text{op}})$ is noetherian.*

Corollary 8.1.4. *Let M be a finitely generated \mathbf{FS}^{op} -module defined over a field \mathbf{k} . Then the Hilbert series*

$$H_M(t) = \sum_{n \geq 0} \dim_{\mathbf{k}} M([n]) \cdot t^n$$

has the form $f(t)/g(t)$, where $f(t)$ and $g(t)$ are polynomials, and $g(t)$ factors as $\prod_{j=1}^r (1-jt)^{e_j}$ for some r and $e_j \geq 0$. If M is generated in degree $\leq d$, then we can take $r = d$.

Proof. If $\Phi: \mathbf{OS}^{\text{op}} \rightarrow \mathbf{FS}^{\text{op}}$ is the forgetful functor, then $H_M(t) = H_{\Phi^*(M)}(t)$, so the result follows from Theorems 8.1.1 and 6.3.2. To prove the last statement, we note that it follows from §8.2 that for a finite set $x = [n]$, the lingual structure on $|S_x|$ (in the notation of §6.3) is built on the set $\{1, \dots, n\}$. Now use Corollary 3.3.8. \square

Remark 8.1.5. Using partial fraction decomposition, a function $f(t)/(\prod_{j=1}^r (1-jt)^{e_j})$ can be written as a sum $\sum_{j=1}^r f_j(t)/(1-jt)^{e_j}$ for polynomials $f_j(t) \in \mathbf{Q}[t]$. Corollary 8.1.4 implies that if M is a finitely generated \mathbf{FS}^{op} -module over a field \mathbf{k} , then there exist polynomials p_1, \dots, p_r so that the function $n \mapsto \dim_{\mathbf{k}} M([n])$ agrees with $\sum_{j=1}^r p_j(n)j^n$ for $n \gg 0$. \square

Remark 8.1.6. Let Γ be the category of finite sets with a basepoint and basepoint-preserving functions. It is shown in [Pi] that $\text{Rep}_{\mathbf{k}}(\mathbf{FS}^{\text{op}})$ is equivalent to the category of functors $\Gamma^{\text{op}} \rightarrow \text{Mod}_{\mathbf{k}}$ that send the one-point set to the zero module. \square

We record the following more general form of the corollary for later use.

Corollary 8.1.7. *Let M be a finitely generated $(\mathbf{FS}^{\text{op}})^r$ -representation over a field \mathbf{k} . Then the Hilbert series*

$$H_M(t) = \sum_{\mathbf{n} \in \mathbf{N}^r} \dim_{\mathbf{k}} M([\mathbf{n}]) \cdot \mathbf{t}^{\mathbf{n}}$$

is a rational function $f(\mathbf{t})/g(\mathbf{t})$, where $g(\mathbf{t})$ factors as $\prod_{i=1}^r \prod_{j=1}^{\infty} (1-jt_i)^{e_{i,j}}$, where all but finitely many $e_{i,j}$ are 0. Equivalently, the exponential Hilbert series

$$H'_M(t) = \sum_{\mathbf{n} \in \mathbf{N}^r} \dim_{\mathbf{k}} M([\mathbf{n}]) \cdot \frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!},$$

is a polynomial in the t_i and the e^{t_i} . (Here $\mathbf{n}! = n_1! \cdots n_r!$.)

Proof. Proposition 6.3.3 shows that $(\mathbf{OS}^{\text{op}})^r$, given the product norm over $I = \{1, \dots, r\}$, is O-lingual. However, to get the stated result we need to extract slightly more information from the proofs. The languages appearing in the proof of Proposition 6.3.3 all have the form $\mathcal{L}_1 \cdots \mathcal{L}_r$, where \mathcal{L}_i is an ordered language normed over $\mathbf{N}^{\{i\}} \subset \mathbf{N}^I$. It follows that the repeatable subsets of \mathcal{L} are singletons. Thus, when applying Theorem 3.3.7 to the relevant languages, each λ_i is a non-negative multiple of a single t_j . \square

Remark 8.1.8. It is possible to set-up a more complicated theory of O-lingual normed categories, by keeping track of a partition of the set I that is compatible with the norm in a certain sense, and show that Hilbert series respect this partition, in an appropriate sense. The above corollary is the only place where we really need this more complicated theory, so we simply explained the ideas in its proof rather than developing the theory fully. \square

8.2. Proof of Theorem 8.1.1. Let Σ be a finite set. Let $s: [n] \rightarrow \Sigma$ and $t: [m] \rightarrow \Sigma$ be two elements of Σ^* . We define $s \leq t$ if there exists an order-preserving injection $\varphi: [n] \rightarrow [m]$ such that for all $i \in [n]$ we have $s_i = t_{\varphi(i)}$, and for all $j \in [m]$ there exists $j' \leq j$ in the image of φ with $t_j = t_{j'}$. This defines a partial order on Σ^* .

Proposition 8.2.1. *The poset Σ^* is noetherian and every ideal is an ordered language.*

Proof. Suppose that Σ^* is not noetherian. We use the notion of bad sequences from the proof of Theorem 2.5.

Given $x \in \Sigma^*$ call a value of x *exceptional* if it appears exactly once. If x has a non-exceptional value, then let $m(x)$ denote the index, counting from the end, of the first non-exceptional value. Note that $m(x) \leq \#\Sigma$. Also, $\ell(x) > \#\Sigma$ implies that x has some non-exceptional value, so $m(x_i)$ is defined for $i \gg 0$. (Note: $\ell(x_i) \rightarrow \infty$, as there are only finitely many sequences of a given length.) So we can find an infinite subsequence of x_1, x_2, \dots on which m is defined and is constant, say equal to m_0 . We can find a further infinite subsequence where the value in the position m_0 is constant. Call this subsequence x_{i_1}, x_{i_2}, \dots and let y_{i_j} be the sequence x_{i_j} with the m_0 th position (counted from the end) deleted. By construction, the sequence $x_1, \dots, x_{i_1-1}, y_{i_1}, y_{i_2}, \dots$ is not bad and so some pair is comparable. Note that $y_{i_j} \leq x_{i_j}$, so x_i and y_{i_j} are incomparable, and so we have $j < k$ such that $y_{i_j} \leq y_{i_k}$. But this implies $x_{i_j} \leq x_{i_k}$, contradicting that x_1, x_2, \dots forms a bad sequence. Thus Σ^* is noetherian.

We now show that a poset ideal of Σ^* is an ordered language. It suffices to treat the case of principal ideals; the principal ideal generated by $w = w_1 \cdots w_n$ is the language

$$w_1 \Pi_1^* w_2 \Pi_2^* \cdots w_n \Pi_n^*,$$

where $\Pi_i = \{w_1, \dots, w_i\}$, which is clearly ordered. \square

Proof of Theorem 8.1.1. Clearly, $\mathcal{C} = \mathbf{OS}^{\text{op}}$ is directed. Let n be a non-negative integer, and regard $x = [n]$ as an object of \mathcal{C} . A morphism $f: [m] \rightarrow [n]$ can be regarded as a word of length m in the alphabet $\Sigma = [n]$. In this way, we have an injective map $i: |\mathcal{C}_x| \rightarrow \Sigma^*$. This map is strictly order-preserving with respect to the order on Σ^* defined above, and so Proposition 8.2.1 implies that $|\mathcal{C}_x|$ is noetherian. The lexicographic order on words induces an admissible order on $|\mathcal{C}_x|$. For $f \in |\mathcal{C}_x|$ we have $m = \nu(f) = \ell(i(f))$, and so we have a lingual structure on $|\mathcal{C}_x|$ if we norm words in Σ^* by their length. Finally, an ideal of $|\mathcal{C}_x|$ gives an ordered language over Σ by Proposition 8.2.1, and so this is an O-lingual structure. \square

8.3. Linear categories. Let R be a finite commutative ring. A linear map between free R -modules is **splittable** if the image is a direct summand. Let \mathbf{VA}_R (resp. \mathbf{VI}_R) be the category whose objects are finite rank free R -modules and whose morphisms are splittable maps (resp. splittable injections).

Theorem 8.3.1. *The categories \mathbf{VI}_R and \mathbf{VA}_R are quasi-Gröbner.*

Proof. Define a functor $\Phi: \mathbf{FS}^{\text{op}} \rightarrow \mathbf{VI}_R$ by $S \mapsto \text{Hom}_R(R[S], R) = R[S]^*$. It is clear that Φ is essentially surjective. We claim that Φ satisfies property (F). Fix $U \in \mathbf{VI}_R$. Pick a finite set S and a splittable injection $f: U \rightarrow R[S]^*$. Dualize this to get a surjection $R[S] \rightarrow U^*$. Letting $T \subseteq U^*$ be the image of S under this map, the map factorizes as $R[S] \rightarrow R[T] \rightarrow U^*$ where the first map comes from a surjective function $S \rightarrow T$. So we can take y_1, y_2, \dots to be the set of subsets of U^* which span U^* as an R -module (there are finitely many of them since R is finite) and $f_i: U \rightarrow R[T]^*$ to be the dual of the natural map

$R[T] \rightarrow U^*$. This establishes the claim. Also, as in the proof of Theorem 7.4.4, the inclusion functor $\mathbf{VI}_R \rightarrow \mathbf{VA}_R$ satisfies property (F). Since \mathbf{FS}^{op} is quasi-Gröbner (Theorem 8.1.2), the theorem follows from Proposition 5.2.6. \square

Remark 8.3.2. The idea of using the functor Φ in the above proof was communicated to us by Aurélien Djament after a first version of these results was circulated. The original version of the above proof involved working with a version of \mathbf{VI}_R consisting of spaces with ordered bases and upper-triangular linear maps. This idea is no longer needed to prove the desired properties for \mathbf{VI}_R , but is still needed in [PS] to prove noetherianity of a related category \mathbf{VIC}_R (splittable maps plus a choice of splitting). \square

Corollary 8.3.3. *If \mathbf{k} is left-noetherian then $\text{Rep}_{\mathbf{k}}(\mathbf{VI}_R)$ and $\text{Rep}_{\mathbf{k}}(\mathbf{VA}_R)$ are noetherian.*

Corollary 8.3.4. *Let M be a finitely generated \mathbf{VI}_R -module or \mathbf{VA}_R -module over a field \mathbf{k} . Then the Hilbert series*

$$H_M(t) = \sum_{n \geq 0} \dim M_{\mathbf{k}}([n]) \cdot t^n$$

is a rational function of the form $f(t)/g(t)$ where $g(t)$ factors as $\prod_{j=1}^r (1 - jt)^{e_r}$, for some r .

Remark 8.3.5. When $R = \mathbf{k}$ is a finite field, Corollary 8.3.3 proves the Lannes–Schwartz artinian conjecture [K2, Conjecture 3.12]. This is also a consequence of the results in [PS], and so a similar, but distinct, proof of the artinian conjecture appears there as well. The ideas for proving this conjecture were only made possible by work from both projects. \square

Remark 8.3.6. While preparing this article, we learned that [GL] proves that $\text{Rep}_{\mathbf{k}}(\mathbf{VI}_R)$ is noetherian in the special case that R is a field and \mathbf{k} is a field of characteristic 0. \square

Remark 8.3.7. Consider the category $\text{Rep}_{\mathbf{F}_q}(\mathbf{VA}_{\mathbf{F}_q})$. Since $\mathbf{VA}_{\mathbf{F}_q}$ is like a linear version of \mathbf{FA} , it is tempting to find a linear analogue of Theorem 8.4.4 below, but this is not possible. An explicit description of the submodule lattice of $P_{\mathbf{F}_q}^{\vee}$ is given in [K4, Theorem 6.4]. In particular, one can read off from this description that the Krull dimension of $P_{\mathbf{F}_q}$ is 1. It is conjectured that the Krull dimension of P_W is $\dim(W)$ in general [K4, Conjecture 6.8]. \square

8.4. Additional results. Recall that \mathbf{FA} is the category of finite sets.

Theorem 8.4.1. *The category \mathbf{FA}^{op} is quasi-Gröbner.*

Proof. The proof is similar to that of Theorem 7.4.4; the role of \mathbf{FI}_1 is played by \mathbf{FS}^{op} . \square

Corollary 8.4.2. *If \mathbf{k} is left-noetherian then $\text{Rep}_{\mathbf{k}}(\mathbf{FA}^{\text{op}})$ is noetherian.*

Proposition 8.4.3. *The category \mathbf{FA} satisfies property (D) (see Definition 5.3.3).*

Proof. Let x be a given finite set. Let $y = 2^x$ be the power set of x , and let $S \subset \text{Hom}(x, y)$ be the set of maps f for which $a \in f(a)$ for all $a \in x$. Let $f: x \rightarrow z$ be given. Define $g: z \rightarrow y$ by $g(a) = f^{-1}(a)$. If $f': x \rightarrow z$ is an arbitrary map then $g(f'(a)) = f^{-1}(f'(a))$ contains a if and only if $f(a) = f'(a)$. Thus $g_*(f')$ belongs to S if and only if $f = f'$. We conclude that $g_*^{-1}(S) = \{f\}$, which establishes the proposition. \square

We can now give our improvement of Corollary 7.4.5:

Theorem 8.4.4. *If \mathbf{k} is a field, then every finitely generated \mathbf{FA} -module is artinian, and so has a finite composition series. In other words, $\text{Rep}_{\mathbf{k}}(\mathbf{FA})$ has Krull dimension 0.*

Proof. We can use Proposition 5.3.5 since \mathbf{FA} is quasi-Gröbner (Theorem 7.4.4), \mathbf{FA}^{op} is quasi-Gröbner (Theorem 8.4.1), and \mathbf{FA} satisfies property (D) (Proposition 8.4.3). \square

Remark 8.4.5. $\text{Rep}_{\mathbf{k}}(\mathbf{FA}^{\text{op}})$ is not artinian: the principal projective P_S in \mathbf{FA} grows like a polynomial of degree $|S|$, i.e., $|\text{Hom}_{\mathbf{FA}}(S, T)| = |T|^{|S|}$, but the principal projectives in \mathbf{FA}^{op} grow like exponential functions, so the duals of projective objects in \mathbf{FA}^{op} cannot be finitely generated \mathbf{FA} -modules. In particular, \mathbf{FA}^{op} does not satisfy property (D). \square

Remark 8.4.6. In [Do, Theorem 1.3] and [Pa, Corollary 4.4], the functions $n \mapsto |F([n])|$ coming from functors $F: \mathbf{FA}^{\text{op}} \rightarrow \mathbf{FA}$ are classified. \square

Proposition 8.4.7. *The category \mathbf{FS}^{op} satisfies property (F).*

Proof. Let x_1 and x_2 be given finite sets. Given surjections $f_1: y \rightarrow x_1$ and $f_2: y \rightarrow x_2$, let y' be the image of y in $x_1 \times x_2$, and let $g: y \rightarrow y'$ be the quotient map. Then $f_i = p_i \circ g$, where $p_i: x_1 \times x_2 \rightarrow x_i$ is the projection map. Since there are only finitely many choices for (y', p_1, p_2) up to isomorphism, the result follows. \square

Corollary 8.4.8. *For any commutative ring \mathbf{k} , the pointwise tensor product of finitely generated representations of \mathbf{FS}^{op} is finitely generated.*

Proposition 8.4.9. *Let S be a finite set and let M be the \mathbf{FS}^{op} -module defined by $M(T) = \mathbf{k}[\text{Hom}_{\mathbf{FA}}(T, S)]$. Then M is finitely generated, and hence noetherian.*

Proof. Let S_1, \dots, S_n be the subsets of S . Then $M(T) = \bigoplus_{i=1}^n \mathbf{k}[\text{Hom}_{\mathbf{FS}}(T, S_i)]$. The i th summand is exactly the principal projective at S_i . \square

9. Δ -MODULES

We now apply our methods to the theory of Δ -modules, originally introduced by the second author in [Sn]. In §9.1, we recall the definitions, the results of [Sn], and state our new results. In §9.2, we explain how our results yield new information on syzygies of Segre embeddings. The remainder of §9 is devoted to the proofs.

9.1. Background and statement of results. A Δ -module is a sequence $(F_n)_{n \geq 0}$, where F_n is an S_n -equivariant polynomial functor $\text{Vec}^n \rightarrow \text{Vec}$, equipped with transition maps

$$(9.1.1) \quad F_n(V_1, \dots, V_{n-1}, V_n \otimes V_{n+1}) \rightarrow F_{n+1}(V_1, \dots, V_{n+1})$$

satisfying certain compatibilities. (See §9.3 for background on polynomial functors.) To state these compatibilities succinctly, it is convenient to introduce an auxiliary category Vec^Δ . Objects of Vec^Δ are finite collections of vector spaces $\{V_i\}_{i \in I}$. A morphism $\{V_i\}_{i \in I} \rightarrow \{W_j\}_{j \in J}$ is a surjection $f: J \rightarrow I$ together with, for each $i \in I$, a linear map $\eta_i: V_i \rightarrow \bigotimes_{f(j)=i} W_j$. A Δ -module is then a polynomial functor $F: \text{Vec}^\Delta \rightarrow \text{Vec}$, where polynomial means that the functor $F_n: \text{Vec}^n \rightarrow \text{Vec}$ given by $(V_1, \dots, V_n) \mapsto F(\{V_i\}_{i \in [n]})$ is polynomial for each n . We write Mod_Δ for the category of Δ -modules.

Let F be a Δ -module. An **element** of F is an element of $F(\{V_i\})$ for some object $\{V_i\}$ of Vec^Δ . Given a set S of elements of F , the submodule of F **generated** by S is the smallest Δ -submodule of F containing S . Then F is **finitely generated** if it is generated by a finite set of elements, and F is **noetherian** if every Δ -submodule of F is finitely generated.

Let Λ be the ring of symmetric functions (see [St2, Ch. 7]). Given a finite length polynomial representation V of $\mathbf{GL}(\infty)^n$ over \mathbf{k} , we define its character $\tilde{\text{ch}}(V) \in \Lambda^{\otimes n}$ by

$$\tilde{\text{ch}}(V) = \sum_{(\lambda_1, \dots, \lambda_n)} \dim_{\mathbf{k}}(V_{\lambda_1, \dots, \lambda_n}) \cdot m_{\lambda_1} \otimes \cdots \otimes m_{\lambda_n},$$

where the sum is over all n -tuples of partitions, $V_{\lambda_1, \dots, \lambda_n}$ is the $(\lambda_1, \dots, \lambda_n)$ -weight space of V , and m_{λ} is the monomial symmetric function indexed by λ . We let $\text{ch}(V)$ be the image of $\tilde{\text{ch}}(V)$ in $\text{Sym}^n(\Lambda)$, a polynomial of degree n in the m_{λ} . If V is an S_n -equivariant representation then $\tilde{\text{ch}}(V)$ belongs to $(\Lambda^{\otimes n})^{S_n}$, and so passing to $\text{ch}(V)$ does not lose information. If $F: \text{Vec}^n \rightarrow \text{Vec}$ is a finite length polynomial functor, define $\text{ch}(F) := \text{ch}(F(\mathbf{k}^{\infty}, \dots, \mathbf{k}^{\infty}))$.

Suppose now that $F = (F_n)_{n \geq 0}$ is a Δ -module, and each F_n has finite length. We define two notions of **Hilbert series** of F :

$$H'_F(\mathbf{m}) = \sum_{n \geq 0} \frac{\text{ch}(F_n)}{n!}, \quad H_F(\mathbf{m}) = \sum_{n \geq 0} \text{ch}(F_n).$$

These are elements of $\widehat{\text{Sym}}(\Lambda_{\mathbf{Q}}) \cong \mathbf{Q}[[\mathbf{m}]]$. If F is finitely generated, then each F_n is finite length; moreover, only finitely many of the m_{λ} appear in $H'_F(\mathbf{m})$ and $H_F(\mathbf{m})$.

In [Sn], a Δ -module is called “small” if it appears as a submodule of a Δ -module (F_n) generated by F_1 , and the following abstract result is proved:

Theorem 9.1.2 (Snowden). *Suppose \mathbf{k} has characteristic 0 and F is a finitely generated small Δ -module. Then F is noetherian and $H_F(\mathbf{m})$ is a rational function of the m_{λ} .*

Our main result, which we prove below, is the following:

Theorem 9.1.3. *Let F be a finitely generated Δ -module. Then F is noetherian and $H'_F(\mathbf{m})$ is a polynomial in the m_{λ} and the $e^{m_{\lambda}}$.*

This result improves Theorem 9.1.2 in three ways:

- (1) there is no restriction on the characteristic of \mathbf{k} ;
- (2) there is no restriction to small Δ -modules; and
- (3) the Hilbert series statement is significantly stronger.

To elaborate on (3), our result implies that the denominator of $H_F(\mathbf{m})$ factors into linear pieces of the form $1 - \alpha$, where α is an \mathbf{N} -linear combination of the m_{λ} . This stronger result on Hilbert series answers [Sn, Question 5] affirmatively, and even goes beyond what is asked there. (Note that the question in [Sn] is phrased in terms of Schur functions rather than monomial symmetric functions, but they are related by a change of basis with integer coefficients [St2, Corollary 7.10.6].)

9.2. Applications. The motivating example of a Δ -module comes from the study of syzygies of the Segre embedding. Define the **Segre product** of two graded \mathbf{k} -algebras A and B to be the graded \mathbf{k} -algebra $A \boxtimes B$ given by

$$(A \boxtimes B)_n = A_n \otimes_{\mathbf{k}} B_n.$$

Given finite-dimensional \mathbf{k} -vector spaces V_1, \dots, V_n , put

$$R(V_1, \dots, V_n) = \text{Sym}(V_1) \boxtimes \cdots \boxtimes \text{Sym}(V_n), \quad S(V_1, \dots, V_n) = \text{Sym}(V_1 \otimes \cdots \otimes V_n).$$

Then R is an S -algebra, and applying Proj gives the Segre embedding

$$i(V_1, \dots, V_n): \mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_n) \rightarrow \mathbf{P}(V_1 \otimes \cdots \otimes V_n).$$

Fix an integer $p \geq 0$, and define

$$(9.2.1) \quad F_n(V_1, \dots, V_n) = \mathrm{Tor}_p^{S(V_1, \dots, V_n)}(R(V_1, \dots, V_n), \mathbf{k}).$$

This is the space of p -syzygies of the Segre. The factorization

$$i(V_1, \dots, V_{n+1}) = i(V_1, \dots, V_{n-1}, V_n \otimes V_{n+1}) \circ (\mathrm{id} \times i(V_n, V_{n+1})),$$

combined with general properties of Tor , yield transition maps as in (9.1.1). In this way, the sequence $F = (F_n)_{n \geq 0}$ naturally has the structure of a Δ -module.

Theorem 9.2.2. *For every $p \geq 0$ and field \mathbf{k} , the Δ -module F defined by (9.2.1) is finitely presented and $H'_F(\mathbf{m})$ is a polynomial in the m_λ and the e^{m_λ} .*

Proof. The Tor modules are naturally $\mathbf{Z}_{\geq 0}$ -graded and are concentrated between degrees p and $2p$: to see this, we first note that the Segre variety has a quadratic Gröbner basis and the dimensions of these Tor modules are bounded from above by those of the initial ideal. The Taylor resolution of a monomial ideal gives the desired bounds [Ei, Exercise 17.11].

Using the Koszul resolution of \mathbf{k} as an S -module, each graded piece of F can be realized as the homology of a complex of finitely generated Δ -modules; now use Theorem 9.1.3. \square

Finite generation means that there are finitely many p -syzygies of Segres that generate all p -syzygies of all Segres under the action of general linear groups, symmetric groups, and the maps (9.1.1), i.e., pullbacks along Segre embeddings. The statement about Hilbert series means that, with a single polynomial, one can store all of the characters of the \mathbf{GL} actions on spaces of p -syzygies. We refer to the introduction of [Sn] for a more detailed account.

Remark 9.2.3. The syzygies of many varieties related to the Segre, such as secant and tangential varieties of the Segre, also admit the structure of a Δ -module. See [Sn, §4]. In fact, the argument in Theorem 9.2.2 almost goes through: one can show that each graded piece of each Tor module is finitely generated and has a rational Hilbert series, but there are few general results about these Tor modules being concentrated in finitely many degrees.

The results that we know of are in characteristic 0: [Ra] proves that the ideal of the secant variety of the Segre is generated by cubic equations, and [OR] proves that the tangential variety of the Segre is generated by equations of degree ≤ 4 . \square

9.3. Polynomial functors. We give some background on polynomial functors. This material is standard (and first appeared in [FS]), but we could not find a convenient reference for multivariate functors, so we give a self-contained treatment here. Fix an infinite field \mathbf{k} and let Vec be the category of finite-dimensional vector spaces over \mathbf{k} .

A **(strict) polynomial functor** $F: \mathrm{Vec} \rightarrow \mathrm{Vec}$ of degree d is a functor such that for all V and W , the map

$$\mathrm{Hom}_{\mathbf{k}}(V, W) \rightarrow \mathrm{Hom}_{\mathbf{k}}(F(V), F(W))$$

is given by homogeneous polynomial functions of degree d .

Lemma 9.3.1. *Let F be a polynomial functor of degree d taking values in finite-dimensional vector spaces. The function $n \mapsto \dim_{\mathbf{k}} F(\mathbf{k}^n)$ is weakly increasing and is a polynomial of degree $\leq d$. In particular, if $F(\mathbf{k}^d) = 0$, then $F = 0$.*

Proof. This is similar to the proof of Theorem 7.4.6. Define $h_F(n) = \dim_{\mathbf{k}} F(\mathbf{k}^n)$. Define ΣF by $V \mapsto F(V \oplus \mathbf{k})$. There is a natural splitting $\Sigma F = F \oplus \Delta F$, so $h_{\Delta F}(n) = h_F(n+1) - h_F(n)$, and $h_F(n)$ is a weakly increasing function. It follows from the definition that ΔF is a polynomial functor of degree $< d$, so $\Delta^{d+1} F = 0$. Thus $h_{\Delta F}(n)$ is a polynomial of degree

$\leq d$. For the last sentence, if $F(\mathbf{k}^d) = 0$, then the above says that $h_F(n) = 0$ for $n = 0, \dots, d$. Since $h_F(n)$ is a polynomial of degree $\leq d$, it must be identically 0. \square

A polynomial functor is a direct sum of polynomial functors of degree d for various d . Since \mathbf{k} is infinite, a degree d polynomial map of finite-dimensional vector spaces $U \rightarrow U'$ is the same as an element of $\text{Sym}^d(U^*) \otimes U'$, or equivalently, a linear map $D^d(U) \rightarrow U'$, where D is the divided power functor. Hence we can rephrase the definition as follows. Let $D^d(\text{Vec})$ be the category whose objects are finite-dimensional \mathbf{k} -vector spaces and

$$\text{Hom}_{D^d(\text{Vec})}(U, U') = D^d \text{Hom}_{\mathbf{k}}(U, U').$$

Then a polynomial functor of degree d is a linear functor $D^d(\text{Vec}) \rightarrow \text{Vec}$. (If we allow \mathbf{k} to be finite, then this would be used as the definition of polynomial functor.)

We extend this definition to multivariate functors. A **(strict) polynomial functor** $F: \text{Vec}^n \rightarrow \text{Vec}$ of multidegree $\mathbf{d} = (d_1, \dots, d_n)$ is a functor such that for all V_1, \dots, V_n and W_1, \dots, W_n , the map

$$\text{Hom}_{\mathbf{k}}(V_1, W_1) \times \dots \times \text{Hom}_{\mathbf{k}}(V_n, W_n) \rightarrow \text{Hom}_{\mathbf{k}}(F(V_1, \dots, V_n), F(W_1, \dots, W_n))$$

is given by homogeneous polynomial functions of multidegree \mathbf{d} . A polynomial functor is a direct sum of polynomial functors of multidegree \mathbf{d} for various \mathbf{d} . As above, define a category $D^{\mathbf{d}}(\text{Vec})$ whose objects are n -tuples of vector spaces $\underline{V} = (V_1, \dots, V_n)$ and morphisms are

$$\text{Hom}_{D^{\mathbf{d}}(\text{Vec})}(\underline{V}, \underline{W}) = D^{d_1}(\text{Hom}_{\mathbf{k}}(V_1, W_1)) \otimes \dots \otimes D^{d_n}(\text{Hom}_{\mathbf{k}}(V_n, W_n)).$$

So polynomial functors of multidegree \mathbf{d} are equivalent to linear functors $D^{\mathbf{d}}(\text{Vec}) \rightarrow \text{Vec}$.

Given vector spaces $\underline{U} = (U_1, \dots, U_n)$, define a polynomial functor $P_{\underline{U}}$ by

$$P_{\underline{U}}(\underline{V}) = \text{Hom}_{D^{\mathbf{d}}(\text{Vec})}(\underline{U}, \underline{V}).$$

For a partition $\lambda = (\lambda_1, \dots, \lambda_s)$, set $D^{\lambda}(V) = D^{\lambda_1}(V) \otimes \dots \otimes D^{\lambda_s}(V)$. For a sequence of partitions $\Lambda = (\lambda^1, \dots, \lambda^n)$, define $D^{\Lambda}: \text{Vec}^n \rightarrow \text{Vec}$ by

$$D^{\Lambda}(V_1, \dots, V_r) = D^{\lambda^1}(V_1) \otimes \dots \otimes D^{\lambda^n}(V_n).$$

Proposition 9.3.2. *The D^{Λ} are projective generators for the category of polynomial functors $\text{Vec}^n \rightarrow \text{Vec}$.*

Proof. Every polynomial functor $\text{Vec}^n \rightarrow \text{Vec}$ naturally breaks up as a direct sum of polynomial functors of a given multidegree, so it suffices to fix a multidegree \mathbf{d} .

The functor $P_{\underline{U}}$ is projective by Yoneda's lemma. When $n = 1$, there is a natural map $F(U) \otimes P_U \rightarrow F$ whose image F' satisfies $F'(U) = F(U)$. So if $\dim(U) \geq d$, then $F/F' = 0$ by Lemma 9.3.1, and hence P_U is a projective generator in this case. Similarly, $P_{\underline{U}}$ is a projective generator if $\dim(U_i) \geq d_i$ for all i . Since $P_{\underline{U}}$ is isomorphic to a direct sum of functors of the form D^{Λ} , they are also projective generators. \square

Finally, if V_i are finite-dimensional and $V = \text{colim } V_i$, then we extend the polynomial functors by $F(V) := \text{colim } F(V_i)$ and similarly for multivariate polynomial functors.

9.4. Proof of Theorem 9.1.3: noetherianity. Let N be a positive integer and define a functor $\Phi_N: \mathbf{FS}^{\text{op}} \rightarrow \text{Vec}^\Delta$ as follows. Given a finite set S , we let $\Phi_N(S)$ be the family $\{V_i\}_{i \in S}$ where $V_i = \mathbf{k}^N$ for all i . Given a surjection $f: S \rightarrow T$, we let $\Phi_N(f)$ be the map $\Phi_N(T) \rightarrow \Phi_N(S)$ that is f on index sets, and where the linear map $\eta_t: \mathbf{k}^N \rightarrow \bigotimes_{f(s)=t} \mathbf{k}^N$ takes the basis vector e_i to $\bigotimes_{f(s)=t} e_i$. We thus obtain a functor $\Phi_N^*: \text{Mod}_\Delta \rightarrow \text{Rep}_\mathbf{k}(\mathbf{FS}^{\text{op}})$.

Let $\mathbf{d} = (d_1, \dots, d_r)$ be a tuple of positive integers. Define a Δ -module $\mathcal{T}^\mathbf{d}$ by

$$\mathcal{T}^\mathbf{d}(\{V_i\}_{i \in I}) = \bigoplus_{\alpha: I \rightarrow [r]} \bigotimes_{i \in I} V_i^{\otimes d_{\alpha(i)}},$$

where the sum is over all surjections α from I to $[r]$.

We now proceed to show that $\mathcal{T}^\mathbf{d}$ is a noetherian Δ -module and that every finitely generated Δ -module is a subquotient of a finite direct sum of $\mathcal{T}^\mathbf{d}$'s. This will show that finitely generated Δ -modules are noetherian.

Lemma 9.4.1. *For all N , the representation $\Phi_N^*(\mathcal{T}^\mathbf{d})$ of \mathbf{FS}^{op} is noetherian.*

Proof. Put $M = \Phi_N^*(\mathcal{T}^\mathbf{d})$. We have

$$M(S) = \bigoplus_{\alpha: S \rightarrow [r]} \bigotimes_{i \in S} (\mathbf{k}^N)^{\otimes d_{\alpha(i)}}.$$

Let $\mathcal{F}(d)$ be the set of all functions $[d] \rightarrow [N]$, which naturally indexes a basis of $(\mathbf{k}^N)^{\otimes d}$. The space $M(S)$ has a basis indexed by pairs (α, β) , where α is a surjection $S \rightarrow [r]$ and β is a function assigning to each element $i \in S$ an element of $\mathcal{F}(d_{\alpha(i)})$. Write $e_{(\alpha, \beta)}$ for the basis vector corresponding to (α, β) . If $f: T \rightarrow S$ is a surjection and $f^*: M(S) \rightarrow M(T)$ is the induced map then $f^*(e_{(\alpha, \beta)}) = e_{(f^*(\alpha), f^*(\beta))}$. A pair (α, β) defines a function $S \rightarrow [r] \times \mathcal{F}$, where $\mathcal{F} = \mathcal{F}(d_1) \amalg \dots \amalg \mathcal{F}(d_r)$. The previous remark shows that M is a submodule of the \mathbf{FS}^{op} -module defined by $T \mapsto \mathbf{k}[\text{Hom}_{\mathbf{FA}}(T, [r] \times \mathcal{F})]$, which is noetherian (Proposition 8.4.9), so the result follows. \square

Proposition 9.4.2. *The Δ -module $\mathcal{T}^\mathbf{d}$ is noetherian.*

Proof. Let M be a submodule of $\mathcal{T}^\mathbf{d}$. We claim that M is generated by its restriction to the subcategory $\Phi_N(\mathbf{FS}^{\text{op}})$, where $N = \max(d_i)$. To see this, pick an element $v \in M_n(\mathbf{k}^{s_1}, \dots, \mathbf{k}^{s_n})$ for some s_i . Then v is a linear combination of weight vectors, so we may assume that v is a weight vector. A weight vector in $\mathcal{T}^\mathbf{d}(\mathbf{k}^{s_1}, \dots, \mathbf{k}^{s_n})$ uses at most N basis vectors from each space. Thus there is an element $g \in \prod_{i=1}^n \mathbf{GL}(s_i)$ such that $gv \in \mathcal{T}^\mathbf{d}(\mathbf{k}^N, \dots, \mathbf{k}^N)$. This proves the claim. It follows that if $M \subsetneq M' \subseteq \mathcal{T}^\mathbf{d}$ then $\Phi_N^*(M) \subsetneq \Phi_N^*(M')$. Now use Lemma 9.4.1. \square

We now start on the second part of our plan. Let \mathbf{T} be the category whose objects are sequences $F = (F_n)_{n \geq 0}$ where F_n is a polynomial functor $\text{Vec}^n \rightarrow \text{Vec}$, with no extra data. Given an object F of this category, we define a Δ -module $\Gamma(F)$ by

$$\Gamma(F)(\{V_i\}_{i \in I}) = \bigoplus_{0 \leq n \leq |I|} \bigoplus_{\alpha: I \rightarrow [n]} F_n(U_1, \dots, U_n),$$

where $U_j = \bigotimes_{\alpha(i)=j} V_i$ and the second sum is over all surjective functions $\alpha: I \rightarrow [n]$.

Proposition 9.4.3. (a) Γ is the left adjoint of the forgetful functor $\text{Mod}_\Delta \rightarrow \mathbf{T}$.
 (b) Γ is an exact functor.

Proof. (a) Pick $G \in \text{Mod}_\Delta$ and $F \in \mathbf{T}$ and $f \in \text{Hom}_\mathbf{T}(F, G)$. Suppose that $\{V_i\}_{i \in I}$ is an object of Mod_Δ and $\alpha: I \rightarrow [n]$ is a surjection; let U_j be as above. We have maps

$$F_n(U_1, \dots, U_n) \rightarrow G_n(U_1, \dots, U_n) \rightarrow G(\{V_i\}),$$

where the first is f and the second uses functoriality of G with respect to the morphism $\{U_j\}_{j \in [n]} \rightarrow \{V_i\}_{i \in I}$ in Vec^Δ corresponding to α (and the identity maps). Summing over all choices of α gives a map $(\Gamma F)(\{V_i\}) \rightarrow G(\{V_i\})$, and thus a map of Δ -modules $\Gamma(F) \rightarrow G$.

(b) A Δ -module is a functor $\text{Vec}^\Delta \rightarrow \text{Vec}$, so exactness can be checked pointwise. Similarly, exactness of a sequence of objects in \mathbf{T} can be checked pointwise. So let $0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow 0$ be an exact sequence in \mathbf{T} and pick $\{V_i\}_{i \in I} \in \text{Vec}^\Delta$. The sequence

$$(\Phi F_1)(\{V_i\}) \rightarrow (\Phi F_2)(\{V_i\}) \rightarrow (\Phi F_3)(\{V_i\})$$

is the direct sum (over all surjective functions $\alpha: I \twoheadrightarrow [n]$) of sequences of the form

$$F_1(U_1, \dots, U_n) \rightarrow F_2(U_1, \dots, U_n) \rightarrow F_3(U_1, \dots, U_n)$$

The latter are exact by assumption, so our sequence of interest is also exact. \square

Define \mathcal{D}^Δ to be the Δ -module $\Gamma(\mathcal{D}^\Delta)$, where \mathcal{D}^Δ (defined in §9.1) is regarded as an object of \mathbf{T} concentrated in degree r .

Proposition 9.4.4. *The \mathcal{D}^Δ are projective generators for Mod_Δ .*

Proof. By Proposition 9.3.2, the functors \mathcal{D}^Δ are projective generators for \mathbf{T} . Since Φ is the left adjoint of an exact functor, it follows that \mathcal{D}^Δ are projective generators. \square

Corollary 9.4.5. *A finitely generated Δ -module is a subquotient of a finite direct sum of $\mathcal{T}^\mathbf{d}$.*

Proof. Given $\mathbf{d} = (d_1, \dots, d_r)$, let $\mathbf{T}^\mathbf{d}: \text{Vec}^r \rightarrow \text{Vec}$ be the functor given by

$$\mathbf{T}^\mathbf{d}(V_1, \dots, V_r) = V_1^{\otimes d_1} \otimes \dots \otimes V_r^{\otimes d_r}.$$

Then $\mathcal{T}^\mathbf{d} = \Gamma(\mathbf{T}^\mathbf{d})$. Since \mathcal{D}^Δ is a subobject of $\mathbf{T}^\mathbf{d}$, where $d_i = |\lambda^i|$, it follows that \mathcal{D}^Δ is a submodule of $\mathcal{T}^\mathbf{d}$. Now use Corollary 9.4.5. \square

Corollary 9.4.5 and Proposition 9.4.2 prove the noetherianity statement in Theorem 9.1.3.

9.5. Proof of Theorem 9.1.3: Hilbert series. By Corollary 9.4.5, the Grothendieck group of finitely generated Δ -modules is spanned by the classes of submodules of $\mathcal{T}^\mathbf{d}$. Therefore, it suffices to analyze the Hilbert series of a submodule M of $\mathcal{T}^\mathbf{d}$.

Let $N = \max(d_i)$, and write $\Phi = \Phi_N$ (§9.4). The space $\Phi^*(M)(S)$ has an action of the group $\mathbf{GL}(N)^S$ that we exploit. Let $\lambda_1, \dots, \lambda_n$ be the partitions of size at most N . These are the only partitions that appear as weights in $(\mathbf{k}^N)^{\otimes d_i}$. Let (S_1, \dots, S_n) be a tuple of finite (possibly empty) sets and let S be their disjoint union. Define $\Psi(M)(S_1, \dots, S_n)$ to be the subspace of $\Phi^*(M)(S)$ where the torus in the i th copy of $\mathbf{GL}(N)$ acts by weight λ_j , where $i \in S_j$. If $f_i: S_i \rightarrow T_i$ are surjections, and $f: S \rightarrow T$ is their disjoint union, then the map $f^*: \Phi^*(M)(T) \rightarrow \Phi^*(M)(S)$ carries $\Psi(M)(T_1, \dots, T_n)$ into $\Psi(M)(S_1, \dots, S_n)$. Thus $\Psi(M)$ is an $(\mathbf{FS}^{\text{op}})^n$ -module. If M' is a submodule of $\Psi(M)$ then $S \mapsto \bigoplus_{S=S_1 \amalg \dots \amalg S_n} M'(S_1, \dots, S_n)$ is an \mathbf{FS}^{op} -submodule of $\Phi^*(M)$. Since $\Phi^*(M)$ is noetherian, it follows that $\Psi(M)$ is noetherian. In particular, $\Psi(M)$ is finitely generated.

Set $V = M_e(\mathbf{k}^N, \dots, \mathbf{k}^N)$. The weight space V_{μ_1, \dots, μ_e} is isomorphic to $\Psi(M)([e_1], \dots, [e_n])$ where $e_i = \#\{j \mid \mu_j = \lambda_i\}$. The coefficient of $m_{\lambda_1}^{e_1} \cdots m_{\lambda_n}^{e_n}$ in $H'_M(\mathbf{m})$ is then

$$\frac{1}{e!} \sum_{\substack{\mu_1, \dots, \mu_e \\ \#\{j \mid \mu_j = \lambda_i\} = e_i}} \dim_{\mathbf{k}}(V_{\mu_1, \dots, \mu_e}) = \frac{1}{e_1! \cdots e_n!} \dim_{\mathbf{k}}(\Psi(M)([e_1], \dots, [e_n])).$$

It follows that $H'_M(\mathbf{m})$ is equal to $H'_{\Psi(M)}(\mathbf{t})$, and so the theorem follows from Corollary 8.1.7.

10. ADDITIONAL EXAMPLES

10.1. Categories of G -sets. The proofs of the results in this section appear in [SS4].

Let G be a group. A G -map $S \rightarrow T$ between finite sets S and T is a pair (f, σ) consisting functions $f: S \rightarrow T$ and $\sigma: S \rightarrow G$. Given G -maps $(f, \sigma): S \rightarrow T$ and $(g, \tau): T \rightarrow U$, their composition is the G -map $(h, \eta): S \rightarrow U$ with $h = g \circ f$ and $\eta(x) = \sigma(x)\tau(f(x))$, the product taken in G . In this way, we have a category \mathbf{FA}_G whose objects are finite sets and whose morphisms are G -maps. The automorphism group of $[n]$ in \mathbf{FA}_G is $S_n \wr G = S_n \ltimes G^n$. Let \mathbf{FS}_G (resp. \mathbf{FI}_G) denote the subcategory of \mathbf{FA}_G containing all objects but only those morphisms (f, σ) with f surjective (resp. injective).

Remark 10.1.1. The category $\mathbf{FI}_{\mathbf{Z}/2\mathbf{Z}}$ is equivalent to the category \mathbf{FI}_{BC} defined in [Wi, Defn. 1.2]. Representations of $\prod_{n=0}^{\infty} \mathbf{FS}_n^{\text{op}}$ are equivalent to Δ -modules (in good characteristic). Thus the representation theory \mathbf{FI}_G and $\mathbf{FS}_G^{\text{op}}$ generalizes many examples of interest. This motivates the results stated below. \square

Recall that a group G is **polycyclic** if it has a finite composition series $1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_r = G$ such that G_i/G_{i-1} is cyclic for $i = 1, \dots, r$, and it is **polycyclic-by-finite** if it contains a polycyclic subgroup of finite index.

Theorem 10.1.2. *If G is finite then \mathbf{FI}_G , $\mathbf{FS}_G^{\text{op}}$, \mathbf{FA}_G , and $\mathbf{FA}_G^{\text{op}}$ are quasi-Gröbner. Finitely generated modules over any of these categories have rational Hilbert series.*

If G is polycyclic-by-finite and \mathbf{k} is left-noetherian, then $\text{Rep}_{\mathbf{k}}(\mathbf{FI}_G)$ and $\text{Rep}_{\mathbf{k}}(\mathbf{FA}_G)$ are noetherian.

Let M be a finitely generated $\mathbf{FS}_G^{\text{op}}$ -module over an algebraically closed field \mathbf{k} . Then $M([n])$ is a finite-dimensional representation of G^n . Let $[M]_n$ be the image of the class of this representation under the map $\mathcal{R}_{\mathbf{k}}(G^n) = \mathcal{R}_{\mathbf{k}}(G)^{\otimes n} \rightarrow \text{Sym}^n(\mathcal{R}_{\mathbf{k}}(G))$ ($\mathcal{R}_{\mathbf{k}}(H)$ is the Grothendieck group of all finitely generated $\mathbf{k}[H]$ -modules). One can recover the isomorphism class of $M([n])$ as a representation of G^n from $[M]_n$ due to the S_n -equivariance. If $\{L_j\}$ are the irreducible representations of the G , then $[M]_n$ is a polynomial in corresponding variables $t_j = [L_j]$. Define the **enhanced Hilbert series** of M (see §1.5.2) by

$$\mathbf{H}_M(t) = \sum_{n \geq 0} [M]_n \in \widehat{\text{Sym}}(\mathcal{R}_{\mathbf{k}}(G)_{\mathbf{Q}}) \cong \mathbf{Q}[[t_j]].$$

Let N be the exponent of G (LCM of all orders of elements) and fix a primitive N th root of unity ζ_N . The following is a simplified version of our main theorem on Hilbert series.

Theorem 10.1.3. *Notation as above. Then $\mathbf{H}_M(\mathbf{t}) = f(\mathbf{t})/g(\mathbf{t})$ where $f(\mathbf{t}) \in \mathbf{Q}(\zeta_N)[\mathbf{t}]$ and $g(\mathbf{t}) = \prod_k (1 - \lambda_k)$ where each λ_k is a $\mathbf{Z}[\zeta_N]$ -linear combination of the t_i .*

10.2. An example with non-regular languages. Let \mathbf{OI}_d^- be the subcategory of \mathbf{OI}_d containing all objects but only those morphisms (f, g) for which all fibers of g have the same size. The inclusion $\mathbf{OI}_d^- \subset \mathbf{OI}_d$ satisfies property (S) (Definition 5.3.1), so \mathbf{OI}_d^- is Gröbner by Proposition 5.3.2 and Theorem 7.1.1. Endow \mathbf{OI}_d^- with the restricted norm from \mathbf{OI}_d . Obviously, $\mathbf{OI}_1^- = \mathbf{OI}_1$, which is O-lingual (Theorem 7.1.1). We now examine $d > 1$.

Proposition 10.2.1. *The normed category \mathbf{OI}_2^- is UCF-lingual.*

Proof. Let $\mathcal{C} = \mathbf{OI}_2^-$ and let $\mathcal{C}' = \mathbf{OI}_2$. Let $x = [n]$ be an object of \mathcal{C} and let $\Sigma = \{0, 1, 2\}$. We regard $|\mathcal{C}_x|$ as a subset of $|\mathcal{C}'_x|$ with the induced order, which is admissible by Proposition 5.3.2. Define the map $i: |\mathcal{C}'_x| \rightarrow \Sigma^*$ as in the proof of Theorem 7.1.1.

If T is an ideal of $|\mathcal{C}_x|$ and S is the ideal of $|\mathcal{C}'_x|$ it generates, then $T = S \cap |\mathcal{C}_x|$. Thus $i(T)$ is the intersection of the regular language $i(S)$ with the language $\mathcal{L} = i(|\mathcal{C}_x|)$. The language \mathcal{L} consists of words in Σ that contain exactly n 0's and use the symbols 1 and 2 the same number of times. This is a deterministic context-free (DCF) language (the proof is similar to [HU, Exercise 5.1]), so the same is true for $i(T)$ [HU, Thm. 10.4]. Finally, DCF implies UCF (see the proof of [HU, Thm. 5.4]). \square

Corollary 10.2.2. *If M is a finitely-generated representation of \mathbf{OI}_2^- then $H_M(t)$ is an algebraic function of t .*

Example 10.2.3. Let $d \geq 1$ be arbitrary. Let $M_d \in \text{Rep}_{\mathbf{k}}(\mathbf{OI}_d^-)$ be the principal projective at $[0]$, and let H_d be its Hilbert series. The space $M_d([n])$ has for a basis the set of all strings in $\{1, \dots, d\}$ of length n in which the numbers $1, \dots, d$ occur equally. The number of strings in which i occurs exactly n_i times is the multinomial coefficient $\binom{n_1 + \dots + n_d}{n_1, \dots, n_d}$.

It follows that $H_d(t) = \sum_{n \geq 0} \frac{(dn)!}{n!^d} t^{dn}$. So $H_1(t) = (1 - t)^{-1}$ and $H_2(t) = (1 - 4t^2)^{-1/2}$ but $H_d(t)$ is not algebraic for $d > 2$ [WS, Thm. 3.8]. \square

Corollary 10.2.4. *If $d > 2$ then \mathbf{OI}_d^- is not UCF-lingual.*

Let \mathbf{FI}_d^- be the subcategory of \mathbf{FI}_d defined in a way similar to \mathbf{OI}_d^- with the induced norm. Then \mathbf{FI}_d^- is quasi-Gröbner. The above results and examples show that the Hilbert series of a finitely generated \mathbf{FI}_2^- -module is algebraic, but that this need not be the case for \mathbf{FI}_d^- -modules with $d > 2$. We have shown, by very different means, that for $d > 2$ the Hilbert series are always D-finite (in characteristic 0).

The $\text{tca Sym}(\text{Sym}^2(\mathbf{C}^\infty))$ is the coordinate ring of infinite size symmetric matrices, so has a natural rank stratification, and a module is **bounded** if it lives in one of the finite strata. The above results on \mathbf{FI}_2^- imply that a module supported in rank ≤ 2 have algebraic Hilbert series. This motivates the following conjecture, which we have verified for rank ≤ 3 :

Conjecture 10.2.5. *A bounded $\text{Sym}(\text{Sym}^2(\mathbf{C}^\infty))$ -module has an algebraic Hilbert series.*

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