# The Jacquet-Langlands Correspondence for GL(2)

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# Abstract

Let $F$ be a finite extension of $\mathbb{Q}_p$, let $G$ be the group $\mathrm{GL}_n(F)$ and let $G'$ be the multiplicative group of a central simple division algebra algebra over $F$ of rank $n^2$. The Jacquet-Langlands correspondence is a natural bijection between the set of isomorphism classes of finite dimensional irreducible representations of $G'$ and the set of isomorphism classes of (essentially) square-integrable irreducible admissible representations of $G$ (all of which are infinite dimensional). This purely local result does not have a local proof for $n > 2$. We give a new purely local proof in the $n = 2$ case which should generalize at least to $n = 3$.

Our proof relies heavily on the Fourier transform. Let $X'$ be the space of monic degree two polynomials over $F$ with non-zero constant term which are either irreducible or have a doubled root. We identify $X'$ with the space of conjugacy classes in $G'$ and also the space of elliptic conjugacy classes in $G$. Using the Fourier transform on the $2 \times 2$ matrix algebra we construct a Fourier transform on $X'$ and show that this transform determines which functions on $X'$ are characters of cuspidal representations of $G$. Using the Fourier transform on the non-split quaternion algebra we construct another Fourier transform on $X'$, and show that it determines which functions on $X'$ are characters of irreducible representations of $G'$. Finally, we show that the two Fourier transforms on $X'$ agree (up to a sign) which gives the correspondence.

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To Professor Kadir Aziz,

who has touched me with his kindness and generosity.

# Contents

# Chapter 1

# Introduction

**(1.1)** Let $F$ be a finite extension of $\mathbb{Q}_p$, let $G$ be the group $\mathrm{GL}_2(F)$ and let $G'$ be the group of units in the unique non-split quaternion algebra over $F$. We write $\mathrm{Irr}_G^\circ$ for the set of irreducible admissible cuspidal representations of $G$ and $\mathrm{Irr}_{G'}^\circ$ for the set of irreducible representations of $G'$ of dimension at least two. (All representations are on complex vector spaces.) The Jacquet-Langlands correspondence states that there is a bijection[1]

$$\mathrm{Irr}_G^\circ \to \mathrm{Irr}_{G'}^\circ$$

characterized by $\pi \to \pi'$ if $\chi_\pi(g) = -\chi_{\pi'}(g')$ whenever $g$ is a regular elliptic element of $G$ and $g'$ is an element of $G'$ with the same characteristic polynomial as $g$. Here $\chi_\pi$ and $\chi_{\pi'}$ denote the characters of $\pi$ and $\pi'$. Furthermore, if $\pi$ corresponds to $\pi'$ then $d_\pi = d_{\pi'}$ and $\epsilon(s, \pi, \psi) = \epsilon(s, \pi', \psi)$ where $d_\pi$ is the formal degree of $\pi$, $d_{\pi'}$ the degree of $\pi'$ and the $\epsilon$'s are the usual $\epsilon$-factors. The purpose of this thesis is to detail a new proof of this correspondence.

**(1.2)** We now briefly review the history of the correspondence. As given above, it was first established by Jacquet and Langlands in the book [JL]. They established the correspondence by using a Weil representation to construct a representation of $G$ given a representation of $G'$, and then proving that this construction satisfies the requisite properties. They also proved a global version of the correspondence by making use of the Selberg trace formula. A version of the correspondence, both local and global, for $\mathrm{GL}_n$ was established by Rogawski [Rog] and independently by Deligne, Kazhdan

[1] In fact, the correspondence holds on a slightly larger set of representations, those which are essentially square integrable. We will restrict ourselves to the cuspidal case, however.

and Vigneras [DKV]. Both proofs work by first establishing the global result via trace formula techniques and then obtaining the local version by embedding into a global situation. For $n \geq 3$ there is currently no purely local proof of the local correspondence[2]. The original proof of Jacquet and Langlands does not generalize, as the existence of the Weil representation for $\mathrm{GL}_2$ is a consequence of the exceptional isomorphism $\mathrm{GL}_2 = \mathrm{GSp}_2$. We should mention, however, that there is a purely local *construction* of the correspondence. This construction, which makes use of non-abelian Lubin-Tate theory, is due to some combination of Carayol, Deligne and Drinfeld. A proof that this construction works was given by Harris and Taylor in their book [HT]. Their proof is global. For more details, see the introduction of [HT].

**(1.3)** Our proof of the correspondence is purely local and makes no use of the Weil representation. It does, however, use the local converse theorem, which states: if $\pi$ and $\pi'$ are two cuspidal representations of $G$ with the same central character and for which $\epsilon(s, \eta\pi, \psi) = \epsilon(s, \eta\pi', \psi)$ for all characters $\eta$ of $F^\times$ then $\pi \cong \pi'$. This form of the converse theorem is valid for $\mathrm{GL}_2$ and $\mathrm{GL}_3$ but not for $\mathrm{GL}_n$ with $n \geq 4$ (in general one has to twist by representations of $\mathrm{GL}_m$ with $m > 1$). No other steps in our proof are specific to $\mathrm{GL}_2$, however (at least in theory). We therefore feel that our method should generalize directly to $\mathrm{GL}_3$, and perhaps, with some modification, to other $\mathrm{GL}_n$. In fact, our method of proof does reduce the proof of the correspondence for $\mathrm{GL}_3$ to a rather elementary integral identity which makes no direct reference to representation theoretic concepts. We are not at this time able to establish this identity, however.

**(1.4)** We should also mention that we only prove the correspondence for $F/\mathbb{Q}_p$ with $p$ odd. We make the restriction that $p$ be odd mainly for convenience: many computations become much more simple in this situation. However, we believe that this restriction is unnecessary and that our approach will work just as well when $p = 2$: we expect the computations that arise to be feasible, but slightly more complicated than the ones we present here. Another direction of generalization would be to consider that case when $F$ is a local function field; we have not thought about this.

**(1.5)** We now give an outline of our proof. Let $B = M_2(F)$ and let $B'$ be the unique non-split quaternion algebra over $F$, so that $G = B^\times$ and $G' = (B')^\times$. We let $X$ (resp. $X'$) be the space of characteristic polynomials of elements of $B$ (resp. $B'$). Thus $X$ is the space of all monic degree two polynomials over $F$ while $X'$ consists of those monic degree two polynomials which either have a double root or are irreducible. There is a natural inclusion $X' \to X$. We must prove that if

[2]Several experts (for instance G. Henniart) have told me that they believes this to be the case.

$\pi$ is a cuspidal representation of $G$ then $\chi_\pi|_{X'}$ is the character of an irreducible representation of $G'$ (of course, characters are not defined on all of $X$ or $X'$, but a dense subset). The main idea of our approach is to consider $X$ and $X'$ as the fundamental objects and to endow them with enough structure so that one can determine which functions on them are characters of irreducible representations. We then compare this structure to compare characters of representations. It turns out that the extra structure needed is simply a Fourier transform defined on (a certain subspace of) the functions on $X$ or $X'$.

**(1.6)** To be more specific, there are three steps that we carry out on each side (meaning in the split and non-split case):

1. We define the Fourier transform on an appropriate space of functions on $X$ or $X'$.

2. We show that this is enough information to determine which functions are characters.

3. We express the Fourier transform in such a way that it is easy to compare.

After having accomplished these three steps on each side, we compare the two Fourier transforms. These two transforms will be acting on more or less the same function space and we must show that they are more or less equal. This will prove the correspondence.

**(1.7)** We now say a bit more about the first step. Let $\pi : B \to X$ and $\pi' : B' \to X'$ be the characteristic polynomial maps. We will define a certain space of functions $H(X)$ on $X$ and $H(X')$ on $X'$, called the *cuspidal spaces*. Restricting along $X' \to X$ gives an isomorphism $H(X) \to H(X')$. The Fourier transforms on $X$ and $X'$ will be defined on these cuspidal spaces and map them into themselves. (In fact, the Fourier transform will also be defined on the slightly larger space of functions with regular elliptic support.) To define the Fourier transform, we use the usual Fourier transform on $B$ or $B'$ and push-forward. In other words, the Fourier transform is defined by requiring it to commute with $\pi_*$ or $\pi'_*$. In the non-split case, it is straightforward to see that this is well-defined and has the desired properties, owing to the fact that $\pi'$ is a proper map. We denote the resulting Fourier transform by $\mathscr{F}_{X',\psi}$. In the split case, $\pi$ is not proper and things become more difficult. We end up defining two Fourier transforms, $\mathscr{F}^{(1)}_{X,\psi}$ and $\mathscr{F}^{(2)}_{X,\psi}$. The first of these is defined on the space of functions with regular elliptic support and is better suited for the comparison step, but its analytic properties (such as $L^2$-continuity) and interaction with representation theory are not clear. The second is defined on the cuspidal space and has better analytic and representation theoretic properties, but could not be used to make the comparison. A key result (§4.4.8) shows that $\mathscr{F}^{(1)}_{X,\psi}$

and $\mathscr{F}^{(2)}_{X,\psi}$ agree on a dense subspace of the cuspidal space. The main comparison step (discussed below) shows that $\mathscr{F}^{(1)}_{X,\psi}$ and $\mathscr{F}_{X',\psi}$ agree. From this we see that $\mathscr{F}^{(2)}_{X,\psi}$ and $\mathscr{F}_{X',\psi}$ agree on a dense subspace and as both are continuous they must be equal.

**(1.8)** We now say more about the second step. For an additive character $\psi$ of $F$ and a character $\eta$ of $F^\times$ we define an operator $A_{\psi,\eta}$ on the cuspidal space of $X$ or $X'$ by the formula

$$A_{\psi,\eta}f = \eta^{-1}\mathscr{F}_\psi(|\cdot|_F^{-2}\eta^{-1}f^\vee).$$

Here $f^\vee$ is the function $x \mapsto f(x^{-1})$ and $\eta$ is regarded as a function on $X$ by composing with the determinant. The form of this operator is motivated by the local functional equation. In fact, the local functional equation shows that if $\pi$ is a cuspidal representation then $\chi_\pi$ is an eigenvector of each of the operators $A_{\psi,\eta}$. The eigenvalue is closely related to the $\epsilon$-factor $\epsilon(s,\eta\pi,\psi)$. Using this insight, we analyze the cuspidal space as a module over the algebra generated by the $A_{\psi,\eta}$. We find that it is semi-simple, multiplicity free and that its simple constituents are naturally indexed by the unramified twist classes of cuspidal representations of $G$ or $G'$. The multiplicity freeness here is equivalent to the local converse theorem mentioned above. We take this result further in the non-split case and give a precise criterion for determining when a function on $X'$ is a character of an irreducible representation.

**(1.9)** We now elaborate on the third step. The Fourier transforms we define on $X$ and $X'$ would be difficult to compare directly. To compare them, we break them into simpler pieces. The main idea is to regard $X$ or $X'$ as being built out of the degree two field extensions $K$ of $F$ and try to relate the Fourier transform on $X$ or $X'$ to the Fourier transform on these fields. This, it turns out, is not difficult to do. If $f$ is a function on $K$ (which we can essentially regard as a function on $X$) then its Fourier transform is by definition $\pi_*(\mathscr{F}_{B,\psi}\widetilde{f})$ where $\widetilde{f}$ is a function on $B$ for which $\pi_*\widetilde{f} = f$. It turns out that one can essentially take for $\widetilde{f}$ the function $f \otimes \delta$ where $\delta$ is the $\delta$-function in the direction orthogonal to $K$ (with respect to the trace pairing). The Fourier transform of $f \otimes \delta$ is $(\mathscr{F}_{K,\psi}f) \otimes 1$ (up to a constant), where here $\mathscr{F}_{K,\psi}f$ is the Fourier transform on $K$. We can thus factor the Fourier transform on $X$ as a Fourier transform on $K$ followed by the operation $f \mapsto \pi_*(f \otimes 1)$. We denote this operation by $\overline{p}_*$. The same analysis holds on the non-split case and we denote the operation $f \mapsto \pi'_*(f \otimes 1)$ by $\overline{p}'_*$. We thus see that to compare the Fourier transforms on $X$ and $X'$ it suffices to compare $\overline{p}_*$ and $\overline{p}'_*$.

**(1.10)** We now discuss the comparison of $\overline{p}_*$ and $\overline{p}'_*$. Our comparison of these two operators is by brute force calculation. We show that $\overline{p}_*$ and $\overline{p}'_*$ are both given by integrating against a kernel which is expressed in terms of the integral $I_2$ defined in §3.2. The ultimate comparison amounts to the fact that

$$I_2(ax, b) + I_2(x, b)$$

is independent of $x$ if $(b, -a/b) = -1$, where $(,)$ denotes the Hilbert symbol. We prove this fact by explicit evaluation of $I_2$.

**(1.11)** We now say a word about future directions for the ideas presented here. Naturally, as already indicated, the most obvious problem to attempt next is the Jacquet-Langlands correspondence for $\mathrm{GL}_3$. Everything presented here should carry over directly to $\mathrm{GL}_3$ except for our comparison of $\overline{p}_*$ and $\overline{p}'_*$. We have not yet found a way to carry out this comparison for $\mathrm{GL}_3$. Moving beyond $\mathrm{GL}_3$, much of what we do here in fact works for $\mathrm{GL}_n$: one can still define Fourier transforms and factor them via $\overline{p}_*$ like operators. A natural problem is to try to compare the Fourier transforms (or, equivalently, the $\overline{p}_*$ operators) coming from division algebras and $\mathrm{GL}_n$. For $n > 3$ this would not imply the Jacquet-Langlands correspondence, but it would give some sort of first order approximation to it. For further discussion along these lines, see §7. Looking in a different direction, it may be possible to prove other instances of Langlands functoriality using our approach: in particular, we have in mind base change for $\mathrm{GL}_2$.

# Chapter 2

# Notation

Throughout $F$ denotes a fixed extension of $\mathbb{Q}_p$ of finite degree, where $p$ is an odd prime. The following are the most important pieces of notation on the split side:

- $B$ is the matrix algebra $M_2(F)$.
- $G$ is the group of units in $B$, namely $\mathrm{GL}_2(F)$.
- $X$ is the set of all monic degree two polynomials.
- $\widetilde{X}$ is the disjoint union of the four degree two étale algebras over $F$ (the three quadratic field extensions of $F$ and the split algebra $F \oplus F$).
- $K$ typically denotes one of the four degree two algebras over $F$.
- $\pi : B \to X$ is the map which assigns to a matrix its characteristic polynomial.
- $p : \widetilde{X} \to X$ is the map which assigns to an element of $\widetilde{X}$ its characteristic polynomial.
- $i : \widetilde{X} \to B$ is a fixed map such that $i|_K : K \to B$ is an injection of algebras, for each component $K$ of $\widetilde{X}$.

The following are the most important pieces of notation on the non-split side:

- $B'$ is the unique non-split quaternion algebra over $F$.
- $G'$ is the group of units in $B'$.
- $X'$ is the set of all monic degree two polynomials which are either irreducible or have a double root.

- $\widetilde{X}'$ is the disjoint union of the three quadratic extensions of $F$.
- $K$ typically denotes one of the three quadratic field extensions of $F$.
- $\pi' : B' \to X'$ is the map which assigns to an element of $B'$ its characteristic polynomial.
- $p' : \widetilde{X}' \to X'$ is the map which assigns to an element of $\widetilde{X}'$ its characteristic polynomial.
- $i' : \widetilde{X}' \to B'$ is a fixed map such that $i'|_K : K \to B'$ is an injection of algebras, for each component $K$ of $\widetilde{X}'$.

Notation related to $F$:

- $\mathscr{O}_F$ is the ring of integers in $F$.
- $q$ is the cardinality of the residue field $\kappa_F$ of $\mathscr{O}_F$.
- $\mathfrak{p}_F$ is the maximal ideal of $\mathscr{O}_F$.
- $U_F$ is the group of units of $\mathscr{O}_F$.
- $U_F^{(n)}$ is the group $1 + \mathfrak{p}^n$.
- $\varpi_F$ is a uniformizer for $\mathscr{O}_F$.
- $\eta : F^\times \to \{\pm 1\}$ is 1 on squares and $-1$ on non-squares.
- $(,)$ is the Hilbert symbol on $F$.

More notation on the split side:

- We write elements of $X$ as $x^2 - tx + \nu$. We regard $t$ and $\nu$ as functions $X \to F$. They give an isomorphism $X \to F^2$.
- We let $\Delta : X \to F$ be the function $t^2 - 4\nu$. We write still $\Delta$ for its pull-back to $B$, $\widetilde{X}$ or $K$.
- $X_{\text{reg}}$ is the set of polynomials in $X$ with distinct roots.
- $X_{\text{ell}}$ is the set of polynomials in $X$ which are either irreducible or have a double root.
- $X_{\text{re}}$ is the set $X_{\text{reg}} \cap X_{\text{ell}}$.
- $X_{\text{ns}}$ is the subset of $X$ where $\nu \neq 0$ (ns meaning "non-singular").
- $X_n$ is the subset of $X$ where $\nu$ has valuation $n$.

- $X_{*,n}$ is $X_* \cap X_n$; for example, we have $X_{\mathrm{re},n}$.
- We write $B_*$, $\widetilde{X}_*$ or $K_*$ for the inverse image of $X_*$. For example, $B_{\mathrm{ns}} = G$. Also, $G_n$ denotes the set of elements in $G$ whose determinant has valuation $n$.
- $Z$ is the center of $G$. It is isomorphic to $F^\times$.

More notation on the non-split side:

- We use much of the same notation for $B'$ as for $F$. Thus $\mathscr{O}_{B'}$ is the maximal order in $B'$, $\varpi_{B'}$ is a uniformizer, *etc.*
- We regard $X'$ as a subset of $X$ in the obvious way and use much of the same notation, *e.g.*, $t$, $\nu$, $\Delta$, $X'_{\mathrm{ns}}$, *etc.* Note that $X' = X'_{\mathrm{ell}} = X_{\mathrm{ell}}$.
- We write $B'_*$, $\widetilde{X}'_*$ or $K_*$ for the inverse of $X'_*$. For example, $B'_{\mathrm{ns}} = G'$ and $G'_0 = U_{B'}$.
- $Z'$ is the center of $G'$. It is isomorphic to $F^\times$.

Degree two étale algebras:

- We use the letter $K$ to denote degree two étale algebras over $F$. There are four: the three quadratic fields extensions of $F$ and the split algebra $F \oplus F$.
- We write $\mathbf{d}_K$ for the discriminant of $K$, which we treat as an element of $F$. This is defined to be 1 in the split case.
- We write $d_K$ for $|\mathbf{d}_K|_F$. This is 1 if $K$ is split or an unramified field extension and $q^{-1}$ otherwise.

Norms, traces, absolute values:

- We let $\mathbf{N}$ (resp. tr) be the norm (resp. trace) map on $B$, $B'$ or $K$ to $F$. In all cases $\mathbf{N}\,x = x\overline{x}$ (resp. $\operatorname{tr} x = x + \overline{x}$), where $\overline{x}$ is the conjugate of $x$. We also, at times, use the same notation on $X$ or $X'$; of course, $\mathbf{N} = \nu$ and $\operatorname{tr} = t$ in those settings.
- For a topological ring $A$ we let $|\cdot|_A$ be the absolute value given by $|a|_A = d(ax)/dx$ where $dx$ is a Haar measure on $A$. If $A$ is $B$ or $B'$ then $|x|_A = |\,\mathbf{N}\,x|_F^2$. If $A = K$ is a degree two étale algebra then $|x|_A = |\,\mathbf{N}\,x|_F$.

Haar measures:

- If $A$ is one of the algebras $F$, $K$, $B$, or $B'$ then we let $d\mu_A$ be the unique Haar measure on $A$ which gives maximal orders volume 1.
- For such $A$ we let $d\mu_{A^\times}$ be $|\cdot|_A^{-1} d\mu_A$. It is a Haar measure on $A^\times$.

We call the above Haar measures the *normalized Haar measures*. Let $Y$ be a topological space. All function spaces below deal with complex valued functions.

- $\mathscr{S}(Y)$ is the space of Schwartz (=locally constant and compact support) functions on $Y$.
- $\mathscr{C}^\infty(Y)$ is the space of smooth (=locally constant) functions on $Y$.
- We typically put support conditions in subscripts. For example, $\mathscr{S}_{\mathrm{re}}(X)$ (resp. $\mathscr{C}^\infty_{\mathrm{re}}(X)$) denotes the subspace of $\mathscr{S}(X)$ (resp. $\mathscr{C}^\infty(X)$) consisting of those functions whose support is contained in $X_{\mathrm{re}}$. Note $\mathscr{S}_{\mathrm{re}}(X) = \mathscr{S}(X_{\mathrm{re}})$ but $\mathscr{C}^\infty_{\mathrm{re}}(X) \neq \mathscr{C}^\infty(X_{\mathrm{re}})$.
- We will define more function spaces below. The most important of these are the cuspidal spaces $H(X)$ and $H(X')$.

Let $Y$ be a topological space with a measure $d\mu$.

- We write $L^2(Y)$ for the standard $L^2$ function space.
- We let $\|\cdot\|_Y$ be the $L^2$-norm.
- For $f, g \in L^2(Y)$ we put $\langle f, g\rangle_Y = \int_Y fg d\mu$.
- For $f, g \in L^2(Y)$ we put $(f, g)_Y = \langle f, \overline{g}\rangle_Y$ where $\overline{g}$ is the conjugate of $g$. Note $(f, f)_Y = \|f\|_Y^2$.

Fourier transforms:

- $\psi = \psi_F$ is a non-trivial additive character on $F$.
- If $A$ is one of the algebras $B$, $B'$ or $K$ we let $\psi_A$ be the character of $A$ given by $\psi_F \circ \mathrm{tr}_{A/F}$.
- We let $m = m(\psi)$ be the largest integer for which $\psi$ is trivial on $\mathfrak{p}^{-m}$.
- When we have defined a Fourier transform on a space $Y$ with respect to $\psi$ we denote it by something like $\mathscr{F}_{Y,\psi}$. For example, $\mathscr{F}_{F,\psi}$ is the usual Fourier transform on $F$.

Representations:

- $\mathrm{Irr}^\circ_G$ is the set of isomorphism classes of irreducible admissible representations of $G$ which are cuspidal.
- $\mathrm{Irr}^\circ_{G'}$ is the set of isomorphism classes of finite dimensional irreducible "cuspidal" representations of $G'$, where here "cuspidal" simply means having dimension at least two.
- $\overline{\mathrm{Irr}}^\circ_G$ is the quotient of $\mathrm{Irr}^\circ_G$ by the twisting action of the group of unramified characters; similarly for $G'$.
- $\mathrm{Irr}^\circ_{G,\omega}$ is the subset of $\mathrm{Irr}^\circ_G$ consisting of those representations with central character $\omega$; similarly for $G'$.
- $\overline{\mathrm{Irr}}^\circ_{G,\omega}$ is the image in $\overline{\mathrm{Irr}}^\circ_G$ of $\mathrm{Irr}^\circ_{G,\omega}$. Similarly for $G'$. The map $\mathrm{Irr}^\circ_{G,\omega} \to \overline{\mathrm{Irr}}^\circ_{G,\omega}$ is 2-1.
- $\xi$ denotes the character of $F^\times$ (or $G$ or $G'$ by composing with the norm) given by $x \mapsto (-1)^{\mathrm{val}\, x}$.
- A representation $\pi$ of $G$ or $G'$ is *even* if $\xi \otimes \pi \cong \pi$ and *odd* otherwise.
- For a representation $\pi$ of $G$ or $G'$ we denote by $n(\pi)$ its conductor.
- For a representation $\pi$ of $G$ of $G'$ we denote by $\omega_\pi$ its central character.

# Chapter 3

# Some integrals

**(3.1)** For $a, b \in F^\times$ define

$$I_1(a,b) = \left|\frac{b}{a}\right|_F^{1/2} \int_F (1+\eta(a+bx^2))dx.$$

Here $dx$ is the normalized Haar measure on $F$. If $b$ is not a square then the integrand has compact support and so the integral makes sense. One easily sees that its value only depends on $a$ and $b$ modulo squares. We now compute its value.

**Proposition.** *Let $a$ and $b$ be as above. Then*

$$I_1(a,b) = \begin{cases} 1+q^{-1} - \frac{1+\eta(-b/a)}{q+1} & \operatorname{val} a \textit{ and } \operatorname{val} b \textit{ even} \\ 0 & \operatorname{val} a \textit{ odd and } \operatorname{val} b \textit{ even} \\ (1+\eta(a))q^{-1/2} & \operatorname{val} a \textit{ even and } \operatorname{val} b \textit{ odd} \\ \frac{1+\eta(-b/a)}{q+1} & \operatorname{val} a \textit{ and } \operatorname{val} b \textit{ odd} \end{cases}$$

*Proof.* Without loss of generality, we assume $a$ and $b$ have valuation 0 or 1. We proceed by cases.

*Case 1: $a$ and $b$ have valuation 0.* For $a + bx^2$ to be a square we must have $x \in \mathscr{O}_F$. We thus regard $I_1$ as an integral over $\mathscr{O}_F$ and then break it up over the cosets of $\mathfrak{p}$ as follows:

$$I_1(a,b) = \epsilon \int_{\pm\alpha+\mathfrak{p}} (1+\eta(a+bx^2))dx + \sum_{x\in S} \int_{x+\mathfrak{p}} 2dx.$$

Here $S$ is the set of $x$ in the residue field $\kappa$ such that $a+bx^2$ is a non-zero square, $\epsilon$ is 1 if $a+bx^2=0$

has a solution in $\kappa$ and zero otherwise and $\alpha$ is a solution to $a + bx^2 = 0$ if one exists. The second term above is of course equal to $2q^{-1}\#S$. We thus have to compute $\#S$ and the integral.

We begin with the computation of $\#S$. Consider the projective variety over $\kappa$ defined by $y^2 = az^2 + bx^2$. This is smooth and has a rational point and so is isomorphic to $\mathbb{P}^1$. It therefore has $q+1$ solutions in $\kappa$. As $b$ is not a square, there are no solutions when $z = 0$. Thus $y^2 = a + bx^2$ has $q+1$ solutions. Now, if $\epsilon = 0$ then there are no solutions with $y = 0$. Therefore, if $(x, y)$ is a solution then $(x, -y)$ is a distinct solution; thus $\#S = (q + 1)/2$. On the other hand, if $\epsilon = 1$ then there are two solutions with $y = 0$; removing these, we find $\#S = (q - 1)/2$. Thus we have

$$\#S = (q + 1)/2 - \epsilon$$

in all situations.

We now handle the integral, assuming $\alpha$ exists. First we lift $\alpha$ to a solution to $a+bx^2 = 0$ in $\mathscr{O}_F$. We now make the change of variables $x = \alpha+y$, so that the integral takes place with $y \in \mathfrak{p}$. We have $a + bx^2 = 2b\alpha y + by^2$. As the first term has strictly smaller valuation, we find $\eta(a + bx^2) = \eta(2b\alpha y)$. We therefore have

$$\int_{\alpha+\mathfrak{p}} (1 + \eta(a + bx^2))dx = \int_{\mathfrak{p}} (1 + \eta(2b\alpha y))dy = \sum_{n=1}^{\infty} \int_{\varpi^n U_F} (1 + \eta(2b\alpha y))dy.$$

Now, for $2b\alpha y$ to be a square $y$ must have even valuation. Thus only the even terms in the above series contribute. The function $y \mapsto \eta(2b\alpha y)$ is a non-trivial character of $\varpi^{2n}U_F$ and so has integral zero. Thus the above series is equal to

$$\sum_{n=1}^{\infty} \mathrm{Vol}(\varpi^{2n}U_F) = \frac{q^{-2}}{1 + q^{-1}}.$$

Of course, we get the same result for $-\alpha$ as for $\alpha$.

Putting everything together, we find

$$I_1(a, b) = 2\epsilon\frac{q^{-2}}{1 + q^{-1}} + 2q^{-1}((q + 1)/2 - \epsilon) = 1 + q^{-1} - \frac{2\epsilon}{q + 1}$$

The identity $2\epsilon = 1 + \eta(-b/a)$ gives the stated result.

*Case 2: $a$ has valuation 1 and $b$ valuation 0.* It is impossible for $a + bx^2$ to be a square and so the integrand is identically zero.

*Case 3: $a$ has valuation 0 and $b$ valuation 1.* In this case, $a + bx^2$ is a square if and only if $a$ is a square and $x$ belongs to $\mathscr{O}_F$. We thus find

$$I_1(a,b) = q^{-1/2} \int_{\mathscr{O}_F} (1+\eta(a))dx = (1+\eta(a))q^{-1/2}.$$

*Case 4: a and b have valuation 1.* For $a+bx^2$ to be a square it is necessary for $x$ to have valuation 0. Since $a+bx^2$ must have even valuation, and it has valuation at least 1, we find $a+bx^2 = 0$ modulo $\mathfrak{p}^2$, which implies $x^2 = -a/b$ modulo $\mathfrak{p}$. Thus if $-a/b$ is not a square then $I_1(a,b) = 0$. We therefore assume from now on that $-a/b = \alpha^2$. Of course, $\alpha$ belongs to $U_F$.

Now, for $a+bx^2$ to be a square we must have $x = \pm\alpha$ modulo $\mathfrak{p}$. Thus

$$I_1(a,b) = \int_{\pm\alpha+\mathfrak{p}} (1+\eta(a+bx^2))dx.$$

We consider the $+\alpha$ integral, the other one going much the same. Make the change of variables $x = \alpha + y$ so that the integral varies over $y \in \mathfrak{p}$. We have $a+bx^2 = 2b\alpha y + by^2$. As the first term is dominant, $\eta(a+bx^2) = \eta(2b\alpha y)$ and so the integral equals

$$\int_{\mathfrak{p}} (1+\eta(2b\alpha y))dy = \sum_{n=1}^{\infty} q^{-n} \int_{U_F} (1+\eta(2b\alpha\varpi^n y))dy$$

As $b$ has valuation 1, only the terms with $n$ odd contribute. As in Case 1, when $n$ is odd $\eta$ is a non-trivial character and its integral vanishes. We thus find that the above equals

$$\sum_{n=0}^{\infty} q^{-(2n+1)} \operatorname{Vol}(U_F) = \frac{1}{q+1}.$$

The $-\alpha$ integral is equal to this as well. Thus $I_1(a,b)$ is $2/(q+1)$ if $-a/b$ is a square and zero otherwise. $\square$

**(3.2)** For $a, b \in F^\times$ define

$$I_2(a,b) = |b|_F^{1/2} \int_F \frac{1+\eta(a+bx^2)}{|a+bx^2|_F^{1/2}} dx.$$

Here $dx$ is the normalized Haar measure on $F$. If $b$ is not a square then the integrand has compact support and so the integral makes sense. One easily finds that $I_2(a,b)$ only depends on $a$ and $b$ modulo squares. We now explicitly compute its value.

**Proposition.** *Let $a$ and $b$ be as above. Then*

$$I_2(a,b) = \begin{cases} 1+q^{-1} & \operatorname{val} a \text{ and } \operatorname{val} b \text{ even} \\ 0 & \operatorname{val} a \text{ odd and } \operatorname{val} b \text{ even} \\ (1+\eta(a))q^{-1/2} & \operatorname{val} a \text{ even and } \operatorname{val} b \text{ odd} \\ (1+\eta(-a/b))q^{-1/2} & \operatorname{val} a \text{ and } \operatorname{val} b \text{ odd} \end{cases}$$

*Proof.* Without loss of generality, we assume $a$ and $b$ have valuation 0 or 1. We proceed by cases, much like the proof in §3.1.

*Case 1: $a$ and $b$ have valuation 0.* The same reasoning as in Case 1 in §3.1 gives

$$I_2(a,b) = \epsilon \int_{\pm\alpha+\mathfrak{p}} \frac{1+\eta(a+bx^2)}{|a+bx^2|_F^{1/2}} dx + \sum_{x\in S} \int_{x+\mathfrak{p}} 2dx$$

using the same notation as there. The second term is $2q^{-1}((q+1)/2-\epsilon)$, as it was in §3.1.

We now compute the integral, which is different from the one occurring in §3.1. Write $x = \alpha + y$ so that $\eta(a+bx^2) = \eta(2b\alpha y)$ and $|a+bx^2|_F = |y|_F$. We then have

$$\int_{\alpha+\mathfrak{p}} \frac{1+\eta(a+bx^2)}{|a+bx^2|_F^{1/2}} dx = \int_{\mathfrak{p}} \frac{1+\eta(2b\alpha y)}{|y|_F^{1/2}} dy = \sum_{n=1}^{\infty} q^{-n/2} \int_{U_F} (1+\eta(2b\alpha\varpi^n y))dy$$

Only the terms with $n$ even contribute. When $n$ is even the $\eta$ term is a non-trivial character and thus has integral zero. We thus find that the integral is equal to

$$\sum_{n=1}^{\infty} q^{-(2n)/2} \operatorname{Vol}(U_F) = q^{-1}.$$

The $-\alpha$ integral is equal to $q^{-1}$ as well.

Putting it all together, we find

$$I_2(a,b) = 2\epsilon q^{-1} + 2q^{-1}((q+1)/2-\epsilon) = 1+q^{-1}.$$

which completes this case.

*Case 2: $a$ has valuation 1 and $b$ valuation 0.* As in Case 2 of §3.1, the integrand vanishes identically.

*Case 3: $a$ has valuation 0 and $b$ valuation 1.* This proceeds like Case 3 of §3.1. Note that if $a+bx^2$ is a square then $|a+bx^2|_F = |a|_F = 1$.

*Case 4: a has valuation 1 and b valuation 0.* As in Case 4 of §3.1 we find that $I_2(a,b)=0$ unless $-a/b$ is a square, in which case

$$I_2(a,b)=q^{-1/2}\int_{\pm\alpha+\mathfrak{p}}\frac{1+\eta(a+bx^2)}{|a+bx^2|_F^{1/2}}dx$$

where $\alpha^2=-a/b$. We evaluate the $+\alpha$ integral. Writing $x=\alpha+y$ gives $\eta(a+bx^2)=\eta(2b\alpha y)$ and $|a+bx^2|_F=|by|_F$. Therefore the integral is equal to

$$\int_{\mathfrak{p}}\frac{1+\eta(2b\alpha y)}{|by|_F^{1/2}}dy=\sum_{n=1}^{\infty}q^{-(n-1)/2}\int_{U_F}(1+\eta(2b\alpha\varpi^n y))dy.$$

Only the terms with $n$ odd contribute and, as usual, in these terms the $\eta$ term vanishes. We thus obtain

$$\sum_{n=0}^{\infty}q^{-n}\operatorname{Vol}(U_F)=1.$$

The $-\alpha$ integral is the same, so $I_2(a,b)=2q^{-1/2}$. Thus $I_2(a,b)=(1+\eta(-a/b))q^{-1/2}$ in all cases, as stated. □

**(3.3)** For $a,b,c\in F^\times$ define

$$I_3(a,b,c)=\left|\frac{bc}{a}\right|_F^{1/2}\int_{F^2}\frac{1+\eta(a+bx^2+cy^2)}{|a+bx^2+cy^2|_F^{1/2}}dxdy$$

where $dx$ and $dy$ are normalized Haar measures on $F$. If $(b,c)=-1$ then the integrand has compact support and so the integral makes sense. One easily finds that $I_3(a,b,c)$ only depends on $a$, $b$ and $c$ modulo squares. We now explicitly compute its value.

**Proposition.** *Let a, b and c be as above. Then*

$$I_3(a,b,c)=\begin{cases}(1+\eta(a))q^{-1} & \operatorname{val}a \text{ even}, \operatorname{val}b \text{ odd}, \operatorname{val}c \text{ odd}\\ (1+q^{-1})q^{-1/2} & \operatorname{val}a \text{ odd}, \operatorname{val}b \text{ odd}, \operatorname{val}c \text{ odd}\\ (1+q^{-1})q^{-1/2} & \operatorname{val}a \text{ even}, \operatorname{val}b \text{ odd}, \operatorname{val}c \text{ even}\\ (1+\eta(-a/b))q^{-1} & \operatorname{val}a \text{ odd}, \operatorname{val}b \text{ odd}, \operatorname{val}c \text{ even}\end{cases}$$

*Proof.* To begin with, we have

$$I_3(a,b,c)=\left|\frac{b}{a}\right|_F^{1/2}\int_F I_2(a+bx^2,c)dx.$$

We now proceed in cases.

*Case 1:* $\operatorname{val} b$ *and* $\operatorname{val} c$ *both odd.* Using our formula for $I_2$, we find

$$I_3(a,b,c) = q^{-1/2}\left|\frac{b}{a}\right|_F^{1/2}\int_F (1+\eta(a+bx^2))dx + q^{-1/2}\left|\frac{b'}{a'}\right|_F^{1/2}\int_F (1+\eta(a'+b'x^2))dx.$$

Here $a' = -a/c$ and $b' = -b/c$. We can rewrite this as

$$I_3(a,b,c) = q^{-1/2}I_1(a,b) + q^{-1/2}I_1(a',b').$$

Using our formula for $I_1$ gives the stated result.

*Case 2:* $\operatorname{val} b$ *odd and* $\operatorname{val} c$ *even.* Using our formula for $I_2$ we find

$$I_3(a,b,c) = (1+q^{-1})\left|\frac{b}{a}\right|^{1/2}\operatorname{Vol}(\Omega)$$

where $\Omega$ is the set of $x$ for which $a+bx^2$ has even valuation. If $a$ has even valuation then $a+bx^2$ has even valuation if and only if $|bx^2| < |a|$ and so $\operatorname{Vol}(\Omega) = |a/b|_F^{1/2}q^{-1/2}$, which gives the stated result. Now say that $a$ has odd valuation. Then $a+bx^2$ will have even valuation for some $x$ if and only if $-a/b$ is a square. Assume this is the case and write $\alpha^2 = -a/b$. Then $a+bx^2$ has even valuation if and only if $x = \pm\alpha(1+\epsilon\varpi^{2k+1})$ for $\epsilon \in U_F$ and a non-negative integer $k$. We thus find

$$\operatorname{Vol}(\Omega) = 2|\alpha|_F\sum_{k=0}^{\infty}\operatorname{Vol}(\varpi^{2k+1}U_F) = \left|\frac{a}{b}\right|^{1/2}\frac{2}{q+1}$$

and the stated result follows. □

# Chapter 4

# The split side

The goals of §4 are as follows:

- Define a Fourier transform $\mathscr{F}^{(1)}_{X,\psi}$ on the space of Schwartz functions on $X$ with regular elliptic support.
- Factor $\mathscr{F}^{(1)}_{X,\psi}$ into two steps, the first of which involves the Fourier transform on quadratic extensions of $F$ and the second of which is a relatively easy operation.
- Define a Fourier transform $\mathscr{F}^{(2)}_{X,\psi}$ on the space of cuspidal functions $H(X)$.
- Relate the two Fourier transforms.
- Use $\mathscr{F}^{(2)}_{X,\psi}$ to define a family of operators $\mathscr{A}$ on $H(X)$ and determine the structure of $H(X)$ as an $\mathscr{A}$-module in terms of the representation theory of $G$.

The first two goals are accomplished in §4.2 while the final three are accomplished in §4.4. The odd numbered sections §4.1 and §4.3 carry out a number of rather routine calculations. The reader should keep the following diagram in mind throughout the section.

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\;\;i\;\;} & B \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & X & \end{array}$$

Recall that $B$ is the matrix algebra $M_2(F)$, $X$ is the space of monic degree two polynomials over $F$, $\widetilde{X}$ is the disjoint union of the four degree two étale algebras over $F$, $p$ and $\pi$ are the characteristic polynomial maps and $i$ is a chosen map which restricts to an algebra injection on each component of $\widetilde{X}$.

## 4.1 Measures and push-forwards

**(4.1.1)** The purpose of §4.1 is to define measures on the spaces $B$, $K$, $K^\perp$, $X$ and $\widetilde{X}$, push-forwards along the maps $p$ and $\pi$ and compute all of these things explicitly. Here is an overview of the section:

- In §4.1.2 and §4.1.3 we introduce a certain class of bases of $B$ and $K$ which we call *standard bases*.
- In §4.1.4 we give an elementary change of variables formula that we will often need.
- In §4.1.5 we define the push-forward $p_*$.
- In §§4.1.6–4.1.10 we define and compute the measures on $B$, $K$, $K^\perp$, $X$ and $\widetilde{X}$. The measures on the first three spaces are just Haar measures, on the latter two they are defined in a somewhat *ad hoc* manner, but are motivated by the Weyl integration formula.
- In §4.1.11 we define the map $\pi_*$. Roughly, $\pi_*(f)$ is defined to be the Radon-Nikodym derivative of $\pi_*(fd\mu_B)$ with respect to $d\mu_X$.
- In §§4.1.12–4.1.15 we relate $\pi_*$ to certain orbital integrals and $\pi^*\pi_*$ to certain averaging operators.
- In §4.1.16 we produce natural liftings of functions on $X$ to functions on $B$. These will be important when we factor the Fourier transform.

**(4.1.2)** By a *standard basis* of $B$ we mean a basis $1$, $i$, $j$, $k$ of $B$ as an $F$-vector space where:

- $1$ is the unit of $B$.
- $i$, $j$ and $k$ anti-commute.
- $i$, $j$ and $k$ square to elements of $F$.
- $ij = k$.

We will typically write $i^2 = \alpha$, $j^2 = \beta$ and $k^2 = \gamma$. The above conditions imply $\alpha\beta = -\gamma$. Given a standard basis and an element $x$ of $B$ we write $x = x_0 + ix_1 + jx_2 + kx_3$.

**(4.1.3)** Let $K$ be a degree two étale algebra over $F$. By a *standard basis* of $K$ we mean a basis $1$, $i$ of $K$ as an $F$-vector space where $1$ is the unit and $i^2$ belongs to $F$. We will typically write $i^2 = \alpha$. Given a standard basis and an element $x$ of $K$ we write $x = x_0 + ix_1$.

**(4.1.4)** Before continuing, we give the following elementary change of variables formula, which we shall have often have the occasion to employ.

**Proposition.** *Let $f \in \mathscr{S}(F)$. Then*

$$\int_F f(x)dx = \int_F f(\pm\sqrt{x})\frac{1+\eta(x)}{|x|_F^{1/2}}dx$$

*where $f(\pm\sqrt{x}) = \frac{1}{2}(f(\sqrt{x}) + f(-\sqrt{x}))$ and $dx$ is a Haar measure on $F$.*

**(4.1.5)** For a function $f$ on $\widetilde{X}$ we define $p_* f$ to be the function on $X$ given by

$$(p_* f)(x) = \frac{1}{\# p^{-1}(x)} \sum_{p(y)=x} f(y)$$

Of course, we have $p_* p^* f = f$. In general, if $f$ belongs to $\mathscr{S}(\widetilde{X})$ then $p_* f$ will not belong to $\mathscr{S}(X)$. However, since $p|_{\widetilde{X}_{\text{reg}}} : \widetilde{X}_{\text{reg}} \to X_{\text{reg}}$ is étale, $p_*$ does induce a map

$$p_* : \mathscr{S}_{\text{reg}}(\widetilde{X}) \to \mathscr{S}_{\text{reg}}(X).$$

In fact, $p_*$ and $p^*$ give mutually inverse isomorphisms between $\mathscr{S}_{\text{reg}}^{\text{inv}}(\widetilde{X})$ and $\mathscr{S}_{\text{reg}}(X)$ where the former space is the subspace of $\mathscr{S}_{\text{reg}}(\widetilde{X})$ consisting of those functions which are Galois invariant.

**(4.1.6)** Recall that $d\mu_B$ is the Haar measure on $B$ which assigns volume 1 to any maximal order. We now compute it in a standard basis.

**Proposition.** *Identifying $B$ with $F^4$ via a standard basis, we have*

$$d\mu_B(x) = |\alpha\beta\gamma|_F^{1/2} dx_0 dx_1 dx_2 dx_3$$

*where $dx_i = d\mu_F(x_i)$ are normalized Haar measures on $F$.*

*Proof.* We first remark that the proposition is true for a standard basis 1, $i$, $j$ and $k$ if and only if it is so for the basis 1, $ai$, $bj$, $ck$ where $a$, $b$ and $c$ belong to $F^\times$. Similarly, it is true for 1, $i$, $j$ and $k$ if and only if it is for 1, $\sigma i \sigma^{-1}$, $\sigma j \sigma^{-1}$ and $\sigma k \sigma^{-1}$ with $\sigma \in B^\times$. We are thus free to scale and conjugate our basis.

Consider the case where one of $\alpha$, $\beta$ or $\gamma$ is a square. It suffices to treat the case where $\alpha$ is. By rescaling, we may then assume $\alpha = 1$. It is then not hard to see that we can conjugate our basis so that

$$i = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \qquad j = \begin{pmatrix} & 1 \\ \beta & \end{pmatrix}, \qquad k = \begin{pmatrix} & 1 \\ -\beta & \end{pmatrix}.$$

From this, we see that an element $x_0 + ix_1 + jx_2 + kx_3$ belongs to $M_2(\mathscr{O}_F)$ if and only if each of

$$x_0 + x_1, \qquad x_0 - x_1, \qquad x_2 + x_3, \qquad \beta(x_2 - x_3)$$

belongs to $\mathscr{O}_F$. As the measure $dx_0dx_1dx_2dx_3$ gives this set volume $|\beta|_F^{-1}$ it follows that

$$|\beta|_F dx_0dx_1dx_2dx_3$$

is the normalized Haar measure on $B$. Finally, observe that $|\beta|_F = |\alpha\beta\gamma|_F^{1/2}$. The proposition is thus established in this case.

Now consider the case where $\alpha$, $\beta$ and $\gamma$ are all non-squares. It follows that they must all belong to the square class of $-1$, which is therefore not a square. By rescaling we may then assume $\alpha = \beta = \gamma = -1$. It is then not hard to see that we may conjugate our basis so that

$$i = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \qquad j = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \qquad k = \begin{pmatrix} b & -a \\ -a & -b \end{pmatrix}$$

with $a^2 + b^2 = -1$. From this, we see that an element $x = x_0 + ix_1 + jx_2 + kx_3$ belongs to $M_2(\mathscr{O}_F)$ if and only if each of

$$x_0 + ax_2 + bx_3, \qquad x_1 + bx_2 - ax_3, \qquad -x_1 + bx_2 - ax_3, \qquad x_0 - ax_2 - bx_3$$

belongs to $\mathscr{O}_F$. Clearly, this is equivalent to each of

$$x_0, \qquad x_1, \qquad ax_2 + bx_3, \qquad bx_2 - ax_3$$

belonging to $\mathscr{O}_F$. Now, note that $a$ and $b$ must each have valuation 0. Therefore,

$$\begin{aligned} &ax_2 + bx_3 \text{ and } bx_2 - ax_3 \text{ belong to } \mathscr{O}_F \\ \Longleftrightarrow\ &a^2x_2 + abx_3 \text{ and } b^2x_2 - abx_3 \text{ belong to } \mathscr{O}_F \\ \Longleftrightarrow\ &(a^2+b^2)x_2 \text{ and } abx_3 \text{ belong to } \mathscr{O}_F \\ \Longleftrightarrow\ &x_2 \text{ and } x_3 \text{ belong to } \mathscr{O}_F \end{aligned}$$

We have therefore shown that $x$ belongs to $M_2(\mathscr{O}_F)$ if and only if each $x_i$ belongs to $\mathscr{O}_F$. Thus $dx_0dx_1dx_2dx_3$ is the normalized Haar measure on $B$. Since $|\alpha\beta\gamma|_F^{1/2} = 1$ this proves the proposition.

□

**(4.1.7)** Let $K$ be a degree two étale algebra over $F$. We have defined $d\mu_K$ to be the Haar measure on $K$ which gives volume 1 to the unique maximal order $\mathscr{O}_K$ of $K$. We now compute this measure in a standard basis.

**Proposition.** *Identifying $K$ with $F^2$ via a standard basis we have*

$$d\mu_K(x) = |\alpha/\mathbf{d}_K|_F^{1/2} dx_0dx_1$$

*where $dx_i = d\mu_F(x_i)$ are normalized Haar measures on $F$.*

*Proof.* We have $\mathscr{O}_K = \mathscr{O}_F + c\mathscr{O}_F i$ where $c = \sqrt{\mathbf{d}_K/\alpha}$. Under the isomorphism $K = F^2$ the lattice $\mathscr{O}_K$ corresponds to $\mathscr{O}_F \oplus c\mathscr{O}_F$. The measure $dx_0dx_1$ on $F^2$ gives this lattice volume $|c|_F$. Therefore the normalized Haar measure $d\mu_K$ on $K$ corresponds to the measure $|c|_F^{-1}dx_0dx_1$ on $F^2$. □

**(4.1.8)** Let $K \subset B$ be a degree two étale algebra over $F$ and let $K^\perp$ denote its orthogonal complement. There is a unique Haar measure $d\mu_{K^\perp}$ on $K$ such that $d\mu_B = d\mu_K d\mu_{K^\perp}$. We now compute $d\mu_{K^\perp}$ in coordinates.

**Proposition.** *Let 1, $i$, $j$, $k$ be a standard basis for $B$ such that 1, $i$ is a standard basis for $K$. Then $j$, $k$ is a basis for $K^\perp$ and under the resulting identification $K^\perp = F^2$ we have*

$$d\mu_{K^\perp} = |\mathbf{d}_K\beta\gamma|_F^{1/2} dx_2dx_3$$

*where $dx_i = d\mu_F(x_i)$ is a normalized Haar measure on $F$.*

*Proof.* This follows immediately from the computations of §4.1.6 and §4.1.7. □

**(4.1.9)** We now define a measure $d\mu_X$ on $X$ by

$$d\mu_X = |\Delta|_F^{1/2} A(\Delta) d\mu_F(\nu) d\mu_F(t)$$

where we have identified $X$ with $F^2$ via $(t, \nu)$ and $A$ is given by:

$$A(\Delta) = \begin{cases} 2 & \Delta \text{ has even valuation} \\ q^{1/2} + q^{-1/2} & \Delta \text{ has odd valuation} \end{cases}$$

This formula may look somewhat arbitrary. However, we shall see that it is quite a natural choice of measure. Perhaps the most convincing reason for this is that the push-forward measure $\pi'_* d\mu_{B'}$ is given by the same formula — see §5.1.8. (One cannot form the push-forward $\pi_* d\mu_B$ since the map $\pi_*$ is not proper.)

**(4.1.10)** We define a measure on $\widetilde{X}$ by $d\mu_{\widetilde{X}} = p^* d\mu_X$. Thus, by definition, we have

$$\int_{\widetilde{X}} f \, d\mu_{\widetilde{X}} = \int_X (p_* f) \, d\mu_X.$$

We now compute this measure in coordinates.

**Proposition.** *We have $d\mu_{\widetilde{X}}|_K = \frac{1}{2}(1 + d_K)|\Delta|_F d\mu_K$. Explicitly, this means that for $f \in \mathscr{S}(\widetilde{X})$ we have*

$$\int_{\widetilde{X}} f \, d\mu_{\widetilde{X}} = \tfrac{1}{2} \sum_K (1 + d_K) \int_K f \, |\Delta|_F d\mu_K$$

*where the sum is the four degree two étale algebras $K$.*

*Proof.* We first compute $(p_K)_* d\mu_K$. Pick a normalized basis for $K$ so that $\alpha = d_K$. For $f \in \mathscr{S}(X)$ we have

$$\int_K (p_K^* f) \, d\mu_K = \int_{F^2} f(2x_0, x_0^2 - d_K x_1^2) dx_0 dx_1 = d_K^{-1} \int_{F^2} f(t, \nu) \frac{1 + \eta(\Delta/\mathbf{d}_K)}{|\Delta/\mathbf{d}_K|_F^{1/2}} d\nu dt$$

where the measures above are the normalized Haar measures on $F$. In the first step we have used §4.1.7 and in the second §4.1.4. Replacing $f$ by $|\Delta|_F^{1/2} f$ gives

$$d_K^{1/2} \int_K (p_K^* f) |\Delta|_F^{1/2} d\mu_K = \int_{F^2} f(t, \nu)(1 + \eta(\Delta/\mathbf{d}_K)) d\nu dt$$

Summing over $K$ and using the fact that $\sum_K(1+\eta(\Delta/\mathbf{d}_K)) = 2$ gives

$$\int_X f(t,\nu)d\nu dt = \tfrac{1}{2}\sum_K d_K^{1/2}\int_K (p_K^* f)|\Delta|_F^{1/2}dx.$$

Replacing $f$ by $f|\Delta|_F^{1/2}A(\Delta)$ and using the fact that for $(t,\nu)\in \operatorname{im} p_K$ we have $A(\Delta)d_K^{1/2} = 1+d_K$ gives

$$\int_X f\, d\mu_X = \tfrac{1}{2}\sum_K (1+d_K)\int_K (p_K^* f)|\Delta|_F\, d\mu_K.$$

Finally, replacing $f$ with $p_* f$ gives

$$\int_{\widetilde{X}} f\, d\mu_{\widetilde{X}} = \tfrac{1}{2}\sum_K (1+d_K)\int_K (p_K^* p_* f)|\Delta|_F\, d\mu_K.$$

The result follows since $p_K^* p_* f$ and $f$ have the same integral over $K$. □

**(4.1.11)** We now study the push-forward map $\pi_*$ on functions. For a measurable function $f$ of compact support on $B$ we can form the push-forward measure $\pi_*(f d\mu_B)$ on $X$. It follows from general theory that this measure is absolutely continuous with respect to $d\mu_F(t)d\mu_F(\nu)$. It follows immediately from this that $\pi_*(f d\mu_B)$ is absolutely continuous with respect to $d\mu_X$ on $X_{\text{reg}}$. We define $\pi_* f$ to be the function on $X_{\text{reg}}$ given by the Radon-Nikodym derivative of $\pi_*(f d\mu_B)$ with respect to $d\mu_X$.

**Proposition.** *We have the following:*

1. *If $f$ belongs to $\mathscr{S}_{\text{reg}}(B)$ then $\pi_* f$ belongs to $\mathscr{S}_{\text{reg}}(X)$.*
2. *If $f$ belongs to $\mathscr{S}(B)$ then $\pi_* f$ belongs to $L^2(X)$.*
3. *For $f\in\mathscr{S}(B)$ and $g\in\mathscr{S}(X)$ we have $\langle \pi_* f, g\rangle_X = \langle f, \pi^* g\rangle_B$.*
4. *We have*

$$(\pi_* f)(t,\nu) = \frac{1}{|\Delta|_F^{1/2}A(\Delta)}\int_{F^2} f(\tfrac{1}{2}t + x_1 i + x_2 j \pm \sqrt{u}\, k)\frac{1+\eta(u)}{|u|_F^{1/2}}dx_1 dx_2$$

*where 1, $i$, $j$, $k$ is a standard basis for $B$, $dx_1$ and $dx_2$ are normalized Haar measures on $F$ and*

$$u = \frac{\Delta/4 - \alpha x_1^2 - \beta x_2^2}{\gamma}.$$

*Proof.* If $f$ belongs to $\mathscr{S}_{\text{reg}}(B)$ then $\pi_* f$ clearly has compact support; it is locally constant because

$\pi : B_{\mathrm{reg}} \to X_{\mathrm{reg}}$ is smooth. This proves (1). For (2) note that $\pi_* f$ can be written as $|\Delta|_F^{-1/2} f'$ where $f'$ is a continuous function of compact support on $X$. Thus $|\pi_* f|^2 d\mu_X = A(\Delta)|\Delta|_F^{-1/2}|f'|^2 d\nu dt$ and the result follows from the local integrability of $|\Delta|_F^{-1/2}$ on $F^2$. As for (3), we have

$$\begin{aligned} \langle \pi_* f, g\rangle_X &= \int_X g(x)(\pi_* f)(x) d\mu_X(x) = \int_X g(x)(\pi_*(f d\mu_B))(x) \\ &= \int_B (\pi^* g)(x) f(x) d\mu_B(x) = \langle f, \pi^* g\rangle_B. \end{aligned}$$

Finally, we come to (4). Let $f$ be a function on $B$ and $g$ a Schwartz function on $X$. We then have, by §4.1.6,

$$\langle f, \pi^* g, \rangle_B = q|\alpha\beta\gamma|_F^{1/2} \int_{F^4} f(x_0 + ix_1 + jx_2 + kx_3) g(t,\nu) dx_0 dx_1 dx_2 dx_3.$$

We now apply §4.1.4 to change the $x_3$ integral to an integral over $\nu$. The result is

$$q\left|\frac{\alpha\beta}{\gamma}\right|^{1/2} \int_{F^4} f(\tfrac{1}{2}t + ix_1 + jx_2 \pm \sqrt{u}\, k) g(t,\nu) \frac{1+\eta(u)}{|u|_F^{1/2}} dx_1 dx_2 dt d\nu$$

where $u$ is as in the statement of the proposition. As $\alpha\beta = -\gamma$ the absolute value in front of the integral is equal to 1. We thus have

$$\langle f, \pi^* g\rangle_B = \int_{F^2} f'(t,\nu) g(t,\nu) A(\Delta)|\Delta|_F^{1/2} dt d\nu = \langle f', g, \rangle_X$$

where

$$f'(t,\nu) = \frac{q}{A(\Delta)|\Delta|_F^{1/2}} \int_{F^2} f(\tfrac{1}{2}t + ix_1 + jx_2 \pm \sqrt{u}\, k) \frac{1+\eta(u)}{|u|_F^{1/2}} dx_1 dx_2.$$

We have thus shown

$$\langle \pi_* f, g\rangle_X = \langle f', g\rangle_X$$

for any $g$, which proves $\pi_* f = f'$ □

**(4.1.12)** We now recall the Weyl integration formula for $G$. First we define some measures. We define $d\mu_G = |\cdot|_B^{-1} d\mu_B$. It is a Haar measure on the group $G$. If $K$ is a degree two étale extension of $F$ and $T$ its group of units then we put $d\mu_T = |\cdot|_K^{-1} d\mu_K$. Finally, if we embed $K$ into $B$ so that $T$ can be regarded as a maximal torus of $G$ then we put $d\mu_{G/T} = d\mu_G / d\mu_T$. It is the unique left-invariant measure on $G/T$ having the property that

$$\int_G f(g)d\mu_G(g) = \int_{G/T}\int_T f(gt)d\mu_T(t)d\mu_{G/T}(g)$$

for any $f \in \mathscr{S}(G)$. The Weyl integration formula is then the identity

$$\int_G f(g)d\mu_G(g) = \tfrac{1}{2}\sum_T \int_T \int_{G/T} \left|\frac{\Delta(t)}{\det t}\right|_F f(gtg^{-1})d\mu_{G/T}(g)d\mu_T(t)$$

valid for any $f \in \mathscr{S}(G)$, where the sum is over a set of representatives of the conjugacy classes of maximal tori in $G$. We can also phrase the formula in a more additive manner, as follows:

$$\int_B f(x)d\mu_B(x) = \tfrac{1}{2}\sum_K \int_K \int_{G/T} |\Delta(x)|_F f(\sigma x\sigma^{-1})d\mu_{G/T}(\sigma)d\mu_K(x)$$

where $f$ belongs to $\mathscr{S}(B)$, the sum is over isomorphism classes of degree two étale algebras over $F$ (regarded as embedding into $B$) and $T = K^\times$.

**(4.1.13)** We now give a formula for $\pi_*$ as an orbital integral.

**Proposition.** *Let $f$ belong to $\mathscr{S}(B)$, let $x$ be a regular element of $G$, let $K$ be the maximal commutative subalgebra of $B$ containing $x$ and let $T = K^\times$. Then*

$$(\pi^*\pi_* f)(x) = \frac{1}{1+d_K}\int_{G/T} f(\sigma x\sigma^{-1})d\mu_{G/T}(\sigma).$$

*Proof.* Let $g$ belong to $\mathscr{S}(X)$. Then

$$\begin{aligned}\langle \pi^* g, f\rangle_B =& \int_B (\pi^* g)(x)f(x)d\mu_B(x)\\ =& \tfrac{1}{2}\sum_K \int_K |\Delta(x)|_F (p_K^* g)(x)\left[\int_{G/T} f(\sigma x\sigma^{-1})d\mu_{G/T}(\sigma)\right]d\mu_K(x)\end{aligned}$$

where in the second step we applied the Weyl integration formula of §4.1.12 and used the fact that for $x \in K \subset B$ we have $(\pi^* g)(\sigma x\sigma^{-1}) = (p_K^* g)(x)$. On the other hand, by §4.1.10 we have

$$\begin{aligned}\langle g, \pi_* f\rangle_X =& \int_X g(x)(\pi_* f)(x)d\mu_X(x)\\ =& \tfrac{1}{2}\sum_K (1+d_K)\int_K |\Delta(x)|_F (p_K^* g)(x)(\pi^*\pi_* f)(x)d\mu_K(x).\end{aligned}$$

Here we are using the fact that for $x \in K \subset B$ we have $(p_K^*\pi_* f)(x) = (\pi^*\pi_* f)(x)$. Comparing the two expressions gives the stated result. □

**(4.1.14)** We now give an improvement of the previous formula at the elliptic elements. First some definitions. We write $Z$ for the center of the group $G$; it is identified with $F^\times$. We let $d\mu_Z = |\cdot|_F^{-1} d\mu_F$ and we let $d\mu_{G/Z}$ be the quotient measure $d\mu_G/d\mu_Z$. Our formula is then:

**Proposition.** *Let $f$ belong to $\mathscr{S}(B)$, let $x$ be a regular elliptic element of $G$, let $K$ be the maximal commutative subalgebra of $B$ containing $x$ and let $T = K^\times$. Then*

$$(\pi^*\pi_* f)(x) = c \int_{G/Z} f(\sigma x \sigma^{-1}) d\mu_{G/Z}(\sigma)$$

*where $c = \frac{1}{2} q/(q+1)$.*

*Proof.* We have

$$\begin{aligned}\int_{G/Z} f(\sigma x \sigma^{-1}) d\mu_{G/Z}(\sigma) &= \int_{G/T} \int_{T/Z} f(\sigma x \sigma^{-1}) d\mu_{T/Z}(\sigma') d\mu_{G/T}(\sigma) \\ &= (1 + d_K) \operatorname{Vol}(T/Z)(\pi^*\pi_* f)(x)\end{aligned}$$

where in the second step we used §4.1.13. We therefore need only show

$$\operatorname{Vol}(T/Z) = 2 \cdot \frac{1 + q^{-1}}{1 + d_K}.$$

Now, if $K/F$ is unramified then $K^\times/F^\times = U_K/U_F$ and so $\operatorname{Vol}(T/Z) = \operatorname{Vol}(U_K)/\operatorname{Vol}(U_F)$. By our normalizations, $\operatorname{Vol}(U_K) = 1 - q^{-2}$ and $\operatorname{Vol}(U_F) = 1 - q^{-1}$ and so we have the stated result. If $K/F$ is ramified then $U_K/U_F$ has index two in $K^\times/F^\times$. Thus $\operatorname{Vol}(T/Z) = 2\operatorname{Vol}(U_K)/\operatorname{Vol}(U_F)$. As $\operatorname{Vol}(U_K) = \operatorname{Vol}(U_F) = 1 - q^{-1}$ we have the stated result. □

**(4.1.15)** Let $U$ be a compact open subset of $G/Z$. For a function $f$ on $B$ we put

$$\operatorname{avg}_U f = c \int_{G/Z} f^\sigma \, d\mu_{G/Z}(\sigma).$$

Here $f^\sigma$ is the function $x \mapsto f(\sigma x \sigma^{-1})$. We also define

$$\operatorname{avg} f = c \int_{G/Z} f^\sigma \, d\mu_{G/Z}(\sigma).$$

We regard $\operatorname{avg} f$ as a function on the regular elliptic elements of $B$ and extend it by zero to all of $B$. The integral defining $\operatorname{avg} f$ will not exist for all functions $f$; it does, however, for all $f$ in $\mathscr{S}_{\mathrm{re}}(B)$.

In fact, for $f \in \mathscr{S}_{\mathrm{re}}(B)$ the results of §4.1.14 gives $\operatorname{avg} f = \pi^*(\pi_* f)$. One easily verifies that $\operatorname{avg}_U$ is adjoint to $\operatorname{avg}_{U^{-1}}$ and that avg is self-adjoint (on $\mathscr{S}_{\mathrm{re}}(B)$). Furthermore, we have:

**Proposition.** *Let $f$ belong to $\mathscr{S}_{\mathrm{re}}(B)$ and let $W$ be a compact subset of $B_{\mathrm{re}}$. Then for $U$ sufficiently large we have $(\operatorname{avg} f)(x) = (\operatorname{avg}_U f)(x)$ for all $x \in W$. Thus $\operatorname{avg}_U f \to \operatorname{avg} f$ pointwise as $U \to G/Z$*

*Proof.* The map $i : G/Z \times W \to G$ given by $(\sigma, x) \mapsto \sigma x \sigma^{-1}$ is proper since $W$ is a compact set of regular elliptic elements. Thus $i^{-1}(\operatorname{supp} f)$ is a compact subset of $G/Z \times W$ and so we can pick a compact subset $V$ of $G/Z$ such that $V \times W$ contains it. We then have

$$\int_{G/Z} f(\sigma x \sigma^{-1}) d\mu_{G/Z}(\sigma) = \int_V f(\sigma x \sigma^{-1}) d\mu_{G/Z}(\sigma)$$

for all $x \in W$. Thus $(\operatorname{avg} f)(x) = (\operatorname{avg}_U f)(x)$ whenever $U$ contains $V$. $\square$

**(4.1.16)** We now show how one can lift regular Schwartz functions on $X$ to regular Schwartz functions on $B$. This is one of the key ingredients that goes into the factorization of the Fourier transform we will give later in this section.

**Proposition.** *Let $K \subset B$ be a degree two étale algebra and let $f$ belong to $\mathscr{S}_{\mathrm{reg}}(K)$. For any sufficiently small compact open set $\mathfrak{a}$ of $K^\perp$ containing 0 we have*

$$\pi_*(f \otimes \delta_{\mathfrak{a}}) = \frac{2}{1 + d_K} |\Delta|_F^{-1} (p_K)_* f.$$

*Here $\delta_{\mathfrak{a}} = \frac{1}{\mathrm{Vol}(\mathfrak{a})} \chi_{\mathfrak{a}}$ where $\chi_{\mathfrak{a}}$ is the characteristic function of $\mathfrak{a}$ and $\mathrm{Vol}(\mathfrak{a})$ is the volume of $\mathfrak{a}$ with respect to $d\mu_{K^\perp}$.*

*Proof.* Pick a standard basis for $B$ so that 1, $i$ is a standard basis for $K$. Take $\mathfrak{a}$ to be the set of $x \in K^\perp$ with $x_2 \in \mathfrak{p}^n$ and $x_3 \in \mathfrak{p}^m$, for fixed integers $n$ and $m$. (It suffices to consider such sets for $\mathfrak{a}$.) We have $\mathrm{Vol}(\mathfrak{a}) = |\mathbf{d}_K \beta\gamma|_F^{1/2} q^{-n-m}$. Using our formula for $\pi_*$ from §4.1.11, we find

$$(\pi_*(f \otimes \delta_{\mathfrak{a}})(t, \nu) = \frac{1}{\mathrm{Vol}(\mathfrak{a})} \cdot \frac{1}{A(\Delta)|\Delta|_F^{1/2}} \int_{F^2} f(\tfrac{1}{2}t + ix_1) \chi_{\mathfrak{p}^n}(x_2) \chi_{\mathfrak{p}^m}(\sqrt{u}) \frac{1 + \eta(u)}{|u|_F^{1/2}} dx_1 dx_2$$

where

$$u = \frac{\Delta/4 - \alpha x_1^2 - \beta x_2^2}{\gamma}.$$

Denote by $I$ the integral in the above expression. We now use §4.1.4 to change variables so that we integrate over $u$ instead of $x_1$. Putting

$$v = \Delta/4\alpha + \beta u - \beta\alpha^{-1}x_2^2$$

we obtain

$$I = |\beta|_F \int_{F^2} f(\tfrac{1}{2}t \pm i\sqrt{v})\chi_{\mathfrak{p}^n}(x_2)\chi_{\mathfrak{p}^m}(\sqrt{u})\frac{1+\eta(v)}{|v|_F^{1/2}}\frac{1+\eta(u)}{|u|_F^{1/2}}dudx_2.$$

Pick an integer $N$ such that $f(x+iy)$ only depends on $y$ modulo $\mathfrak{p}^N$. Since $f$ is regular, $f(x+iy)$ will vanish for $y \in \mathfrak{p}^N$. We now take $n$ and $m$ so large that $\beta\alpha^{-1}\mathfrak{p}^{2n} \subset \mathfrak{p}^{2N}$ and $\beta\mathfrak{p}^m \subset \mathfrak{p}^{2N}$. We thus have $v = \Delta/4\alpha$ modulo $\mathfrak{p}^{2N}$. Thus if $\Delta/4\alpha$ belongs to $\mathfrak{p}^{2N}$ then the integral will vanish. This shows that $\pi_*(f \otimes \delta_{\mathfrak{a}})$ has regular support. Now, if $\Delta/4\alpha$ does not belong to $\mathfrak{p}^{2N}$ then we see that $v$ is a square if and only if $\Delta/4\alpha$ is and also $|v|_F = |\Delta/\alpha|_F$. Thus if $\Delta/4\alpha$ is not a square then the integral vanishes. This shows that $\pi_*(f \otimes \delta_{\mathfrak{a}})$ is supported on $\operatorname{im} p_K$. Now assume that $\Delta/4\alpha$ is a square, so that $v$ is as well. The value of $f(\frac{1}{2}t \pm i\sqrt{v})$ is then independent of $u$ and $x_2$ as they vary in $\mathfrak{p}^{2m}$ and $\mathfrak{p}^n$. We thus find $f(\frac{1}{2}t \pm i\sqrt{u}) = ((p_K)_*f)(t,\nu)$. Therefore

$$I = \frac{2|\beta|_F}{|\Delta/\alpha|_F^{1/2}}((p_K)_*f)(t,\nu)\int_{F^2}\chi_{\mathfrak{p}^n}(x_2)\chi_{\mathfrak{p}^m}(\sqrt{u})\frac{1+\eta(u)}{|u|_F^{1/2}}dudx_2$$

The $x_2$ integral here is just $q^{-n}$. The $u$ integral is easily evaluated and found to be $q^{-m}$. We thus obtain

$$I = \frac{2|\beta|_F q^{-n-m}}{|\Delta/\alpha|_F^{1/2}}((p_K)_*f)(t,\nu).$$

Putting this into our formula for $(\pi_*(f \otimes \delta_{\mathfrak{a}}))(t,\nu)$ and using our formula for $\mathrm{Vol}(\mathfrak{a})$ gives

$$\pi_*(f \otimes \delta_{\mathfrak{a}}) = \frac{1}{A(\Delta)|\Delta|_F^{1/2}} \cdot \frac{1}{|\mathbf{d}_K\beta\gamma|_F^{1/2}q^{-n-m}} \cdot \frac{2|\beta|_F q^{-n-m}}{|\Delta/\alpha|_F^{1/2}}(p_K)_*f$$

which after simplification gives

$$\pi_*(f \otimes \delta_{\mathfrak{a}}) = \frac{2}{A(\Delta)d_K^{1/2}|\Delta|_F}(p_K)_*f.$$

Finally the identity $A(\Delta)d_K^{1/2} = 1 + d_K$ gives the stated result. □

**Corollary.** *Let $f \in \mathscr{S}_{\mathrm{reg}}(K)$. For any sufficiently small compact open set $\mathfrak{a}$ of $K^\perp$ containing 0 we have*

$$(p_K)_*f = \tfrac{1}{2}(1+d_K)\pi_*(|\Delta|_F f \otimes \delta_{\mathfrak{a}}).$$

*Proof.* Apply the proposition to $|\Delta|_F f$, which belongs to $\mathscr{S}_{\mathrm{reg}}(K)$. ☐

**Corollary.** *The map* $\pi_* : \mathscr{S}_{\mathrm{reg}}(B) \to \mathscr{S}_{\mathrm{reg}}(X)$ *is surjective.*

*Proof.* It suffices to show that for $f \in \mathscr{S}_{\mathrm{reg}}(K)$ and $\mathfrak{a}$ a sufficiently small subset of $K^\perp$ the function $f \otimes \delta_{\mathfrak{a}}$ belongs to $\mathscr{S}_{\mathrm{reg}}(B)$. This was essentially proven in the course of proving the proposition and in any case is easy enough that it is left to the reader. ☐

## 4.2 The Fourier transform $\mathscr{F}^{(1)}_{X,\psi}$

**(4.2.1)** The purpose of §4.2 is to define a Fourier transform on the space of Schwartz functions on $X$ with regular elliptic support and prove a factorization result for it. Here is an overview:

- In §§4.2.2–4.2.4 we recall the Fourier transforms on $B$, $K$ and $K^\perp$.
- In §4.2.5 we establish the fundamental result that allows for the definition of the Fourier transform. This result says, roughly, that the Fourier transform commutes with the operator $\pi^*\pi_*$ on the space of Schwartz functions on $B$ with regular elliptic support.
- In §4.2.6 we define the Fourier transform $\mathscr{F}^{(1)}_{X,\psi}$ by the formula $\mathscr{F}^{(1)}_{X,\psi}(\pi_* f) = \pi_*(\mathscr{F}_{B,\psi} f)$ where $f$ is a Schwartz function on $B$ with regular elliptic support.
- In §4.2.7 we factor the Fourier transform $\mathscr{F}^{(1)}_{X,\psi}$ as $\overline{p}_* \mathscr{F}'_{\widetilde{X}}$, where $\mathscr{F}'_{\widetilde{X}}$ is essentially the usual Fourier transform on the various $K$'s and $\overline{p}_*$ is some operator which has a fairly simple form. This factorization results from taking the Fourier transform of the identity given in §4.1.16.
- In §4.2.8 we compute an explicit formula for $\overline{p}_*$. It is this formula which we will ultimately use in the comparison step.

**(4.2.2)** Let $\psi = \psi_F$ be a non-trivial additive character of $F$. Define $\psi_B$ to be the additive character of $B$ given by $\psi_F \circ \mathrm{tr}_{B/F}$. For a function $f$ on $B$ we put

$$(\mathscr{F}_{B,\psi} f)(x) = q^{-2m} \int_B f(y)\psi_B(xy)\, d\mu_B(y).$$

Here $m = m(\psi)$ is the largest integer for which $\psi$ is trivial on $\mathfrak{p}^{-m}$. It is a standard fact (and easy to prove) that $\mathscr{F}_{B,\psi}$ induces an isomorphism $\mathscr{S}(B) \to \mathscr{S}(B)$ and can be extended to a continuous isomorphism $L^2(B) \to L^2(B)$. By our normalization, $\mathscr{F}_{B,\psi}$ is self-adjoint with respect to $\langle , \rangle_B$, an isometry with respect to $( , )_B$ and has inverse $\mathscr{F}_{B,\overline{\psi}}$,

**(4.2.3)** Let $K/F$ be a degree two étale algebra. Define $\psi_K$ to be the additive character of $K$ given by $\psi_F \circ \mathrm{tr}_{K/F}$. For a function $f$ on $K$ we put

$$(\mathscr{F}_{K,\psi} f)(x) = q^{-m} d_K^{1/2} \int_K f(y)\psi_K(xy) d\mu_K(y)$$

Again, $\mathscr{F}_{K,\psi}$ maps $\mathscr{S}(K)$ into itself isomorphically and extends to a continuous automorphism of $L^2(K)$. By our normalization, $\mathscr{F}_{K,\psi}$ is self-adjoint with respect to $\langle , \rangle_K$, an isometry with respect to $( , )_K$ and has inverse $\mathscr{F}_{K,\overline{\psi}}$.

**(4.2.4)** Let $K \subset B$ be a degree two étale algebra and let $K^\perp$ be its orthogonal complement. For a function $f$ on $K^\perp$ we put

$$(\mathscr{F}_{K^\perp,\psi} f)(x) = q^{-m} d_K^{-1/2} \int_{K^\perp} f(y)\psi_B(xy) d\mu_{K^\perp}(y).$$

As usual, $\mathscr{F}_{K^\perp,\psi}$ takes $\mathscr{S}(K^\perp)$ to itself and extends to $L^2(K^\perp)$. The key property of the above definition is the following: for $f \in \mathscr{S}(K)$ and $g \in \mathscr{S}(K^\perp)$ we have

$$\mathscr{F}_{B,\psi}(f \otimes g) = (\mathscr{F}_{K,\psi} f) \otimes (\mathscr{F}_{K^\perp,\psi} g)$$

where $f \otimes g$ is the function on $B$ given by $(x, y) \mapsto f(x)g(y)$, where $B$ is identified with $K \times K^\perp$.

**(4.2.5)** We now give the fundamental result which will allow us to define a Fourier transform on $X$.

**Proposition.** *For $f, g \in \mathscr{S}_{\mathrm{re}}(B)$ we have*

$$\langle \operatorname{avg} f, \mathscr{F}_{B,\psi} g\rangle_B = \langle \mathscr{F}_{B,\psi} f, \operatorname{avg} g\rangle_B.$$

*Proof.* We have

$$\operatorname{avg}_U(\mathscr{F}_{B,\psi} f) = \int_U (\mathscr{F}_{B,\psi} f)^\sigma \, d\mu_{G/Z}(\sigma) = \int_U \mathscr{F}_{B,\psi}(f^{\sigma^{-1}}) \, d\mu_{G/Z}(\sigma) = \mathscr{F}_{B,\psi}(\operatorname{avg}_{U^{-1}} f).$$

We therefore have

$$\langle \operatorname{avg}_U f, \mathscr{F}_{B,\psi} g\rangle_B = \langle \mathscr{F}_{B,\psi} f, \operatorname{avg}_U g\rangle_B.$$

If we now take $U$ so large such that $\operatorname{avg}_U f = \operatorname{avg} f$ holds on the support of $\mathscr{F}_{B,\psi} g$ and $\operatorname{avg}_{U^{-1}} g = \operatorname{avg} g$ holds on the support of $\mathscr{F}_{B,\psi} f$ then we obtain the stated identity. (It is possible to choose $U$

as such by §4.1.15.) □

**(4.2.6)** We now define a Fourier transform

$$\mathscr{F}^{(1)}_{X,\psi} : \mathscr{S}(X_{\mathrm{re}}) \to \mathscr{C}^{\infty}(X_{\mathrm{re}})$$

by $\mathscr{F}^{(1)}_{X,\psi} f = \pi_*(\mathscr{F}_{B,\psi}\widetilde{f})|_{X_{\mathrm{re}}}$ where $\widetilde{f}$ is any element of $\mathscr{S}_{\mathrm{re}}(B)$ such that $\pi_*\widetilde{f} = f$. The following proposition shows that this is well-defined.

**Proposition.** *The map $\mathscr{F}^{(1)}_{X,\psi}$ is well-defined. Furthermore, the Fourier transform commutes with pull-back in the sense that for $f \in \mathscr{S}_{\mathrm{re}}(X)$ and $g \in \mathscr{S}_{\mathrm{re}}(B)$ we have*

$$\langle \pi^*(\mathscr{F}^{(1)}_{X,\psi} f), g\rangle_B = \langle \mathscr{F}_{B,\psi}(\pi^* f), g\rangle_B.$$

*Here $\mathscr{F}_{B,\psi}(\pi^* f)$ is the Fourier transform of $\pi^* f$ in the sense of distributions.*

*Proof.* Let $\widetilde{f}$ and $g$ be two elements of $\mathscr{S}_{\mathrm{re}}(B)$. We have

$$\langle \pi^*\pi_*\widetilde{f}, \mathscr{F}_{B,\psi} g\rangle_B = \langle \mathscr{F}_{B,\psi}\widetilde{f}, \pi^*\pi_* g\rangle_B = \langle \pi_*\mathscr{F}_{B,\psi}\widetilde{f}, \pi_* g\rangle_X.$$

Here we have used §4.2.5 and the identity $\pi^*\pi_* = \mathrm{avg}$ on $\mathscr{S}_{\mathrm{re}}(B)$, *c.f.* §4.1.15. This shows that if $\pi_*\widetilde{f} = 0$ then $\pi_*(\mathscr{F}_{B,\psi}\widetilde{f})$ pairs to zero with each element of $\mathscr{S}_{\mathrm{re}}(B)$, since every element of $\mathscr{S}_{\mathrm{re}}(B)$ is of the form $\pi_* g$ by §4.1.16. Thus if $\pi_*\widetilde{f} = 0$ then $\pi_*(\mathscr{F}_{\psi}\widetilde{f})|_{X_{\mathrm{re}}} = 0$ and so $\mathscr{F}^{(1)}_{X,\psi}$ is well-defined. Writing $f = \pi_*\widetilde{f}$, the above equation and some adjointness relations give

$$\langle \pi^* f, \mathscr{F}_{B,\psi} g\rangle_B = \langle \pi^*(\mathscr{F}^{(1)}_{X,\psi} f), g\rangle_B,$$

which proves the statement about the Fourier transform commuting with $\pi^*$. □

**(4.2.7)** We now prove a factorization result for the Fourier transform. For a degree two étale algebra $K/F$ let $\mathscr{S}_0(K)$ be the set of functions $f$ in $\mathscr{S}(K)$ which satisfy

$$\int_F f(x_0 + ix_1)\,dx_1 = 0$$

for all $x_0$. In words, these functions have integral 0 on "purely imaginary" vertical strips. The Fourier transform on $K$ gives an isomorphism $\mathscr{F}_{K,\psi} : \mathscr{S}_{\mathrm{reg}}(K) \to \mathscr{S}_0(K)$. We let $\mathscr{S}_0(\widetilde{X}_{\mathrm{ell}})$ be the space of $f \in \mathscr{S}(\widetilde{X}_{\mathrm{ell}})$ for which $f|_K$ belongs to $\mathscr{S}_0(K)$ for all quadratic fields $K$. It will be convenient to have a slight modification of the Fourier transform in what follows. We define the *modified Fourier transform* on $\mathscr{S}(K)$, denoted $\mathscr{F}'_{K,\psi}$ by

$$\mathscr{F}'_{K,\psi} f = \tfrac{1}{2} q^{-m}(d_K^{1/2} + d_K^{-1/2})\mathscr{F}_{K,\psi}(|\Delta|_F f).$$

As multiplication by $|\Delta|_F$ gives an isomorphism $\mathscr{S}_{\mathrm{reg}}(K) \to \mathscr{S}_{\mathrm{reg}}(K)$ the modified Fourier transform still gives an isomorphism $\mathscr{F}'_{K,\psi} : \mathscr{S}_{\mathrm{reg}}(K) \to \mathscr{S}_0(K)$. We let $\mathscr{F}'_{\widetilde{X}_{\mathrm{ell}},\psi}$ be the Fourier transform on $\widetilde{X}_{\mathrm{ell}}$ gotten from the $\mathscr{F}'_{K,\psi}$. We now have our main result:

**Proposition.** *There is a unique map $\overline{p}_* : \mathscr{S}_0(\widetilde{X}_{\mathrm{ell}}) \to \mathscr{C}^\infty(X_{\mathrm{re}})$ such that the diagram*

$$\begin{array}{ccc} \mathscr{S}(\widetilde{X}_{\mathrm{re}}) & \xrightarrow{\ p_*\ } & \mathscr{S}(X_{\mathrm{re}}) \\ {\scriptstyle \mathscr{F}'_{\widetilde{X}_{\mathrm{ell}},\psi}}\big\downarrow & & \big\downarrow{\scriptstyle \mathscr{F}^{(1)}_{X,\psi}} \\ \mathscr{S}_0(\widetilde{X}_{\mathrm{ell}}) & \xrightarrow{\ \overline{p}_*\ } & \mathscr{C}^\infty(X_{\mathrm{re}}) \end{array}$$

*commutes. For $f \in \mathscr{S}_0(K)$ we have*

$$(\overline{p}_K)_* f = \pi_*(f \otimes \chi_{\mathfrak{a}})$$

*where $\mathfrak{a}$ is any sufficiently large compact open subset of $K^\perp$. Here $(\overline{p}_K)_*$ is just the restriction of $\overline{p}_*$ to $\mathscr{S}_0(K)$.*

*Proof.* The map $\overline{p}_*$ exists and is unique since the arrow labeled $\mathscr{F}'_{\widetilde{X}_{\mathrm{ell}},\psi}$ in the diagram is an isomorphism. The point of the proposition is the formula for $\overline{p}_*$. We have

$$\begin{aligned} \mathscr{F}_{X,\psi}((p_K)_* f) &= \tfrac{1}{2}(1 + d_K)\mathscr{F}_{X,\psi}(\pi_*(|\Delta|_F f \otimes \delta_{\mathfrak{a}})) \\ &= \tfrac{1}{2}(1 + d_K)\pi_*(\mathscr{F}_{K,\psi}(|\Delta|_F f) \otimes \mathscr{F}_{K^\perp,\psi}(\delta_{\mathfrak{a}})) \end{aligned}$$

In the first step we used §4.1.16 and in the second §4.2.4, together with the fact that $\mathscr{F}_{X,\psi}\pi_* = \pi_*\mathscr{F}_{B,\psi}$. One easily verifies that $\mathscr{F}_{K^\perp,\psi}(\delta_{\mathfrak{a}}) = q^{-m} d_K^{-1/2} \chi_{\mathfrak{a}'}$ where $\mathfrak{a}'$ is a large compact open. The proposition follows. □

**(4.2.8)** We now explicitly compute the map $\overline{p}_*$.

**Proposition.** *Let $f$ belong to $\mathscr{S}_0(K)$. Then*

$$((\overline{p}_K)_* f)(t,\nu) = \frac{d_K^{1/2}}{|\Delta|_F^{1/2} A(\Delta)} \int_F f(\tfrac{1}{2}t + ix) I_2(\Delta - 4\mathbf{d}_K x^2, \mathbf{d}_K) d\mu_F(x)$$

*where $i \in K$ is such that $i^2 = \mathbf{d}_K$.*

*Proof.* Pick a standard basis for $B$ so that 1, $i$ is a standard basis for $K$. We take this basis so that $\alpha = \mathbf{d}_K$, $\beta = -\mathbf{d}_K$ and $\gamma = \mathbf{d}_K^2$. Let $\mathfrak{a}$ be the open set of $K^\perp$ consisting of those $x$ for which $x_2 \in \mathfrak{p}^{-n}$ and $x_3 \in \mathfrak{p}^{-m}$ for large integers $n$ and $m$. Our formula for $\pi_*$ from §4.1.11 then gives

$$(\pi_*(f \otimes \chi_{\mathfrak{a}}))(t,\nu) = \frac{1}{|\Delta|_F^{1/2} A(\Delta)} \int_{F^2} f(\tfrac{1}{2}t + ix_1)\chi_{\mathfrak{p}^{-n}}(x_2)\chi_{\mathfrak{p}^{-m}}(\sqrt{u})\frac{1+\eta(u)}{|u|_F^{1/2}} dx_1 dx_2$$

with

$$u = \frac{\Delta/4 - \alpha x_1^2 - \beta x_2^2}{\gamma}.$$

Let $t$ and $\nu$ be given. We are free to enlarge $n$ and $m$ as this will not change the value of the integral. The condition that $f(\frac{1}{2}t + ix_1)\chi_{\mathfrak{p}^{-n}}(x_2)$ be non-zero puts a bound on $|u|_F$ (since $f$ has compact support). Thus by taking $m$ to be sufficiently large, the non-vanishing of $f(\frac{1}{2}t + ix_1)\chi_{\mathfrak{p}^{-n}}(x_2)$ will imply $\sqrt{u} \in \mathfrak{p}^{-m}$. We can therefore remove the $\chi_{\mathfrak{p}^{-m}}(\sqrt{u})$ from the integrand without changing the value of the integral.

Now, for $u$ to be a square it must be that $|\beta x_2^2|_F \le |\Delta/4 - \alpha x_1^2|_F$ as $-\beta = \mathbf{d}_K$ is not a square. Thus if $u$ is a square then $|\beta x_2^2|_F \le \max(|\Delta|_F, |\alpha x_1^2|_F)$. As $f(\frac{1}{2}t + ix_1)$ vanishes for $|x_1|_F$ large, we see that the condition that $u$ be a square forces $x_2$ to belong to $\mathfrak{p}^{-n}$ if $n$ is sufficiently large. It follows that we can pick $n$ sufficiently large so that $u$ being a square implies $x_2 \in \mathfrak{p}^{-n}$. Therefore we can remove the $\chi_{\mathfrak{p}^{-n}}(x_2)$ from the integrand without changing the value of the integral.

We have thus shown that

$$(\pi_*(f \otimes \chi_{\mathfrak{a}}))(t,\nu) = \frac{1}{|\Delta|_F^{1/2} A(\Delta)} \int_{F^2} f(\tfrac{1}{2}t + ix_1)\frac{1+\eta(u)}{|u|_F^{1/2}} dx_1 dx_2.$$

The $x_2$ integral is now equal to $|b|_F^{-1/2} I_2(a,b)$ (see §3.2) with

$$a = \frac{\Delta/4 - \alpha x_1^2}{\gamma} = \frac{\Delta - 4\mathbf{d}_K x_1^2}{4\mathbf{d}_K^2}, \qquad b = -\frac{\beta}{\gamma} = \mathbf{d}_K^{-1}.$$

We thus find (using some basic properties of $I_2$)

$$((\overline{p}_K)_* f)(t,\nu) = \frac{d_K^{1/2}}{|\Delta|_F^{1/2} A(\Delta)} \int_F f(\tfrac{1}{2}t + ix) I_2(\Delta - 4\mathbf{d}_K x^2, \mathbf{d}_K) dx,$$

which is the stated result. □

## 4.3 The cuspidal space $H(X)$

**(4.3.1)** The purpose of §4.3 is to compute push-forwards and inner products of matrix coefficients of cuspidal representations and introduce the cuspidal space. Here is an overview:

- Is §4.3.2 and §4.3.3 we introduce the matrix coefficients $\phi_{\pi,v,v^*}$ of cuspidal representations and recall the Schur orthogonality relations.
- In §4.3.4 we introduce certain truncated matrix coefficient functions $\phi_{\pi,v,v^*,n}$ and prove a Schur orthogonality type result for them. The $\phi_{\pi,v,v^*,n}$, unlike the $\phi_{\pi,v,v^*}$, have compact support.
- In §4.3.5 we recall the definition of the character of an irreducible admissible representation.
- In §4.3.6 we relate the push-forwards of matrix coefficients to characters. The result is that $\pi_*(\phi_{\pi,v,v^*,n})$ is given by $d_\pi^{-1}\langle v, v^*\rangle\phi_{\pi,n}$ where $d_\pi$ is the formal degree of $\pi$ and $\phi_{\pi,n}$ is a certain truncation of the character of $\pi$.
- In §4.3.7 we prove a Schur orthogonality type result for the $\phi_{\pi,n}$.
- In §4.3.8 we define the space $H(X)$ of cuspidal functions on $X$ as the $L^2$-closure of the space spanned by the $\phi_{\pi,n}$.
- In §4.3.9 we give a characterization of $H(X)$ which is independent of representation theory.

**(4.3.2)** Let $\pi$ be an cuspidal representation of $G$ (by which we will always mean an irreducible admissible representation which is cuspidal). Let $V$ be the representation space of $\pi$ and $V^\vee$ the space of the contragredient $\pi^\vee$. For $v \in V$ and $v^* \in V^\vee$ we define

$$\phi_{\pi,v,v^*}(g) = \langle \pi(g)v, v^*\rangle.$$

The function $\phi_{\pi,v,v^*}$ is called a *matrix coefficient* of $\pi$. As $\pi$ is cuspidal, such functions have compact support modulo the center. The matrix coefficients satisfy the Schur orthogonality relations, which we now recall. Let $\pi_1$ and $\pi_2$ be cuspidal representations whose central characters are inverse to each other. Then $\phi_{\pi_1,v_1,v_1^*} \cdot \phi_{\pi_2,v_2,v_2^*}$ transforms trivially under the center and defines a Schwartz function on $G/Z$. We then have

$$\langle \phi_{\pi_1,v_1,v_1^*}, \phi_{\pi_2,v_2,v_2^*}\rangle_{G/Z} = 0$$

if $\pi_1$ and $\pi_2$ are not contragredient to each other. If $\pi_1$ and $\pi_2$ are contragredient then

$$\langle \phi_{\pi_1,v_1,v_1^*}, \phi_{\pi_2,v_2,v_2^*} \rangle_{G/Z} = d_{\pi_1}^{-1} \langle v_1, v_2 \rangle \langle v_1^*, v_2^* \rangle.$$

We now explain the right side. First, $d_{\pi_1}$ is a non-negative real number, called the *formal degree* of $\pi_1$. Fix an isomorphism $\alpha : \pi_2 \to \pi_1^\vee$. We then have an isomorphism $(\alpha^\vee)^{-1} : \pi_2^\vee \to \pi_1$. The right side of the above equation is to be interpreted as

$$d_{\pi_1}^{-1} \langle v_1, \alpha(v_2) \rangle \langle v_1^*, (\alpha^\vee)^{-1}(v_2^*) \rangle$$

This is independent of the choice of $\alpha$. If $\pi_2$ is equal to the contragredient of $\pi_1$ then one can take $\alpha$ to be the identity map and the above formula looks a bit more pretty.

**(4.3.3)** Now let $\pi$ be a unitary cuspidal representation. Let $(,)$ be the invariant Hermitian form on the representation space $V$ of $\pi$. We always use the convention that such forms are linear in the first variable and conjugate linear in the second. For $v$ and $v'$ in $V$ we define

$$\phi'_{\pi,v,v'}(g) = (\pi(g)v, v').$$

As $(-, v')$ is an element of the contragredient of $V$ the above is just a matrix coefficient of $\pi$. However it will be convenient to use this kind of matrix coefficients at times. Schur orthogonality for these functions can be written as follows: if $\pi_1$ and $\pi_2$ have the same central character then

$$(\phi'_{\pi_1,v_1,v_1'}, \phi'_{\pi_2,v_2,v_2'})_{G/Z} = 0$$

if $\pi_1$ and $\pi_2$ are not isomorphic. Furthermore,

$$(\phi'_{\pi,v_1,v_1'}, \phi'_{\pi,v_2,v_2'})_{G/Z} = d_\pi^{-1}(v_1, v_2)(v_2', v_1').$$

In particular,

$$\|\phi'_{\pi,v,v'}\|^2_{G/Z} = d_\pi^{-1} |(v, v')|^2$$

These orthogonality relations easily deduced from the ones in §4.3.2.

**(4.3.4)** Let $\pi$ be a cuspidal representation of $G$ on the space $V$. For $v \in V$, $v^* \in V^\vee$ and $n \in \mathbb{Z}$ we let $\phi_{\pi,v,v^*,n}$ be the function which is equal to the matrix coefficient $\phi_{\pi,v,v^*}$ on the locus $G_n$ in

$G$ where the determinant has valuation $n$ and 0 off of this locus. For unitary $\pi$ we define $\phi'_{\pi,v,v',n}$ in the analogous manner. These belong to the Schwartz space $\mathscr{S}(B)$. Recall that $\xi : G \to \{\pm 1\}$ is the character $g \mapsto (-1)^{\mathrm{val}(\det g)}$. We call a representation $\pi$ *even* if $\pi$ is equivalent to $\xi \otimes \pi$ and *odd* otherwise. We now have the following:

**Proposition.** *We have*

$$\langle \phi_{\pi_1,v_1,v_1^*,n}, \phi_{\pi_2,v_2,v_2^*,m} \rangle_G = 0$$

*unless $n = m$ and $\pi_1$ is isomorphic to an unramified twist of $\pi_2^\vee$. We have*

$$\langle \phi_{\pi,v_1,v_1^*,n}, \phi_{\pi^\vee,v_2,v_2^*,n} \rangle_G = \tfrac{1}{2}(1-q^{-1}) d_\pi^{-1} \langle v_1, v_2 \rangle \langle v_1^*, v_2^* \rangle$$

*if $\pi$ is odd, while*

$$\langle \phi_{\pi,v_1,v_1^*,n}, \phi_{\pi^\vee,v_2,v_2^*,n} \rangle_G = \tfrac{1}{2}(1-q^{-1}) d_\pi^{-1} \big( \langle v_1, v_2 \rangle \langle v_1^*, v_2^* \rangle + (-1)^n \langle A v_1, v_2 \rangle \langle A^\vee v_1^*, v_2 * \rangle \big)$$

*if $\pi$ is even, where $A : \xi \otimes \pi \to \pi$ is an intertwining operator with $A^2 = 1$. Similarly, we have*

$$(\phi'_{\pi_1,v_1,v_1',n}, \phi'_{\pi_2,v_2,v_2',m})_G = 0$$

*unless $n = m$ and $\pi_1$ is isomorphic to an unramified twist of $\pi_2$. For $\pi$ unitary cuspidal we have*

$$(\phi'_{\pi,v_1,v_1',n}, \phi'_{\pi,v_2,v_2',n})_G = \tfrac{1}{2}(1-q^{-1}) d_\pi^{-1} (v_1, v_2)(v_2', v_1')$$

*if $\pi$ is odd, while*

$$(\phi'_{\pi,v_1,v_1',n}, \phi'_{\pi,v_2,v_2',n})_G = \tfrac{1}{2}(1-q^{-1}) d_\pi^{-1} \big( (v_1, v_2)(v_2', v_1') + (-1)^n (A v_1, v_2)(A v_2', v_1') \big)$$

*if $\pi$ is even and $A : \xi \otimes \pi \to \pi$ is an intertwining operator with $A^2 = 1$.*

Before proving the proposition we need a lemma.

**Lemma.** *Let $f$ be a function on $G$ which is supported on $G_n$ and invariant under $U_F \subset Z$. Let $f'$ be the function on $G$ which is equal to $f$ on $G_n$, invariant under $Z$ and vanishes off of $G_n Z$. Then*

$$\int_G f \, d\mu_G = (1-q^{-1}) \int_{G/Z} f' \, d\mu_{G/Z}.$$

*Proof.* We have

$$\int_G f d\mu_G = \int_{G/Z} \int_Z f(gz) d\mu_Z(z) d\mu_{G/Z}(g).$$

One finds that

$$\int_Z f(gz) d\mu_Z(z) = \mathrm{Vol}(U_F) f'(g),$$

which proves the lemma. □

We now prove the proposition.

*Proof of proposition.* Put $f = \phi_{\pi_1,v_1,v_1^*,n} \cdot \phi_{\pi_2,v_2,v_2^*,m}$. Note that

$$\langle \phi_{\pi_1,v_1,v_1^*,n}, \phi_{\pi_2,v_2,v_2^*,m} \rangle_G = \int_G f \, d\mu_G$$

Now, $f$ is identically zero unless $n = m$. Thus assume $n = m$. If the central characters of $\pi_1$ and $\pi_2$, restricted to $U_F \subset Z$, are not inverse to each other then $f$ transforms by a non-trivial character under $U_F$ and its integral over $G$ is zero. Thus assume that $\omega_{\pi_1}\omega_{\pi_2} = |\cdot|_F^s$. We now have that $f$ is supported on $G_n$ and invariant by $U_F$. Let $f'$ be the function used in the lemma. One finds that

$$f'(g) = \left( \frac{1 + (-1)^n \xi(g)}{2} \right) q^{-ns/2} |\det g|_F^{-s/2} \phi_{\pi_1,v_1,v_1^*}(g) \phi_{\pi_2,v_2,v_2^*}(g)$$

where here $\xi(g) = (-1)^{\mathrm{val}(\det g)}$. The lemma now gives

$$\begin{aligned}\langle \phi_{\pi_1,v_1,v_1^*,n}, \phi_{\pi_2,v_2,v_2^*,n} \rangle_G = \tfrac{1}{2}(1 - q^{-1}) q^{-ns/2} \big( &\langle \phi_{|\cdot|_F^{-s/2}\pi_1,v_1,v_1^*}, \phi_{\pi_2,v_2,v_2^*} \rangle_{G/Z} \\ &+ (-1)^n \langle \phi_{|\cdot|_F^{-s/2}\xi\pi_1,v_1,v_1^*}, \phi_{\pi_2,v_2,v_2^*} \rangle_{G/Z} \big).\end{aligned}$$

If $\pi_1$ is not an unramified twist of $\pi_2$ then the Schur orthogonality relations of §4.3.2 show that both terms on the right vanish. Now take $\pi_1 = \pi$ and $\pi_2 = \pi^\vee$, so that $s = 0$. The first term in the parentheses is $d_\pi^{-1}\langle v_1, v_2 \rangle \langle v_1^*, v_2^* \rangle$. If $\xi \otimes \pi \neq \pi$ then the second term vanishes; otherwise it is equal to $(-1)^n d_\pi^{-1} \langle A v_1, v_2 \rangle \langle A^\vee v_1^*, v_2^* \rangle$ where $A$ is an endomorphism of $\pi$ satisfying $A(g)\pi(g) = \xi(g)\pi(g)A(g)$ and $A^2 = 1$. This gives the stated formula. The proofs for the $\phi'$ go in the same way. □

**(4.3.5)** Let $\pi$ be an irreducible admissible representation of $G$ on a vector space $V$. For a Schwartz function $\phi$ on $G$ we define an operator $\pi(\phi)$ on $V$ by

$$\pi(\phi)v = \int_G \phi(g)\pi(g)v \, d\mu_G(g).$$

It is not difficult to see that the operator $\pi(\phi)$ has finite rank and thus a well-defined trace. There is a unique continuous function $\chi_\pi$ defined on $G_{\mathrm{reg}}$, called the *character* of $\pi$, which has the property

$$\operatorname{tr}\pi(\phi) = \langle \phi, \chi_\pi \rangle_G.$$

It is easily seen that $\chi_\pi$ is conjugation invariant, and thus defines a continuous function on $X_{\mathrm{reg}}$.

**(4.3.6)** Let $\pi$ be a cuspidal representation of $G$ and let $n$ be an integer. We define $\phi_{\pi,n}$ to be the function on $X_{\mathrm{reg}}$ which is equal to $\chi_\pi$ on the locus $X_{\mathrm{re},n}$ where $\mathbf{N}$ has valuation $n$ and 0 off of this locus. Note that $\phi_{\pi,n}$, by definition, vanishes on regular non-elliptic elements of $X$. We now relate this function to the matrix coefficients of $\pi$.

**Proposition.** *For a cuspidal representation $\pi$ we have*

$$\pi_*(\phi_{\pi,v,v^*,n}) = c d_\pi^{-1} \langle v, v^* \rangle \phi_{\pi,n}.$$

*Similarly, for a unitary cuspidal representation $\pi$ we have*

$$\pi_*(\phi'_{\pi,v,v',n}) = c d_\pi^{-1} (v, v') \phi_{\pi,n}.$$

*Here $c = \frac{1}{2} q/(q+1)$.*

*Proof.* We first show that $\pi_*(\phi_{\pi,v,v^*,n})$ vanishes on regular non-elliptic elements. For this, it suffices to show $(\pi^*\pi_*\phi_{\pi,v,v^*,n})(t) = 0$ for $t$ a regular element of the group $T$ of diagonal matrices with $\det t$ of valuation $n$. Let $N$ be the group of upper triangular unipotent matrices and let $K$ be a maximal compact subgroup of $G$. If $f$ is a compactly supported function on $G/T$ then

$$\int_{G/T} f \, d\mu_{G/T} = \int_K \int_N f(kn) dn dk$$

for suitable Haar measures $dk$ and $dn$ on $K$ and $N$. We thus find

$$(\pi^*\pi_*\phi_{\pi,v,v^*,n})(t) = \int_K \int_N \langle \pi(kntn^{-1}k^{-1})v, v^* \rangle dn dk.$$

Writing $t = \begin{pmatrix} a & \\ & b \end{pmatrix}$ and $n = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ gives $ntn^{-1} = \begin{pmatrix} a & (a-b)x \\ & b \end{pmatrix}$. We may thus write

$$(\pi^*\pi_*\phi_{\pi,v,v^*,n})(t) = \int_K A_k(v^*)dk$$

with

$$A_k(v^*) = \int_F \left\langle \pi \left( k \begin{pmatrix} a & (a-b)x \\ & b \end{pmatrix} k^{-1} \right) v, v^* \right\rangle dx.$$

The function $v^* \mapsto A_k(v^*)$ defines a linear map $V^\vee \to \mathbb{C}$ which is invariant by $kNk^{-1}$. Thus $v^* \mapsto A_k(v^*)$ factors through the Jacquet module of $V^\vee$ which is zero since $V^\vee$ is cuspidal. We thus find that $A_k(v^*) = 0$, which proves that $(\pi^*\pi_*\phi_{\pi,v,v^*,n})(t) = 0$.

We now need to determine $\pi_*(\phi_{\pi,v,v^*,n})$ on $X_{\text{ell}}$. Of course, it is supported on the locus where $\mathbf{N}$ has valuation $n$. Let $\phi$ be a Schwartz function on $B$ whose support is regular elliptic and contained in the locus where the determinant has valuation $n$. We then have

$$\langle \pi_*(\phi_{\pi,v,v^*,n}), \pi_*(\phi) \rangle_X = \langle \phi_{\pi,v,v^*}, \operatorname{avg}(\phi) \rangle_B$$

where we have used §4.1.15 to replace $\pi^*\pi_*\phi$ by $\operatorname{avg}\phi$. Let $U$ be a compact open subset of $G/Z$. Then

$$\begin{aligned} \langle \phi_{\pi,v,v^*}, \operatorname{avg}_U(\phi) \rangle_B &= c \int_U \int_B \langle \pi(x)v, v^* \rangle \phi(\sigma x \sigma^{-1}) d\mu_B(x) d\mu_{G/Z}(\sigma) \\ &= cq^{-2n} \int_U \int_G \langle \pi(x\sigma)v, \pi^\vee(\sigma)v^* \rangle \phi(x) d\mu_G(x) d\mu_{G/Z}(\sigma) \\ &= cq^{-2n} \int_U \langle \pi(\phi)\pi(\sigma)v, \pi^\vee(\sigma)v^* \rangle d_{G/Z}(\sigma) \end{aligned}$$

where $c = \frac{1}{2}q/(q+1)$. The factor of $q^{-2n}$ comes from replacing $d\mu_B$ with $d\mu_G = |\det|_F^{-2} d\mu_B$. Pick a basis $v_i$ of $V$ and let $v_i^*$ be the dual basis of $V^\vee$. (The dual basis of $V$ belongs to the contragredient, rather than the dual, because some twist of $V$ is unitary.) We have

$$\pi(\sigma)v = \sum \langle \pi(\sigma)v, v_i^* \rangle v_i, \qquad \pi^\vee(\sigma)v^* = \sum \langle v_j, \pi^\vee v^* \rangle v_j^*$$

and so

$$\langle \pi(\phi)\pi(\sigma)v, \pi^\vee(\sigma)v^* \rangle = \sum \langle \pi(\phi)v_i, v_j^* \rangle \phi_{\pi,v,v_i^*}(\sigma) \phi_{\pi^\vee,v^*,v_j}(\sigma).$$

We thus find

$$\langle \phi_{\pi,v,v^*}, \operatorname{avg}_U(\phi)\rangle_B = cq^{-2n} \sum \langle \pi(\phi)v_i, v_j^*\rangle \int_U \phi_{\pi,v,v_i^*}(\sigma)\phi_{\pi^\vee,v^*,v_j}(\sigma) d\mu_{G/Z}(\sigma).$$

The quantities above are independent of $U$ for $U$ sufficiently large, and equal to the corresponding quantity with $U$ changed to $G/Z$. We thus find

$$\begin{aligned}
\langle \pi_*(\phi_{\pi,v,v^*}), \pi_*(\phi)\rangle_X &= cq^{-2n} \sum \langle \pi(\phi)v_i, v_j^*\rangle \langle \phi_{\pi,v,v_i^*}, \phi_{\pi^\vee,v^*,v_j}\rangle_{G/Z} \\
&= cq^{-2n} d_\pi^{-1} \langle v, v^*\rangle \sum \langle \pi(\phi)v_i, v_j^*\rangle \langle v_i, v_j^*\rangle \\
&= cq^{-2n} d_\pi^{-1} \langle v, v^*\rangle \sum \langle \pi(\phi)v_i, v_i^*\rangle \\
&= cq^{-2n} d_\pi^{-1} \langle v, v^*\rangle \operatorname{tr} \pi(\phi) \\
&= cq^{-2n} d_\pi^{-1} \langle v, v^*\rangle \langle \chi_\pi, \phi\rangle_G
\end{aligned}$$

In the second step above we used the Schur orthogonality relations of §4.3.2. As $\phi$ is supported on the set where the determinant has valuation $n$, the last line is not changed if we replace $\chi_\pi$ by $\pi^*(\phi_{\pi,n})$. We can also get rid of $q^{-2n}$ by changing back to the measure $d\mu_B$. We thus find

$$\langle \pi_*(\phi_{\pi,v,v^*,n}), \pi_*(\phi)\rangle_X = cd_\pi^{-1} \langle v, v^*\rangle \langle \phi_{\pi,n}, \pi_*(\phi)\rangle_X.$$

This now gives the stated result for $\pi_*(\phi_{\pi,v,v^*,n})$. The unitary case goes similarly. □

**Corollary.** *The function $\phi_{\pi,n}$ belongs to $L^2(X_{\mathrm{re}})$.*

*Proof.* The function $\phi_{\pi,v,v^*,n}$ belongs to $\mathscr{S}(B)$ and so its push-forward by $\pi$ belongs to $L^2(X)$ by §4.1.11. □

**(4.3.7)** We now compute the inner products of the functions $\phi_{\pi,n}$.

**Proposition.** *We have*

$$\langle \phi_{\pi_1,n}, \phi_{\pi_2,m}\rangle_X = 0$$

*unless $n = m$ and $\pi_1$ is isomorphic to an unramified twist of $\pi_2^\vee$. We have*

$$\langle \phi_{\pi,n}, \phi_{\pi^\vee,n}\rangle = (1 - q^{-2})q^{-2n}$$

*if $\pi$ is odd, while*

$$\langle \phi_{\pi,n}, \phi_{\pi^\vee,n}\rangle = 2(1 - q^{-2})q^{-2n}$$

*if $\pi$ is even and $n$ is even. If $\pi$ is even then $\phi_{\pi,n} = 0$ for $n$ odd. Similarly, we have*

$$(\phi_{\pi_1,n}, \phi_{\pi_2,m})_X = 0$$

*unless $n = m$ and $\pi_1$ is isomorphic to an unramified twist of $\pi_1$. For $\pi$ unitary cuspidal we have*

$$\|\phi_{\pi,n}\|_X^2 = (1 - q^{-2})q^{-2n}.$$

*if $\pi$ is odd, while*

$$\|\phi_{\pi,n}\|_X^2 = 2(1 - q^{-2})q^{-2n}.$$

*if $\pi$ is even and $n$ is even.*

*Proof.* It is clear that all these inner products vanish unless $n = m$, so we only work in that situation. Using §4.3.6 we find

$$\langle \phi_{\pi_1,n}, \phi_{\pi_2,n} \rangle_X = \frac{c^{-1} d_{\pi_1}}{\langle v, v^* \rangle} \langle \pi_*(\phi_{\pi_1,v,v^*,n}), \chi_{\pi_2} \rangle_X = \frac{c^{-1} q^{-2n} d_{\pi_1}}{\langle v, v^* \rangle} \operatorname{tr} \pi_2(\phi_{\pi_1,v,v^*,n})$$

where here $v$ and $v^*$ are any vectors such that $\langle v, v^* \rangle$ is non-zero. The factor of $q^{-2n}$ comes in because the pairing $\langle , \rangle_X$ uses an additive measure, while the trace of $\pi(\phi)$ is given by integrating $\phi$ against $\chi_\pi$ using a multiplicative measure. Note that in the above we have used the fact that $\pi_*(\phi_{\pi_1,v,v^*,n})$ vanishes on regular non-elliptic elements, since the character $\chi_{\pi_2}$ does not vanish on such elements. Now, let $u_i$ be a basis for $\pi_2$ and let $u_i^*$ be the dual basis of $\pi_2^\vee$. We then have

$$\begin{aligned} \operatorname{tr} \pi_2(\phi_{\pi_1,v,v^*,n}) &= \sum \langle \pi_2(\phi_{\pi_1,v,v^*,n}) u_i, u_i^* \rangle \\ &= \sum \int_G \phi_{\pi_1,v,v^*,n}(g) \langle \pi_2(g) u_i, u_i^* \rangle d\mu_G \\ &= \sum \langle \phi_{\pi_1,v,v^*,n}, \phi_{\pi_2,u_i,u_i^*} \rangle_G \end{aligned}$$

Of course, we can change the $\phi_{\pi_2,u_i,u_i^*}$ in the above to $\phi_{\pi_2,u_i,u_i^*,n}$ without changing the result. By the Schur orthogonality relations of §4.3.2, we see that the trace vanishes unless $\pi_1$ is isomorphic to an unramified twist of $\pi_2$. Taking $\pi_1 = \pi$ and $\pi_2 = \pi^\vee$, we find

$$\operatorname{tr} \pi_2(\phi_{\pi_1,v,v^*,n}) = \tfrac{1}{2}(1 - q^{-1}) d_\pi^{-1} \sum \langle v, u_i \rangle \langle v^*, u_i^* \rangle = \tfrac{1}{2}(1 - q^{-1}) d_\pi^{-1} \langle v, v^* \rangle$$

if $\pi$ is odd, while we get

$$\begin{aligned}\operatorname{tr}\pi_2(\phi_{\pi_1,v,v^*,n}) =& \tfrac{1}{2}(1-q^{-1})d_\pi^{-1}\sum\left[\langle v,u_i\rangle\langle v^*,u_i^*\rangle+(-1)^n\langle Av,u_i\rangle\langle A^\vee v^*,u_i^*\rangle\right]\\ =& \tfrac{1}{2}(1-q^{-1})d_\pi^{-1}\big(\langle v,v^*\rangle+(-1)^n\langle Av,A^\vee v^*\rangle\big)\\ =& \tfrac{1}{2}(1-q^{-1})d_\pi^{-1}\langle v,v^*\rangle(1+(-1)^n)\end{aligned}$$

if $\pi$ is even and $A : \xi\otimes\pi\to\pi$ is an intertwining operator with $A^2 = 1$. Putting this back into our above formula gives the stated result. The computation for $(,)_X$ goes exactly the same. Note that if $\xi\otimes\pi=\pi$ then the character of $\pi$ is forced to vanish on elements of $G$ whose determinant has odd valuation, and this gives $\phi_{\pi,n}=0$ for $n$ odd. □

**(4.3.8)** We now come to an important definition. We define the *space of cuspidal functions* $H(X)$ to be the closure in $L^2(X_{\mathrm{re}})$ of the space spanned by the $\phi_{\pi,n}$ with $\pi$ a cuspidal representation. We also define $\mathscr{S}^\circ(B)$ to be the subspace of $\mathscr{S}(B)$ spanned by the $\phi_{\pi,v,v^*,n}$. By our above computations, $\pi_*$ carries $\mathscr{S}^\circ(B)$ into $H(X)$ and has dense image. Our definitions of the spaces $H(X)$ and $\mathscr{S}^\circ(B)$ are very representation theoretic. However, one can define these spaces without mentioning representations. For instance, $\mathscr{S}^\circ(B)$ is the subspace of $\mathscr{S}(B)$ consisting of those functions $f$ which are supported on non-singular elements and which satisfy

$$\int_N f(gn)dn = 0$$

for any $g\in G$ and any unipotent subgroup $N\subset G$. (That this description of $\mathscr{S}^\circ(B)$ is equivalent to our definition of $\mathscr{S}^\circ(B)$ follows from Harish-Chandra's Plancherel formula.) We will give a characterization of $H(X)$ below.

**(4.3.9)** We now characterize the space $H(X)$.

**Proposition.** *The space $H(X)$ consists of those elements of $L^2(X_{\mathrm{re}})$ which are orthogonal to functions factoring through $\mathbf{N}$.*

*Proof.* Let $V$ be the subspace of $L^2(X_{\mathrm{re}})$ consisting of those functions which factor through the norm. We first prove the following statement: the functions $\phi_{\pi,n}$, with $\pi$ a special representation, span a dense subspace of $V$. (Here $\phi_{\pi,n}$ is the function on $X_{\mathrm{re}}$ which is equal to $\chi_\pi$ on $X_{\mathrm{re},n}$ and 0 off this set. We extend $\phi_{\pi,n}$ by zero to all of $X$.) First observe that functions of the form $\phi\circ\mathbf{N}$ with $\phi\in\mathscr{S}(F^\times)$ span a dense subspace of $V$. By basic Fourier analysis on $F^\times$, we can write $\phi$ as a sum of functions of the form $\phi_{\eta,n}$ where $\eta$ is a character of $F^\times$ and $\phi_{\eta,n}$ is the function which is

$\eta$ on $\varpi_F^n U_F$ and 0 off that set. Thus the functions $\phi_{\eta,n} \circ \mathbf{N}$ span a dense subspace of $V$. Now, if $\pi$ is the special representation associated to $\eta$ then $\chi_\pi|_{X_{\mathrm{re}}} = (\eta \circ \mathbf{N})|_{X_{\mathrm{re}}}$ (see the discussion following [JL, Theorem 7.7]). This shows that $\phi_{\eta,n} \circ \mathbf{N}$ and $\phi_{\pi,n}$ agree on $X_{\mathrm{re}}$ and the claim follows.

Now, we have only stated the Schur orthogonality relations for cuspidal representations. However, they are valid for all square integrable representations. Thus they hold for cuspidal and special representations. From this it follows that $H(X)$ is orthogonal to $V$, that is, $H(X) \subset V^\perp$. We must prove that this containment is an equality. It suffices, therefore, to show that $H(X) \oplus V$ is equal to all of $L^2(X_{\mathrm{re}})$. In other words, we must show that the $\phi_{\pi,n}|_{X_{\mathrm{re}}}$, as $\pi$ varies over square integrable representations, span a dense subspace of $L^2(X_{\mathrm{re}})$. To do this, it suffices to prove the following statement: if $f \in \mathscr{S}(X_{\mathrm{re}})$ is orthogonal to all the $\phi_{\pi,n}$ with $\pi$ square integrable then $f$ vanishes.

We now prove the above statement. Thus let $f$ be given. Let $\widetilde{f}$ be a function on $B$ with regular elliptic support such that $\pi_*\widetilde{f} = f$. (We can find $\widetilde{f}$ by §4.1.16.) If $\pi$ is an infinite dimensional principal series representation then

$$\operatorname{tr}\pi(\widetilde{f}) = \langle \pi^*(\chi_\pi), \widetilde{f}\rangle_B = \langle \chi_\pi, f\rangle_X = 0$$

since $\chi_\pi$ has non-elliptic support (see [JL, Proposition 7.6]) but $f$ has elliptic support. If $\pi$ is square integrable then

$$\operatorname{tr}\pi(\widetilde{f}) = \langle \chi_\pi, f\rangle_X = \sum_{n\in\mathbb{Z}} \langle \phi_{\pi,n}, f\rangle_X = 0$$

by hypothesis. We thus see that $\operatorname{tr}\pi(\widetilde{f})$ vanishes for any infinite dimensional irreducible admissible representation $\pi$ of $G$. The density of characters (which can be proved using the local trace formula, see [Vig, §3.1]) implies that the integral of $\widetilde{f}$ on any conjugacy class vanishes. We thus see that $f = \pi_*\widetilde{f}$ vanishes, which proves the proposition. □

## 4.4 The Fourier transform $\mathscr{F}^{(2)}_{X,\psi}$ and the $\mathscr{A}$-structure on $H(X)$

**(4.4.1)** In §4.4 we introduce a Fourier transform $\mathscr{F}^{(2)}_{X,\psi}$ on the cuspidal space $H(X)$, use this Fourier transform to define a family of operators $\mathscr{A}$ on $H(X)$ and then determine the structure of $H(X)$ as a module over $\mathscr{A}$. Here is an overview:

- In §4.4.2 we compute the Fourier transform (on $B$) of our truncated matrix coefficients $\phi_{\pi,v,v^*,n}$.
- In §4.4.3 we determine the modulus of $\epsilon$-factors.

- In §4.4.4 we show that the $\epsilon$-factor of $\pi$ can be obtained from integrating one of the truncated character functions $\phi_{\pi,n}$ against an additive character. We will ultimately use this result to determine the proper sign in the Jacquet-Langlands correspondence.

- In §4.4.5 we give some characterizations of even representations.

- In §4.4.6 we show that in certain families of representations the formal degree becomes large. This is an important input into the results of §4.4.8.

- In §4.4.7 we define the Fourier transform $\mathscr{F}^{(2)}_{X,\psi}$ by the formula $\mathscr{F}^{(2)}_{X,\psi}(\pi_* f) = \pi_*(\mathscr{F}_{B,\psi} f)$ where $f$ is a linear combination of truncated matrix coefficients. We prove this is well-defined using the explicit computations of §4.4.2.

- In §4.4.8 we prove that $\mathscr{F}^{(1)}_{X,\psi}$ and $\mathscr{F}^{(2)}_{X,\psi}$ agree on a dense subspace of $H(X)$. This is a very important result as certain properties are easier to establish for $\mathscr{F}^{(1)}_{X,\psi}$ and others for $\mathscr{F}^{(2)}_{X,\psi}$, and this result allows one to transfer these properties. The proof of this result is a bit tricky!

- In §4.4.9 we introduce an operation $f \mapsto f^\vee$ on functions on $X$.

- In §4.4.10 we introduce the operators $A_{\psi,\eta}$ and the algebra $\mathscr{A}$ they generate. We also introduce another algebra of operators $\mathscr{T}$, which is much less interesting but still needed.

- In §4.4.11 we determine the structure of $H(X)$ as a module over the coproduct $\mathscr{A} * \mathscr{T}$. It is semi-simple and multiplicity free. Its simple constituents are in bijective correspondence with unramified twist classes of cuspidal representations.

**(4.4.2)** We now compute the Fourier transform on $\mathscr{S}^\circ(B)$ in terms of the spanning set $\phi_{\pi,v,v^*,k}$.

**Proposition.** *We have*

$$\mathscr{F}_{B,\psi}(\phi_{\pi,v,v^*,k}) = \epsilon(\tfrac{3}{2},\pi,\psi)\phi_{|\cdot|_F^{-2}\pi^\vee,v^*,v,-k-n(\pi)-2m(\psi)}$$

*where $\epsilon(s,\pi,\psi)$ is the $\epsilon$-factor of $\pi$, $n(\pi)$ is the conductor of $\pi$ and $m(\psi)$ is the largest integer for which $\psi_F$ is trivial on $\mathfrak{p}^{-m(\psi)}$. Similarly, for $\pi$ unitary we have*

$$\mathscr{F}_{B,\psi}(\phi'_{\pi,v,v',k}) = \epsilon(\tfrac{3}{2},\pi,\psi)q^{-2(n(\pi)+2m(\psi)+k)}\overline{\phi}'_{\pi,v',v,-k-n(\pi)-2m(\psi)}$$

*where the bar denotes complex conjugation.*

*Proof.* Take $\pi$ in Kirillov form (with respect to the character $\psi$), so that its representation space is $\mathscr{S}(F^\times)$. By [JL, Lemma 13.1.1] we have that the Fourier transform of

$$x \mapsto \phi(\det x)|\det x|_F^{-1}\langle v, \pi^\vee(x)v^*\rangle$$

is equal to

$$x \mapsto \phi'(\det x)|\det x|_F^{-1}\omega_\pi^{-1}(\det x)\langle \pi(x)v, v^*\rangle.$$

Here $\phi$ is an element of $\mathscr{S}(F^\times)$, $\phi'$ is $\pi(w)\phi$ where $w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ and $\omega_\pi$ is the central character of $\pi$. For a character $\eta$ of $F^\times$ and an integer $k$ let $\Phi_{k,\eta}$ be the function which is $\eta$ on $\varpi^k U_F$ and 0 of this set. Applying the above formula with $\phi = \Phi_{k,1}$ gives

$$q^k(\mathscr{F}_{B,\psi}\phi_{\pi^\vee,v^*,v,k})(x) = \phi'(\det x)|\det x|_F^{-1}\omega_\pi^{-1}(\det x)\phi_{\pi,v,v^*}(x)$$

where $\phi' = \pi(w)\Phi_{k,1}$. In [JL] it is explained how to compute $\pi(w)$ in the Kirillov model by using formal Mellin transforms. This involves certain power series which are denoted there by $C(\nu, t)$. For cuspidal representations, these series have only a single non-zero term. Working through the details shows that

$$\pi(w)\Phi_{k,\eta} = C(\eta)\Phi_{N(\eta)-k,\omega\eta^{-1}}$$

where $N(\eta)$ is some integer and $C(\eta)$ some constant. One can relate the series $C(\nu, t)$ to the constants appearing in the local functional equation. The result is

$$\epsilon(s, \omega_\pi^{-1}\eta\pi, \psi) = C(\eta)q^{(s-1/2)N(\eta)}.$$

Now, by [Cas] one knows that $\epsilon(s, \pi, \psi)$ is a constant times $q^{Ns}$ where $N = -n(\pi) - 2m(\psi)$. It follows that $N(\eta) = -n(\eta\pi^\vee) - 2m(\psi)$. (Note that $\omega_\pi^{-1}\pi = \pi^\vee$.) From this we find

$$C(\eta) = \epsilon(s, \eta\pi^\vee, \psi)q^{-(s-1/2)(n(\eta\pi^\vee)+2m(\psi))},$$

for any value of $s$. This gives

$$\pi(w)\Phi_{k,\eta} = \epsilon(s, \eta\pi^\vee, \psi)q^{-(s-1/2)(n(\eta\pi^\vee)+2m(\psi))}\Phi_{-n(\eta\pi^\vee)-2m(\psi)-k,\omega\eta^{-1}}.$$

Taking $\eta = 1$ gives a formula for $\phi'$. We thus have

$$\begin{aligned} & q^k(\mathscr{F}_{B,\psi}\phi_{\pi^\vee,v^*,v,k})(x) \\ =& \epsilon(s,\pi^\vee,\psi)q^{-(s-1/2)(n(\pi)+2m(\psi))}\Phi_{-n(\pi)-2m(\psi)-k,\omega}(\det x)|\det x|_F^{-1}\omega^{-1}(\det x)\phi_{\pi,v,v^*}(x) \\ =& \epsilon(s,\pi^\vee,\psi)q^{-(s-1/2)(n(\pi)+2m(\psi))}q^{-n(\pi)-2m(\psi)-k}\phi_{\pi,v,v^*,-n(\pi)-2m(\psi)-k}(x). \end{aligned}$$

Thus, taking $s = 3/2$, we find

$$\begin{aligned} \mathscr{F}_{B,\psi}(\phi_{\pi^\vee,v^*,v,k}) =& \epsilon(\tfrac{3}{2},\pi^\vee,\psi)q^{-2(n(\pi)+2m(\psi)+k)}\phi_{\pi,v,v^*,-n(\pi)-2m(\psi)-k} \\ =& \epsilon(\tfrac{3}{2},\pi^\vee,\psi)\phi_{|\cdot|_F^{-2}\pi,v,v^*,-n(\pi)-2m(\psi)-k}. \end{aligned}$$

Changing $\pi$ to $\pi^\vee$ now gives the stated result. The identity for $\phi'$ can be derived from the one from for $\phi$ by observing $\phi'_{\pi,v,v',n} = \phi_{\pi,v,v^*,n}$ where $v^* = (-,v')$. □

*Remark.* The proposition shows that the space $\mathscr{S}^\circ(B)$ is closed under $\mathscr{F}_{B,\psi}$. In fact, one can give a more conceptual proof of this using the characterization of $\mathscr{S}^\circ(B)$ as the space of functions whose integrals on unipotent cosets vanish. For this, see [JL, Lemma 13.1.2].

**(4.4.3)** We can now use the fact that the Fourier transform is an isometry to compute the size of $\epsilon$-factors.

**Proposition.** *For $\pi$ unitary we have $|\epsilon(\frac{3}{2},\pi,\psi)| = q^{n(\pi)+2m(\psi)}$.*

*Proof.* As $\mathscr{F}_{B,\psi}$ preserves $\|\cdot\|_B^2$, we find

$$\|\phi'_{\pi,v,v',k}\|_B^2 = |\epsilon(\tfrac{3}{2},\pi,\psi)|^2 q^{-4(n(\pi)+2m(\psi)+k)}\|\phi'_{\pi^\vee,v,v',-k-n(\pi)-2m(\psi)}\|_B^2.$$

We have previously computed these norms (see §4.3.4), but with respect to the multiplicative Haar measure. Changing to the additive Haar measure, we find $\|\phi'_{\pi,v,v',k}\|_B^2 = \frac{1}{2}q^{-2k}(1-q^{-1})d_\pi^{-1}|(v,v')|^2$, if $\pi$ is odd or $\pi$ and $k$ are both even (the difference between this and the previous formula is the factor of $q^{-2k}$ that is now present). For $\pi$ odd, this gives the stated result. If $\pi$ is even, we take $k$ to be even. The left side is non-zero, which implies the right side is non-zero, which implies $n(\pi)$ is even. The stated formula now follows from the computation of the norms. □

**(4.4.4)** Define $\psi_X : X \to \mathbb{C}$ by $\psi_X = \psi_F \circ \mathrm{tr}$.

**Proposition.** *We have*

$$\epsilon(\tfrac{3}{2},\pi,\psi)=cd_\pi^{-1}q^{-2m(\psi)}\langle\phi_{\pi,-n(\pi)-2m(\psi)},\psi_X\rangle_X.$$

*Proof.* Observe that $\pi^*\psi_X=\psi_B$ and $\langle f,\psi_B\rangle_B=q^{2m(\psi)}(\mathscr{F}_{B,\psi}f)(1)$. Thus

$$\begin{aligned}\langle\phi_{\pi,k},\psi_X\rangle_X=&\frac{d_\pi}{c\langle v,v^*\rangle}\langle\phi_{\pi,v,v^*,k},\psi_B\rangle_B\\ =&\frac{d_\pi q^{2m}}{c\langle v,v^*\rangle}(\mathscr{F}_{B,\psi}(\phi_{\pi,v,v^*,k}))(1)\\ =&\frac{d_\pi q^{2m}}{c\langle v,v^*\rangle}\epsilon(\tfrac{3}{2},\pi,\psi)\langle v,v^*\rangle\delta_{k,-n(\pi)-2m(\psi)}\end{aligned}$$

which gives the stated result. Here we have used our formula for the Fourier transform, *c.f.* §4.4.2. □

**(4.4.5)** We now give some characterizations of even representations.

**Proposition.** *Let $\pi$ be a cuspidal representation of $G$. The following are equivalent:*

1. *$\pi$ is even.*
2. *$\xi\otimes\pi=\pi$.*
3. *$\phi_{\pi,k}=0$ for $k$ odd.*
4. *$n(\eta\pi)$ is even for any character $\eta$ of $F^\times$.*

*Proof.* (1) and (2) are equivalent by definition. That (2) implies (3) is immediate, as we have already remarked. Now assume (3) holds. Then $\phi_{\eta\pi,k}=0$ for any character $\eta$ and any odd integer $k$. As $\epsilon(\frac{3}{2},\eta\pi,\psi)$ is non-zero, we see from §4.4.4 that $n(\eta\pi)$ is even. Thus (4) holds. Now assume (4) holds. Then by §4.4.4 we have

$$\epsilon(\tfrac{3}{2},\eta\xi\pi,\psi)=\langle\phi_{\eta\xi\pi,-n(\eta\pi)-2m(\psi)},\psi_X\rangle_X=\langle\phi_{\eta\pi,-n(\eta\pi)-2m(\psi)},\psi_X\rangle_X=\epsilon(\tfrac{3}{2},\eta\pi,\psi)$$

as $\phi_{\xi\pi,k}=\phi_{\pi,k}$ for $k$ even. Since $\xi\pi$ and $\pi$ have the same conductor, we see $\epsilon(s,\eta\xi\pi,\psi)=\epsilon(s,\eta\pi,\psi)$ for all $\eta$ and $\psi$. Thus, by the local converse theorem [JL, Corollary 2.19], we find $\xi\otimes\pi=\pi$ and so (2) holds. □

**(4.4.6)** We now study the behavior of the formal degree in certain families of representations.

**Proposition.** *Let $\omega$ be a character of $F^\times$ and let $S$ be a section of $\mathrm{Irr}^\circ_{G,\omega}\to\overline{\mathrm{Irr}}^\circ_{G,\omega}$. Then $\sum d_\pi^2=\infty$, the sum taken over $\pi\in S$.*

*Proof.* We first note that the truth of the proposition is unchanged if we multiply $\omega$ by the square of another character. We can and do therefore assume that $\omega$ is unitary and its restriction to $U_F$ is non-trivial.

Let $L^2_\omega(X_{\text{ell}})$ be the subspace of $L^2(X_{\text{ell}})$ consisting of those functions which transform under $U_F \subset Z$ by $\omega$. If $\omega$ is not a square then this space has for a basis the functions $\phi_{\pi,k}$ with $\pi \in S$. If $\omega = \omega_0^2$ then the functions $\phi_{\omega_0,k}$, defined to be $\omega_0 \circ \mathbf{N}$ on $X_{\text{ell},k}$ and 0 off this set, also belong to $L^2_\omega(X_{\text{ell}})$ and together with the $\phi_{\pi,k}$ form a basis. These statements follow easily from §4.3.9.

Define a function $F : X_{\text{ell}} \to \mathbb{C}$ by

$$F(x) = c|\,\mathbf{N}\,x|_F^{-1} \int_{U_F} \omega^{-1}(\epsilon)\psi_X(\epsilon x)d\epsilon. \tag{4.1}$$

Here $d\epsilon$ is the Haar measure on $U_F$ with total volume 1. Note that $\psi_X(\epsilon x) = \psi_F(\epsilon \operatorname{tr} x)$. The usual evaluation of Gaussian sums shows that

$$F(x) = C_0\omega(\operatorname{tr} x)|\,\mathbf{N}\,x|_F^{-1}\delta_{\text{val}(\operatorname{tr} x),N} \tag{4.2}$$

where $N$ is an integer and $C_0$ a constant. Let $F_k$ be the function which is $F$ on $X_{\text{ell},k}$ and 0 off of this set. The function $F_k$ belongs to $L^2_\omega(X_{\text{ell}})$. For $\pi \in S$ a simple computation using (4.1) and §4.4.4 gives

$$(\phi_{\pi,k}, F_k)_X = a_\pi d_\pi \delta_{k,-n(\pi)-2m(\psi)}$$

where $a_\pi = q^{-n(\pi)-2m(\psi)}\epsilon(\frac{3}{2}, \pi, \overline{\psi})$. If $\omega$ is a square then using (4.2) and the definition of $d\mu_X$ (*c.f.* §4.1.9) we find

$$(\phi_{\omega_0,k}, F_k)_X = q^k\overline{C}_0 \int_{\varpi^N U_F} \int_{\varpi^k U_F} \omega^{-1}(t)\omega_0(\nu)|\Delta|_F^{1/2} \left(\frac{1-\eta(\Delta)}{2}\right) d\nu dt$$

The $1-\eta(\Delta)$ ensures that the integral is really over a subset of $X_{\text{ell}}$. If $k \neq 2N$ then $|\Delta|$ and $\eta(\Delta)$ are independent of $\nu$ and so the integral over the coset of $U_F$ is zero, as $\omega_0|_{U_F}$ is non-trivial. Thus $(\phi_{\omega_0,k}, F_k)_X$ is non-zero only for $k = 2N$ and so we have

$$F_k = \sum d_\pi a_\pi \frac{\phi_{\pi,k}}{\|\phi_{\pi,k}\|_X} + C_1\delta_{k,2N}\frac{\phi_{\omega_0,k}}{\|\phi_{\omega_0,k}\|_X}$$

where the sum is over those $\pi \in S$ for which $n(\pi) = -k - 2m(\psi)$. Therefore

$$\|F_k\|_X^2 = \sum d_\pi^2 + C_2 \cdot \delta_{k,2N}$$

the sum taken over the same set of $\pi$. (Note $|a_\pi| = 1$.) Summing over $k$ now gives

$$\sum_{k \in \mathbb{Z}} \|F_k\|_X^2 = C_2 + \sum_{\pi \in S} d_\pi^2.$$

The proposition will follow if we can show that the left side is infinite.

We now estimate $\|F_k\|_X^2$. Using (4.2) and the definition of $d\mu_X$ gives

$$\|F_k\|_X^2 = |C_0|^2 q^k \int_{\varpi^N U_F} \int_{\varpi^k U_F} A(\Delta) |\Delta|_F^{1/2} \left( \frac{1 - \eta(\Delta)}{2} \right) d\nu dt$$

where $d\nu$ and $dt$ are additive Haar measures on $F$. The quantity $A(\Delta)$ is never zero and assumes only two values. If $k < 2N$ then $|\Delta|_F^{1/2}$ equals $q^{-k/2}$. We thus find, under this hypothesis,

$$\|F_k\|_X^2 \geq C_3 q^k q^{-k/2} \operatorname{Vol}(\varpi^N U_F) \operatorname{Vol}(\varpi^k U_F) \geq C_4 q^{-k/2}.$$

We therefore see that $\|F_k\|_X$ increases without bound as $k \to -\infty$. This proves the proposition. $\square$

**(4.4.7)** We now define a Fourier transform on the cuspidal space $H(X)$.

**Proposition.** *There is a unique map $\mathscr{F}_{X,\psi}^{(2)} : H(X) \to H(X)$ which is an isometry and satisfies $\mathscr{F}_{X,\psi}(\pi_* f) = \pi_*(\mathscr{F}_{B,\psi} f)$ for $f \in \mathscr{S}^\circ(B)$. Explicitly, we have*

$$\mathscr{F}_{X,\psi}^{(2)}(\phi_{\pi,k}) = \epsilon(\tfrac{3}{2}, \pi, \psi) \phi_{|\cdot|_F^{-2}\pi^\vee, -k-n(\pi)-2m(\psi)}.$$

*Proof.* Let $S$ be a set of cuspidal representations such that any cuspidal representation is an unramified twist of exactly one element of $S$. We can, and do, assume that $S$ consists of unitary representations. The $\phi_{\pi,k}$ with $\pi \in S$ then form a basis for $H(X)$ (excluding the $\phi_{\pi,k}$ with $\pi$ even and $k$ odd) and are mutually orthogonal. From this and our explicit computation of $\pi_*(\phi_{\pi,v,v^*,k})$ (*c.f.* §4.3.6) one easily sees that the kernel of $\pi_* : \mathscr{S}^\circ(B) \to H(X)$ is spanned by functions of the form $\sum a_i \phi_{\pi,v_i,v_i^*,k}$ with $\sum a_i \langle v_i, v_i^* \rangle = 0$. Our explicit computation of $\mathscr{F}_{B,\psi}(\phi_{\pi,v,v^*,k})$ (*c.f.* §4.4.2) thus shows that this kernel is mapped into itself. It follows that we have a unique well-defined linear map

$$\mathscr{F}^{(2)}_{X,\psi} : \pi_*(\mathscr{S}^\circ(B)) \to \pi_*(\mathscr{S}^\circ(B))$$

given by $\mathscr{F}^{(2)}_{X,\psi}(\pi_* f) = \pi_*(\mathscr{F}_{B,\psi} f)$. Our computation of $\mathscr{F}_{B,\psi}(\phi_{\pi,v,v^*,k})$ and $\pi_*(\phi_{\pi,v,v^*,k})$ now gives the stated formula for $\mathscr{F}^{(2)}_{X,\psi}(\phi_{\pi,k})$. To finish off the proof, it suffices to show that $\mathscr{F}^{(2)}_{X,\psi}$ is an isometry on $\pi_*(\mathscr{S}^\circ(B))$. For this, it is enough to show that

$$(\phi_{\pi,k}, \phi_{\pi',k'})_X = (\mathscr{F}^{(2)}_{X,\psi}(\phi_{\pi,k}), \mathscr{F}^{(2)}_{X,\psi}(\phi_{\pi',k'}))_X$$

for $\pi, \pi' \in S$. If $\pi \neq \pi'$ or $k \neq k'$ then the left side is zero and our formula for $\mathscr{F}^{(2)}_{X,\psi}$ shows that the right side is zero. (Here we have used the orthogonality relations of §4.3.7.) We are thus reduced to showing

$$\|\phi_{\pi,k}\|_X^2 = \|\mathscr{F}^{(2)}_{X,\psi}(\phi_{\pi,k})\|_X^2$$

Assume first that $\pi$ is odd. Then the left side is $(1-q^{-2})q^{-2k}$ by §4.3.7. On the other hand,

$$\begin{aligned}
\|\mathscr{F}^{(2)}_{X,\psi}(\phi_{\pi,k})\|_X^2 =& |\epsilon(\tfrac{3}{2}, \pi, \psi)|^2 \|\phi_{|\cdot|_F^{-2}\pi^\vee, -k-n(\pi)-2m(\psi)}\|_X^2 \\
=& |\epsilon(\tfrac{3}{2}, \pi, \psi)|^2 q^{-4(k+n(\pi)+2m(\psi))} \|\phi_{\pi^\vee, -k-n(\pi)-2m(\psi)}\|_X^2 \\
=& |\epsilon(\tfrac{3}{2}, \pi, \psi)|^2 q^{-4(k+n(\pi)+2m(\psi))} (1-q^{-2}) q^{2(k+n(\pi)+2m(\psi))} \\
=& |\epsilon(\tfrac{3}{2}, \pi, \psi)|^2 q^{-2(k+n(\pi)+2m(\psi))} (1-q^{-2}).
\end{aligned}$$

This is equal to $(1-q^{-2})q^{-2k}$ by our computation of $|\epsilon(\frac{3}{2}, \pi, \psi)|$ (*c.f.* §4.4.3). Now consider the case where $\pi$ is even. The key point is that $n(\pi)$ is even by §4.4.5, so that $k$ and $-k-n(\pi)-2m(\psi)$ have the same parity. The above analysis of the odd case thus carries over nearly unchanged to the even case. □

**(4.4.8)** We now compare the two Fourier transforms we have defined. This is a crucial result in our proof of the Jacquet-Langlands correspondence.

**Proposition.** *There exists a subspace $V \subset \mathscr{S}(X_{\mathrm{re}}) \cap H(X)$ which is dense in $H(X)$ and for which $\mathscr{F}^{(1)}_\psi f = \mathscr{F}^{(2)}_\psi f$ for all $f \in V$.*

*Proof.* We take $V$ to be the space spanned by functions of the form

$$\Phi_{\pi,\pi',k} = d_\pi^{-1}\phi_{\pi,k} - d_{\pi'}^{-1}\phi_{\pi',k}$$

where $\pi$ and $\pi'$ are cuspidal representations with the same central character and $k$ is any integer. We must show that $\mathscr{F}_{\psi}^{(1)}$ and $\mathscr{F}_{\psi}^{(2)}$ agree on $V$ (implicit in which is the statement that $V$ is contained in $\mathscr{S}(X_{\mathrm{re}})$) and that $V$ is dense in $H(X)$. We first handle the latter task. Assume that $V$ is not dense in $H(X)$ and let $\phi$ be a non-zero element of the orthogonal complement. We can write $\phi = \sum_{k \in \mathbb{Z}} \sum_{\omega \in \widehat{U}_F} \phi_{\omega,k}$ where $\phi_{\omega,k}$ transforms under $U_F \subset Z$ by the character $\omega$ and is supported on $X_{\mathrm{ell},k}$. Each $\phi_{\omega,k}$ belongs to the orthogonal complement of $V$ in $H(X)$, so it suffices to handle the case where $\phi$ itself transforms under $U_F$ by a character $\omega$ and is supported on $X_{\mathrm{ell},k}$. Extend $\omega$ to a unitary character of $Z$. The map

$$\mathrm{Irr}^{\circ}_{G,\omega} \to \overline{\mathrm{Irr}}^{\circ}_{G,\omega}$$

is 2-1; let $S \subset \mathrm{Irr}^{\circ}_{G,\omega}$ be a section. We can then write $\phi = \sum_{\pi \in S} a_\pi \phi_{\pi,k}$, in a unique way. As $\|\phi_{\pi,k}\|^2 = (1-q^{-2})q^{-2k}$ and the $\phi_{\pi,k}$ are orthogonal to each other (*c.f.* §4.3.7), we see that $\sum_{\pi \in S} |a_\pi|^2$ must converge. However, if $\pi$ and $\pi'$ belong to $S$ then

$$0 = \langle \phi, \Phi_{\pi,\pi',k} \rangle_X = \frac{a_\pi}{d_\pi} - \frac{a_{\pi'}}{d_{\pi'}}.$$

We thus see that $a_\pi = c d_\pi$ for some non-zero constant $c$. As the series $\sum_{\pi \in S} d_\pi^2$ does not converge (*c.f.* §4.4.6), we have a contradiction. Thus $V$ is dense in $H(X)$.

We now show that $\mathscr{F}_{X,\psi}^{(1)}$ and $\mathscr{F}_{X,\psi}^{(2)}$ agree on $V$ and that $V \subset \mathscr{S}(X_{\mathrm{re}})$. Consider the statement

($*$) Given two unitary cuspidal representations $\pi$ and $\pi'$ with the same central character there exists $\widetilde{\Phi}$ in $\mathscr{S}^{\circ}(B) \cap \mathscr{S}_{\mathrm{re}}(B)$ for which $\pi_*(\widetilde{\Phi}) = \Phi_{\pi,\pi',k}$.

It is enough to prove ($*$), as $\pi_*(\widetilde{\Phi})$ belongs to $\mathscr{S}(X_{\mathrm{re}})$ and $\pi_*(\mathscr{F}_{B,\psi}(\widetilde{\Phi}_{\pi,\pi',k}))$ computes both $\mathscr{F}_{X,\psi}^{(1)}(\Phi_{\pi,\pi',k})$ and $\mathscr{F}_{X,\psi}^{(2)}(\Phi_{\pi,\pi',k})$. We now prove ($*$). Thus let $\pi$, $\pi'$ and $k$ be given. To find $\widetilde{\Phi}$ we take $\pi$ and $\pi'$ in their Kirillov form. Thus both $\pi$ and $\pi'$ have for their representation space the Schwartz space $\mathscr{S}(F^\times)$. Furthermore, if for $v_1$ and $v_2$ in $\mathscr{S}(F^\times)$ we put

$$(v_1, v_2) = \int_{F^\times} v_1(x) \overline{v}_2(x) d\mu_{F^\times}(x)$$

then $(,)$ is a Hermitian form which is invariant under both $\pi$ and $\pi'$ (see [JL, Proposition 2.21.2]). Let $v$ be a non-zero element of $\mathscr{S}(F^\times)$. Put

$$\widetilde{\Phi}_0 = \phi'_{\pi,v,v,k} - \phi'_{\pi',v,v,k}$$

Of course, $\widetilde{\Phi}$ belongs to $\mathscr{S}^\circ(B)$. Since the group $P$ of upper triangular matrices in $G$ acts on $\mathscr{S}(F^\times)$ in the same manner under $\pi$ and $\pi'$, we see that $\widetilde{\Phi}$ vanishes on $P$. Let $K$ be a maximal compact subgroup of $G$ and put

$$\widetilde{\Phi} = \frac{1}{c^2 \operatorname{Vol}(K)\|v\|^2} \operatorname{avg}_{KZ/Z}(\widetilde{\Phi}_0)$$

where $c = \frac{1}{2}q/(q+1)$. We have

$$\widetilde{\Phi}(g) = \frac{1}{c \operatorname{Vol}(KZ/Z)\|v\|^2} \int_{KZ/Z} \left[\phi'_{\pi,\pi(g)v,\pi(g)v,k} - \phi'_{\pi',\pi'(g)v,\pi'(g)v,k}\right] d\mu_{G/Z}(g)$$

and so $\widetilde{\Phi}$ vanishes on $P$ for the same reason that $\widetilde{\Phi}_0$ did. Let $g$ be an arbitrary element of $G$ and let $p_0$ be an element of $P$. We can write $g$ as $kp$ where $k \in K$ and $p \in P$. We then have

$$\widetilde{\Phi}(gp_0g^{-1}) = \widetilde{\Phi}(pp_0p^{-1}) = 0.$$

It follows that $\widetilde{\Phi}$ has regular elliptic support since any element which is not regular elliptic belongs to some conjugate of $P$. We have thus shown that $\widetilde{\Phi}$ belongs to $\mathscr{S}^\circ(B) \cap \mathscr{S}_{\mathrm{re}}(B)$. Finally, using §4.3.6, we have

$$\pi_*(\widetilde{\Phi}) = \frac{1}{c\|v\|^2}\pi_*(\widetilde{\Phi}_0) = \Phi_{\pi,\pi',k}.$$

This proves $(*)$ and thus the proposition. □

**(4.4.9)** Define a map $X_{\mathrm{ns}} \to X_{\mathrm{ns}}$, denoted $x \mapsto x^{-1}$, by $(t,\nu) \mapsto (t\nu^{-1}, \nu^{-1})$. This map corresponds to inversion in $G$ if we regard $X_{\mathrm{ns}}$ as the set of semi-simple conjugacy classes in $G$. For a function $f$ on $X$ we let $f^\vee$ denote the function $x \mapsto f(x^{-1})$. (This function is really only defined on $X_{\mathrm{ns}}$.)

**Proposition.** *The map $f \mapsto |\cdot|_F^{-2} f$ is self-adjoint for $\langle , \rangle_X$ and an isometry for $( , )_X$.*

*Proof.* This is a simple computation with $d\mu_X$. □

**(4.4.10)** For a non-trivial additive character $\psi$ of $F$ and a unitary character $\eta$ of $F^\times$ define an operator

$$A_{\psi,\eta} : H(X) \to H(X), \qquad A_{\psi,\eta}(f) = \eta^{-1}\mathscr{F}^{(2)}_{X,\psi}(|\cdot|_F^{-2}\eta^{-1}f^\vee).$$

Here $\eta$ is regarded as a function on $X$ by composing with the norm. Since $|\eta| = 1$, multiplying by $\eta^{-1}$ is an isometry; the Fourier transform $\mathscr{F}^{(2)}_{X,\psi}$ is an isometry by §4.4.7 while $f \mapsto |\cdot|_F^{-2} f$ is an isometry by §4.4.9. It follows that $A_{\psi,\eta}$ is an isometry. We will see shortly that various $A_{\psi,\eta}$ commute. We let $\mathscr{A}$ be the polynomial ring over $\mathbb{C}$ generated by the symbols $A_{\psi,\eta}$, so that the above definitions give $H(X)$ the structure of an $\mathscr{A}$-module.

We will need a few more operators as well. For an integer $n$ we define an operator

$$T_n : H(X) \to H(X)$$

by letting $T_n(f)$ be the function which is equal to $f$ on the locus where $\mathbf{N}$ has valuation $n$ and 0 off this set. Of course, the $T_n$ are idempotent operators. We let $\mathscr{T}$ be the polynomial ring over $\mathbb{C}$ in the $T_n$. We have thus give $H(X)$ the structure of a $\mathscr{T}$-module. The $A_{\psi,\eta}$ and $T_n$ do not commute with each other. We write $\mathscr{A} * \mathscr{T}$ for the coproduct of $\mathscr{A}$ and $\mathscr{T}$ in the category of non-commutative algebras. Thus $H(X)$ is a module over this ring.

**(4.4.11)** We now explicitly calculate the $\mathscr{A}$-module structure on $H(X)$ in terms of the basis $\phi_{\pi,k}$.

**Proposition.** *We have*

$$A_{\psi,\eta}(\phi_{\pi,k}) = \lambda_{\psi,\eta}(\pi)\phi_{\pi,k-n(\eta\pi)-2m(\psi)}$$

*where* $\lambda_{\psi,\eta}(\pi) = \epsilon(\frac{3}{2}, \eta\pi, \psi)$.

*Proof.* This is a simple calculation using the formula for $\mathscr{F}^{(2)}_{X,\psi}(\phi_{\pi,k})$ from §4.4.2. □

**(4.4.12)** We now determine the structure of $H(X)$ as a $(\mathscr{A} * \mathscr{T})$-module. For a cuspidal representation $\pi$, let $V_\pi$ be the closure in $H(X)$ of the space spanned by the $\phi_{\pi,k}$. It is clear that $V_\pi$ does not change if $\pi$ is replaced by an unramified twist. The main structure theorem we are after is the following:

**Proposition.** *We have the following:*

1. *Each $V_\pi$ is stable under $\mathscr{A} * \mathscr{T}$ and simple as an $(\mathscr{A} * \mathscr{T})$-module.*
2. *If $\pi$ and $\pi'$ are distinct elements of $\overline{\mathrm{Irr}}^\circ_G$ then $V_\pi$ and $V_{\pi'}$ are not isomorphic as $(\mathscr{A} * \mathscr{T})$-modules.*
3. *We have $H(X) = \bigoplus V_\pi$, the direct sum taken over $\pi \in \overline{\mathrm{Irr}}^\circ_G$.*

*In particular, $H(X)$ is semi-simple and multiplicity-free as an $(\mathscr{A} * \mathscr{T})$-module.*

*Proof.* It is clear from the formula for $A_{\psi,\eta}(\phi_{\pi,k})$ given in §4.4.11 that $V_\pi$ is stable for the action of $\mathscr{A} * \mathscr{T}$. Now, let $V$ be a non-zero $(\mathscr{A} * \mathscr{T})$-stable subspace of $V_\pi$. Since $V$ is $\mathscr{T}$-stable, it is spanned by the $\phi_{\pi,k}$ which it contains. Say that $V$ contains $\phi_{\pi,k_0} \neq 0$. If $\pi$ is odd then by §4.4.5 we can pick $\eta$ such that $n(\pi\eta)$ is odd. Since $m(\psi)$ takes on every integer value as $\psi$ varies, it follows that we can pick $\psi$ so that $A_{\psi,\eta}(\phi_{\pi,k_0})$ is a non-zero multiply of $\phi_{\pi,k}$, for any given $k$. Thus $V = V_\pi$. If $\pi$ is even then $k_0$ must be even since $\phi_{\pi,k_0} \neq 0$. We can therefore pick $\psi$ appropriately so that $A_{\psi,\eta}(\phi_{\pi,k_0})$ is a non-zero multiple of $\phi_{\pi,k}$, for any given even $k$. Thus $V = V_\pi$. This proves (1).

We now prove (2). Let $\pi$ and $\pi'$ be elements of $\overline{\mathrm{Irr}}^\circ_G$ and let $F : V_\pi \to V_{\pi'}$ be an isomorphism of $(\mathscr{A} * \mathscr{T})$-modules. It suffices to show $\pi = \pi'$. Since $F$ is $\mathscr{T}$-linear, we have $F(\phi_{\pi,k}) = a_k \phi_{\pi',k}$ for some scalar $a_k$. Using the $\mathscr{A}$-linearity of $F$, we find that $n(\eta\pi) = n(\eta\pi')$ for all $\eta$ and that

$$a_{k-n(\eta\pi)-2m(\psi)} = \frac{\lambda_{\psi,\eta}(\pi')}{\lambda_{\psi,\eta}(\pi)} a_k.$$

First consider the case where $\pi$ is odd. The above equation then implies that $a_k = ab^k$ for non-zero constants $a$ and $b$. Scaling $F$ by $a^{-1}$ and replacing $\pi'$ by an unramified twist determined by $b$, it follows that we can take the $a_k$ to all be 1. This shows that $\lambda_{\psi,\eta}(\pi') = \lambda_{\psi,\eta}(\pi)$ for all $\eta$ and $\psi$. Combining this with the equality $n(\eta\pi) = n(\eta\pi')$ shows that $\epsilon(s, \eta\pi, \psi) = \epsilon(s, \eta\pi'\psi)$ for all $\eta$ and $\psi$. The local converse theorem [JL, Corollary 2.19] now implies that $\pi = \pi'$. The case with $\pi$ even is similar: just restrict attention to $k$ even. This completes the proof of (2).

Statement (3) follows immediately from the definitions of $H(X)$ and $V_\pi$. $\square$

# Chapter 5

# The non-split side

The goals of §5 are as follows:

- Define a Fourier transform $\mathscr{F}_{X',\psi}$ on $\mathscr{S}(X')$.
- Factor $\mathscr{F}_{X',\psi}$ into two steps, as we did with $\mathscr{F}^{(1)}_{X,\psi}$.
- Use $\mathscr{F}_{X',\psi}$ to define a family of operators $\mathscr{A}$ on the cuspidal space $H(X')$ and determine the structure of $H(X')$ as an $\mathscr{A}$-module.
- Use the $\mathscr{A}$-structure on $H(X')$ to determine which functions on $X'$ are the characters of cuspidal representations.

The first two goals are accomplished in §5.2. The third and fourth are realized in §5.3 and §5.4, respectively. Section §5.1 carries out a number of routine calculations. The reader should keep the following diagram in mind throughout the section.

$$\begin{array}{ccc} \widetilde{X}' & \xrightarrow{\;i'\;} & B' \\ & {\scriptstyle p'}\searrow \quad \swarrow{\scriptstyle \pi'} & \\ & X' & \end{array}$$

Recall that $B'$ is the unique non-split quaternion algebra over $F$ with center $F$, $X'$ is the space of monic degree two polynomials over $F$ which are either irreducible or have a double root, $\widetilde{X}'$ is the disjoint union of the three quadratic extensions of $F$, $p'$ and $\pi'$ are the characteristic polynomial maps and $i'$ is a chosen map which restricts to an algebra injection on each component of $\widetilde{X}'$.

## 5.1 Measures and push-forwards

**(5.1.1)** The purpose of §5.1 is to define measures on the spaces $B'$, $K$, $K^\perp$, $X'$ and $\widetilde{X}'$, push-forwards along the maps $p'$ and $\pi'$ and compute all of these things explicitly. Here is a more detailed overview:

- In §5.1.2 we define a class of bases on $B'$ which we call the *standard bases*.
- In §5.1.3 we define the push-forward $p'_*$.
- In §5.1.4 we show that $(\pi')^*$ gives an isomorphism between the space of Schwartz functions on $X$ and the space of conjugation invariant Schwartz functions on $B'$.
- In §§5.1.5–5.1.9 we define and compute the measures on $B'$, $K$, $K^\perp$, $X'$ and $\widetilde{X}'$. (Actually, the relevant measures on $K$ were defined and computed in §4.1.7.)
- In §5.1.10 and §5.1.11 we define and study the push-forward $\pi'_*$ and relate $(\pi')^*\pi'_*$ to a certain averaging operator.
- In §5.1.12 we produce certain natural liftings of functions on $X'$ to functions on $B'$. As before, these will be important when we factor the Fourier transform.

**(5.1.2)** By a *standard basis* of $B'$ we mean a basis $1$, $i$, $j$, $k$ of $B'$ as an $F$-vector space where:

- $1$ is the unit of $B'$.
- $i$, $j$ and $k$ anti-commute.
- $i$, $j$ and $k$ square to elements of $F$.
- $ij = k$.

We will typically write $i^2 = \alpha$, $j^2 = \beta$ and $k^2 = \gamma$. The above conditions imply $\alpha\beta = -\gamma$. Given a standard basis and an element $x$ of $B'$ we write $x = x_0 + ix_1 + jx_2 + kx_3$. We have previous defined what is meant by a standard basis of a quadratic extension $K/F$.

**(5.1.3)** For a function $f$ on $\widetilde{X}'$ we define $p'_*f$ to be the function on $X'$ given by

$$(p'_*f)(x) = \frac{1}{\#(p')^{-1}(x)} \sum_{p'(y)=x} f(y)$$

much like our previous definition of $p_*$. The map $p'_*$ has the same properties as $p_*$ (in fact, they are basically the same thing). In particular, $p'_*$ maps $\mathscr{S}_{\mathrm{reg}}(\widetilde{X})$ surjectively onto $\mathscr{S}_{\mathrm{reg}}(X)$.

**(5.1.4)** Write $\mathscr{S}^{\mathrm{inv}}(B')$ for the subspace of $\mathscr{S}(B')$ consisting of those functions which are invariant under the conjugation action of $G'$. The following result is a consequence of $\pi$ being proper and open.

**Proposition.** *The pull-back map $(\pi')^* : \mathscr{S}(X') \to \mathscr{S}^{\mathrm{inv}}(B')$ is an isomorphism. Under this isomorphism, $\mathscr{S}_{\mathrm{reg}}(X')$ is identified with $\mathscr{S}^{\mathrm{inv}}_{\mathrm{reg}}(B')$.*

*Proof.* Since $\pi'$ is proper, it induces a map $(\pi')^* : \mathscr{S}(X') \to \mathscr{S}(B')$. It is clear that this map is injective and has image contained in $\mathscr{S}^{\mathrm{inv}}(B')$. Let $f$ be an element of $\mathscr{S}^{\mathrm{inv}}(B')$. As two elements of $B'$ are conjugate by an element of $G'$ if and only if their images in $X'$ are equal, we see that $f$ can be written as $f' \circ \pi'$ for some function $f'$ on $X'$. Since $\pi'$ is open and $f$ is locally constant we see that $f'$ too is locally constant. It is clear that $f'$ has compact support. Thus $f = (\pi')^* f'$ with $f' \in \mathscr{S}(X')$, which proves the proposition. □

**(5.1.5)** Recall that $d\mu_{B'}$ is the Haar measure on $B'$ giving the unique maximal order $\mathscr{O}_{B'}$ of $B'$ volume 1. We now compute it in a standard basis. This is similar to §4.1.6.

**Proposition.** *Identifying $B'$ with $F^4$ via a standard basis, we have*

$$d\mu_{B'}(x) = q|\alpha\beta\gamma|_F^{1/2} dx_0 dx_1 dx_2 dx_3$$

*where $dx_i = d\mu_F(x_i)$ are normalized Haar measures on $F$.*

*Proof.* It is clear that $d\mu_B(x) = c \cdot dx_0 dx_1 dx_2 dx_3$ for some constant $c$, so we need only compute $x$. We do this by computing the volume of $\mathscr{O}_{B'}$ using the measure $dx_0 dx_1 dx_2 dx_3$. To do this, note that $x \in B'$ belongs to $\mathscr{O}_{B'}$ if and only if $\mathbf{N}\, x$ belongs to $\mathscr{O}_F$. Furthermore, $\mathbf{N}\, x$ belongs to $\mathscr{O}_F$ if and only if each of $x_0^2$, $\alpha x_1^2$, $\beta x_2^2$ and $\gamma x_3^2$ do. We thus have

$$\int_{\mathscr{O}_B} dx_0 dx_1 dx_2 dx_3 = \mathrm{Vol}(\sqrt{\mathscr{O}_F})\,\mathrm{Vol}(\sqrt{\alpha^{-1}\mathscr{O}_F})\,\mathrm{Vol}(\sqrt{\beta^{-1}\mathscr{O}_F})\,\mathrm{Vol}(\sqrt{\gamma^{-1}\mathscr{O}_F})$$

where for $\Omega \subset F$ we use the notation $\sqrt{\Omega}$ to denote the set of elements $x \in F$ for which $x^2$ belongs to $\Omega$ and $\mathrm{Vol}\,\Omega$ to denote the volume of $\Omega$ with respect to the normalized Haar measure. Now, one of $\alpha$, $\beta$ and $\gamma$ has even valuation and the other two odd valuations. Assume $\alpha$ has even valuation. Then $\mathrm{Vol}(\sqrt{\alpha^{-1}\mathscr{O}_F}) = |\alpha|_F^{-1/2}$. As $\beta$ has odd valuation, we find $\mathrm{Vol}(\sqrt{\beta^{-1}\mathscr{O}_F}) = q^{-1/2}|\beta|_F^{-1/2}$, and similarly for $\gamma$. We thus have

$$\int_{\mathscr{O}_{B'}} dx_0 dx_1 dx_2 dx_3 = q^{-1}|\alpha\beta\gamma|_F^{-1/2}.$$

The constant $c$ is the inverse of this quantity, and so the proposition follows. □

**(5.1.6)** Recall that for a quadratic extension $K/F$ we have defined $d\mu_K$ to be the Haar measure on $K$ giving the ring of integers $\mathscr{O}_K$ volume 1. We have previously (in §4.1.7) computed this in a standard basis. We recall the result:

**Proposition.** *Identifying $K$ with $F^2$ via a standard basis we have*

$$d\mu_K = |\alpha/\mathbf{d}_K|_F^{1/2} dx_0 dx_1$$

*where $dx_i = d\mu_F(x_i)$ are normalized Haar measures on $F$.*

**(5.1.7)** Let $K$ be a quadratic extension and choose an embedding $i_K : K \to B'$. Let $K^\perp$ denote the orthogonal complement to $K$ in $B'$ under the trace pairing. There is then a unique Haar measure $d\mu_{K^\perp}$ on $K^\perp$ so that $d\mu_{B'} = d\mu_K d\mu_{K^\perp}$. We now compute $d\mu_{K^\perp}$ in coordinates. (Note that we have previously in §4.1.8 defined a measure $d\mu_{K^\perp}$ but this was for a subspace $K^\perp$ of $B$.)

**Proposition.** *Let 1, $i$, $j$, $k$ be a standard basis for $B'$ so that 1, $i$ is a standard basis for $K \subset B'$. Then $j$, $k$ is a basis for $K^\perp$ and under the resulting identification $K^\perp = F^2$ we have*

$$d\mu_{K^\perp}(x) = q|\mathbf{d}_K\beta\gamma|_F^{1/2} dx_2 dx_3$$

*where $dx_i = d\mu_F(x_i)$ are normalized Haar measures on $F$.*

*Proof.* This follows immediately from §5.1.5 and §5.1.6. □

**(5.1.8)** We define $d\mu_{X'}$ to be the measure on $X'$ given by $\pi'_*(d\mu_{B'})$. We now compute it in coordinates. Note that this computation is one of the reasons for defining the measure $d\mu_X$ as we did in §4.1.9.

**Proposition.** *We have*

$$d\mu_{X'}(x) = |\Delta|_F^{1/2} A(\Delta) d\nu dt$$

*where we have identified $X'$ with its image in $F^2$, $d\nu$ and $dt$ are normalized Haar measures on $F$ and $A(\Delta)$ is as previously (c.f. §4.1.9). Explicitly, the formula for $d\mu_{X'}$ means that for $f \in \mathscr{S}(X')$*

*we have*

$$\int_B ((\pi')^* f)(x) d\mu_{B'}(x) = \int_X f(t,\nu)|\Delta|_F^{1/2} A(\Delta) d\nu dt.$$

*Proof.* Pick a standard basis for $B'$ such that $\alpha$ has even valuation and $\beta$ and $\gamma$ have odd valuation. In the resulting coordinates, we have $t = 2x_0$ and $\nu = x_0^2 - \alpha x_1^2 - \beta x_2^2 - \gamma x_3^2$. By §5.1.5 we have

$$\int_{B'} f(\pi'(x))dx = q|\alpha\beta\gamma|_F^{1/2} \int_{F^4} f(2x_0, x_0^2 - \alpha x_1^2 - \beta x_2^2 - \gamma x_3^2) dx_0 dx_1 dx_2 dx_3.$$

We now apply §4.1.4 and make the change of variables $x_3 = \sqrt{u}$ where

$$u = \frac{\Delta/4 - \alpha x_1^2 - \beta x_2^2}{\gamma}$$

to obtain

$$\int_{B'} f(\pi'(x))dx = q \left|\frac{\alpha\beta}{\gamma}\right|_F^{1/2} \int_{F^4} f(t,\nu) \frac{1+\eta(u)}{|u|_F^{1/2}} dx_1 dx_2 dt d\nu.$$

As $\alpha\beta/\gamma = -1$, the absolute value in front of the integral is equal to one. We can thus write

$$\int_{B'} f(\pi'(x))dx = \int_{F^2} f(t,\nu) A'(\Delta) dt d\nu$$

with

$$A'(\Delta) = q \int_{F^2} \frac{1+\eta(u)}{|u|_F^{1/2}} dx_1 dx_2.$$

Now, one easily sees that if $\Delta = 0$ then the integrand vanishes identically, and so $A'(0) = 0$. If $\Delta \neq 0$ then we have

$$A'(\Delta) = q|\Delta|_F^{1/2} I_3(\Delta/4\gamma, -\alpha/\gamma, -\beta/\gamma).$$

The results of §3.3 now shows that $A'(\Delta) = |\Delta|_F^{1/2} A(\Delta)$ when $\Delta$ is not a square while $A'(\Delta) = 0$ if $\Delta$ is a square. This gives the stated result. □

**(5.1.9)** We define $d\mu_{\widetilde{X}'}$ to be the measure on $\widetilde{X}'$ given by $(p')^* d\mu_{X'}$. We now compute it in coordinates.

**Proposition.** *We have* $d\widetilde{\mu}_{\widetilde{X}}|_K = \frac{1}{2}(1+d_K)|\Delta|_F d\mu_K$. *Explicitly, this means that for* $f \in \mathscr{S}(\widetilde{X}')$ *we have*

$$\int_{\widetilde{X}'} f \, d\mu_{\widetilde{X}'} = \tfrac{1}{2} \sum_K (1+d_K) \int_K f|\Delta|_F d\mu_K$$

*where the sum is over the three degree two field extensions* $K/F$.

*Proof.* This goes just like the proof in §4.1.10. □

**(5.1.10)** For $\sigma \in G'$ and $f \in \mathscr{S}(B')$ we let $f^\sigma$ be the function $x \mapsto f(\sigma x \sigma^{-1})$. Define a map

$$\mathrm{avg} : \mathscr{S}(B') \to \mathscr{S}^{\mathrm{inv}}(B'), \qquad \mathrm{avg}\, f = \frac{1}{\mathrm{Vol}(G'/Z')} \int_{G'/Z'} f^\sigma \, d\sigma$$

where $d\sigma$ is any Haar measure on $G'/Z'$ ($Z'$ being the center of $G'$). The function $\mathrm{avg}\, f$ is easily seen to be independent of the choice of Haar measure. Since $f$ belongs to $\mathscr{S}(B')$ its stabilizer in $G'/Z'$ is open, and so the above integral is really a finite sum. Thus $\mathrm{avg}\, f$ is well-defined and a Schwartz function. It is clear that avg is a projector, that is, $\mathrm{avg}^2 = \mathrm{avg}$.

**(5.1.11)** We now define and study the map $\pi'_*$.

**Proposition.** *We have the following:*

1. *The map* $(\pi')^* : \mathscr{S}(X') \to \mathscr{S}(B')$ *has a unique adjoint* $\pi'_* : \mathscr{S}(B') \to \mathscr{S}(X')$.
2. *We have* $\pi'_*(\pi')^* = \mathrm{id}$ *and* $(\pi')^* \pi'_* = \mathrm{avg}$.
3. *We have* $\|\pi'_* f\|_{X'} \le \|f\|_{B'}$. *The map* $(\pi')^*$ *is an* $L^2$*-isometry.*
4. *The map* $\pi'_*$ *carries* $\mathscr{S}_{\mathrm{reg}}(B')$ *into* $\mathscr{S}_{\mathrm{reg}}(X')$.
5. *Pick a standard basis. For* $f \in \mathscr{S}_{\mathrm{reg}}(B')$ *we have*

$$(\pi'_* f)(t, \nu) = \frac{q}{|\Delta|_F^{1/2} A(\Delta)} \int_{F^2} f(\tfrac{1}{2}t + x_1 i + x_2 j \pm \sqrt{u}\, k) \frac{1 + \eta(u)}{|u|_F^{1/2}} dx_1 dx_2$$

*where* $f(\pm\sqrt{u})$ *means* $\frac{1}{2}(f(\sqrt{u}) + f(-\sqrt{u}))$ *and*

$$u = \frac{\Delta/4 - \alpha x_1^2 - \beta x_2^2}{\gamma}.$$

*Proof.* If an adjoint exists then it is unique since the pairings are non-degenerate. Now, let $f \in \mathscr{S}(X')$. Then the measure $f d\mu_{B'}$ is absolutely continuous with respect to $d\mu_{B'}$. It follows that $\pi'_*(f d\mu_{B'})$ is absolutely continuous with respect to $\pi'_*(d\mu_{B'}) = d\mu_{X'}$. Thus by the Radon-Nikodym theorem we can find a function $\pi'_* f$ on $X'$ such that $\pi'_*(f d\mu_{B'}) = (\pi'_* f) d\mu_{X'}$. It is clear that $\pi'_* f$ has compact support, but maybe not clear that it is locally constant; we will prove that shortly. Nonetheless, the formula

$$\langle f, (\pi')^* g \rangle_{B'} = \langle \pi'_* f, g \rangle_{X'}$$

holds for $f \in \mathscr{S}(B')$ and $g \in \mathscr{S}(X')$.

For $f$ and $g$ in $\mathscr{S}(X')$ we have

$$\begin{aligned}\langle \pi'_*(\pi')^* f, g\rangle_{X'} &= \langle (\pi')^* f, (\pi')^* g\rangle_{B'} = \int_{B'} ((\pi')^* f)((\pi')^* g) d\mu_{B'} \\ &= \int_{B'} ((\pi')^* fg) d\mu_{B'} = \int_{X'} fg\, d\mu_{X'} = \langle f, g\rangle_{X'}.\end{aligned}$$

The non-degeneracy of $\langle , \rangle_{X'}$ now gives $f = \pi'_*(\pi')^* f$ for any $f \in \mathscr{S}(X')$. Now, for $f \in \mathscr{S}(B')$ and $\sigma \in G'$ we have

$$\langle \pi'_*(f^\sigma), g\rangle_{X'} = \langle f^\sigma, (\pi')^* g\rangle_{B'} = \langle f, (\pi')^* g\rangle_{B'} = \langle \pi'_* f, g\rangle_{X'}$$

and so $\pi'_*(f^\sigma) = \pi'_* f$. (The middle equality above follows from the fact that $(\pi')^* g$ belongs to $\mathscr{S}^{\mathrm{inv}}(B')$.) It follows that $\pi'_*(\operatorname{avg} f) = \pi'_* f$. As $\operatorname{avg} f$ belongs to $\mathscr{S}^{\mathrm{inv}}(B')$ we can, by §5.1.4, find $f' \in \mathscr{S}(B')$ such that $\operatorname{avg} f = (\pi')^* f'$. We thus have

$$\pi'_* f = \pi'_*(\operatorname{avg} f) = \pi'_*(\pi')^* f' = f'.$$

In particular, this shows that $\pi'_* f$ belongs to $\mathscr{S}(X')$ and thus establishes (1). Applying $(\pi')^*$ to each side of the above gives $(\pi')^* \pi'_* = \operatorname{avg}$ and thus establishes (2).

We have seen already that for $f, g \in \mathscr{S}(X')$ we have $\langle (\pi')^* f, (\pi')^* g\rangle_{B'} = \langle f, g\rangle_{X'}$ and so $(\pi')^*$ is an $L^2$-isometry. Now let $f$ be an element of $\mathscr{S}(B')$. We then have

$$\begin{aligned}\|\pi'_* f\|^2_{X'} &= \langle \pi'_* f, \pi'_* \overline{f}\rangle_{X'} = \langle \operatorname{avg} f, \overline{f}\rangle_{B'} \\ &= \frac{1}{\mathrm{Vol}(G'/Z')} \int_{G'/Z'} \langle f^\sigma, \overline{f}\rangle_{B'} d\sigma \\ &\leq \frac{1}{\mathrm{Vol}(G'/Z')} \int_{G'/Z'} \|f^\sigma\|_{B'} \|f\|_{B'} d\sigma = \|f\|^2_{B'}\end{aligned}$$

the last step following from $\|f^\sigma\|_{B'} = \|f\|_{B'}$. We thus have $\|\pi'_* f\|_{X'} \leq \|f\|_{B'}$, establishing (3).

Let $f \in \mathscr{S}_{\mathrm{reg}}(B')$. Then for any $\sigma \in G'$ the function $f^\sigma$ still belongs to $\mathscr{S}_{\mathrm{reg}}(B')$. It follows that $\operatorname{avg} f \in \mathscr{S}^{\mathrm{inv}}_{\mathrm{reg}}(B')$. We can thus find $f' \in \mathscr{S}_{\mathrm{reg}}(X')$ such that $\operatorname{avg} f = (\pi')^* f'$. We then have $\pi'_* f = \pi'_* \operatorname{avg} f = f'$ and so $\pi'_* f$ belongs to $\mathscr{S}_{\mathrm{reg}}(X')$. This establishes (4). The proof of (5) goes just like the proof of part (4) of §4.1.11. □

**(5.1.12)** We now prove the following result, which is directly analogous to §4.1.16.

**Proposition.** *Let $f$ belong to $\mathscr{S}_{\mathrm{reg}}(K)$ with $K \subset B'$. For any sufficiently small compact open set* $\mathfrak{a}$

*of $K^\perp$ containing 0 we have*

$$\pi'_*(f \otimes \delta_{\mathfrak{a}}) = \frac{2}{1+d_K}|\Delta|_F^{-1}(p_K)_* f.$$

*Here $\delta_{\mathfrak{a}} = \chi_{\mathfrak{a}}/\operatorname{Vol}(\mathfrak{a})$ where $\chi_{\mathfrak{a}}$ is the characteristic function of $\mathfrak{a}$ and $\operatorname{Vol}(\mathfrak{a})$ the volume of $\mathfrak{a}$ with respect to the normalized Haar measure on $K^\perp$.*

*Proof.* The proof goes exactly like the proof given in §4.1.16. □

**Corollary.** *Let $f \in \mathscr{S}_{\mathrm{reg}}(K)$. For any sufficiently small compact open set $\mathfrak{a}$ of $K^\perp$ containing 0 we have*

$$(p'_K)_* f = \tfrac{1}{2}(1+d_K)\pi'_*(|\Delta|_F f \otimes \delta_{\mathfrak{a}}).$$

*Proof.* The proof is just the like the corresponding corollary in §4.1.16. □

## 5.2 The Fourier transform $\mathscr{F}_{X',\psi}$

**(5.2.1)** The purpose of §5.2 is to introduce a Fourier transform on the Schwartz space $\mathscr{S}(X')$ and prove a factorization result for it. Here is an overview:

- In §5.2.2 and §5.2.3 we recall the Fourier transforms on $B'$ and $K^\perp$.
- In §5.2.4 we define the Fourier transform $\mathscr{F}_{X',\psi}$ by $(\pi')^*(\mathscr{F}_{X',\psi}(f)) = \mathscr{F}_{B',\psi}((\pi')^* f)$. One has to check that this is well-defined, but this is quite straightforward. We verify that $\mathscr{F}_{X',\psi}\pi'_* = \pi'_*\mathscr{F}_{B',\psi}$, which makes $\mathscr{F}_{X',\psi}$ look more like $\mathscr{F}^{(1)}_{X,\psi}$.
- In §5.2.5 we factor the Fourier transform $\mathscr{F}_{X',\psi}$ as $\overline{p}'_*\mathscr{F}'_{\widetilde{X}}$ where $\mathscr{F}'_{\widetilde{X}}$ is essentially the Fourier transform on the various $K$'s (as it was before) and $\overline{p}'_*$ is some fairly simple operation.
- In §5.2.6 we compute an explicit formula for $\overline{p}'_*$.

**(5.2.2)** Let $\psi = \psi_F$ be a non-trivial additive character of $F$. Define $\psi_{B'}$ to be the additive character of $B'$ given by $\psi_F \circ \operatorname{tr}_{B'/F}$. For a function $f$ on $B'$ we put

$$(\mathscr{F}_{B',\psi} f)(x) = q^{-2m-1}\int_{B'} f(y)\psi_{B'}(xy)d\mu_{B'}(y).$$

Here, as before, $\mathfrak{p}^{-m}$ is the conductor of $\psi_F$. Once again, $\mathscr{F}_{B',\psi}$ induces an isomorphism $\mathscr{S}(B') \to \mathscr{S}(B')$ and can be extended to an isometry $L^2(B') \to L^2(B')$. With out normalization, $\mathscr{F}_{B',\psi}$ is self-adjoint with respect to $\langle , \rangle_{B'}$, an isometry with respect to $( , )_{B'}$ and has inverse $\mathscr{F}_{B',\overline{\psi}}$.

**(5.2.3)** Let $K \subset B'$ be a quadratic extension of $F$. We have previously defined a Fourier transform $\mathscr{F}_{K,\psi}$ on $K$. For a function $f$ on $K^{\perp}$ we put

$$(\mathscr{F}_{K^{\perp},\psi} f)(x) = q^{-m-1} d_K^{-1/2} \int_K f(y) \psi_K(xy) dy.$$

Do not confuse this Fourier transform with the one $\mathscr{F}_{K^{\perp},\psi}$ where $K^{\perp}$ is a subspace of $B$. The above Fourier transform has the property that

$$\mathscr{F}_{B',\psi}(f \otimes g) = (\mathscr{F}_{K,\psi} f) \otimes (\mathscr{F}_{K^{\perp},\psi} g)$$

for $f \in \mathscr{S}(K)$ and $g \in \mathscr{S}(K^{\perp})$. Here $f \otimes g$ is the function $(x, y) \mapsto f(x)g(y)$ where $B'$ has been identified with $K \times K^{\perp}$.

**(5.2.4)** The map $\mathscr{F}_{B',\psi}$ carries $\mathscr{S}^{\mathrm{inv}}(B')$ into itself. Using the isomorphism $(\pi')^* : \mathscr{S}(X') \to \mathscr{S}^{\mathrm{inv}}(B')$ of §5.1.4 we transport the Fourier transform to $\mathscr{S}(X')$. That is, we define

$$\mathscr{F}_{X',\psi} : \mathscr{S}(X') \to \mathscr{S}(X'), \qquad \mathscr{F}_{X',\psi}(f) = \pi'_*(\mathscr{F}_{B',\psi}((\pi')^* f)).$$

Note that $\pi'_*$ is the inverse to $(\pi')^*$ by §5.1.11.

**Proposition.** *We have the following:*

1. *$\mathscr{F}_{X',\psi}$ is self-adjoint with respect to $\langle , \rangle_{X'}$.*
2. *$\mathscr{F}_{X',\psi}$ is an isometry for $( , )_{X'}$.*
3. *The inverse of $\mathscr{F}_{X',\psi}$ is $\mathscr{F}_{X',\overline{\psi}}$.*
4. *We have $\pi'_* \mathscr{F}_{B',\psi} = \mathscr{F}_{X',\psi} \pi'_*$.*

*Proof.* (1) follows from the adjointness of $\pi'_*$ and $(\pi')^*$ and the self-adjointness of $\mathscr{F}_{B',\psi}$. (2) follows from the corresponding statement for $\mathscr{F}_{B',\psi}$ and the fact (*c.f.* §5.1.11) that $(\pi')^* : \mathscr{S}(X') \to \mathscr{S}^{\mathrm{inv}}(B')$ is an isometry. (3) is similar. We now prove (4). A simple computation shows that for $f \in \mathscr{S}(B')$ and $\sigma \in G'$ we have $\mathscr{F}_{B',\psi}(f^{\sigma}) = (\mathscr{F}_{B',\psi} f)^{\sigma^{-1}}$. From this, we see that $\mathscr{F}_{B',\psi}(\operatorname{avg} f) = \operatorname{avg}(\mathscr{F}_{B',\psi} f)$. Thus, using §5.1.11 we find

$$\mathscr{F}_{X',\psi}(\pi'_* f) = \pi'_*(\mathscr{F}_{B',\psi}((\pi')^* \pi'_* f)) = \pi'_*(\mathscr{F}_{B',\psi}(\operatorname{avg} f)) = \pi'_*(\operatorname{avg}(\mathscr{F}_{B',\psi} f)) = \pi'_*(\mathscr{F}_{B',\psi} f),$$

which is the stated identity. $\square$

**(5.2.5)** We now come to our main result on the factorization of the Fourier transform. For a quadratic field $K$ we have defined $\mathscr{S}_0(K)$ be the set of functions $f$ which have integral zero on vertical strips (*c.f.* §4.2.7). The Fourier transform gives an isomorphism $\mathscr{F}_{K,\psi} : \mathscr{S}_{\mathrm{reg}}(K) \to \mathscr{S}_0(K)$. We have also defined a modified Fourier transform $\mathscr{F}'_{K,\psi}$, which also gives an isomorphism $\mathscr{S}_{\mathrm{reg}}(K) \to \mathscr{S}_0(K)$. We let $\mathscr{F}'_{\widetilde{X}',\psi}$ be the Fourier transform on $\widetilde{X}'$ gotten from the $\mathscr{F}'_{K,\psi}$. We now have our main result:

**Proposition.** *There is a unique map $\overline{p}'_* : \mathscr{S}_0(\widetilde{X}') \to \mathscr{S}(X')$ such that the diagram*

$$\begin{array}{ccc} \mathscr{S}_{\mathrm{reg}}(\widetilde{X}') & \xrightarrow{p'_*} & \mathscr{S}_{\mathrm{reg}}(X') \\ \downarrow{\scriptstyle \mathscr{F}'_{\widetilde{X}',\psi}} & & \downarrow{\scriptstyle \mathscr{F}_{X',\psi}} \\ \mathscr{S}_0(\widetilde{X}') & \xrightarrow{\overline{p}'_*} & \mathscr{S}(X') \end{array}$$

*commutes. For $f \in \mathscr{S}(K)$ we have*

$$(\overline{p}'_K)_* f = q^{-1} \pi'_*(f \otimes \chi_{\mathfrak{a}})$$

*where $\mathfrak{a}$ is any sufficiently large compact open subset of $K^{\perp}$.*

*Proof.* Again, the existence and uniqueness of $\overline{p}'_*$ is clear since $\mathscr{F}'_{\widetilde{X}',\psi}$ is an isomorphism. To compute the formula for $\overline{p}'_*$, let $f \in \mathscr{S}_{\mathrm{reg}}(K)$ be given. Let $\mathfrak{a}$ be a very small compact open neighborhood of 0 in $K^{\perp}$. By §5.1.12 we have

$$(p'_K)_* f = \tfrac{1}{2}(1 + d_K)\pi'_*(|\Delta|_F f \otimes \delta_{\mathfrak{a}})$$

Now take the Fourier transform of each side. We find

$$\begin{aligned} \mathscr{F}_{X',\psi}((p'_K)_* f) &= \tfrac{1}{2}(1 + d_K)\mathscr{F}_{X',\psi}(\pi'_*(|\Delta|_F f \otimes \delta_{\mathfrak{a}})) \\ &= \tfrac{1}{2}(1 + d_K)\pi'_*(\mathscr{F}_{K,\psi}(|\Delta|_F f) \otimes \mathscr{F}_{K^{\perp},\psi}(\delta_{\mathfrak{a}})). \end{aligned}$$

In the second step we used the identity $\mathscr{F}_{X',\psi}\pi'_* = \pi'_*\mathscr{F}_{B',\psi}$ from §5.2.4 and the identity from §5.2.3 regarding the Fourier transform of a pure tensor. One easily finds $\mathscr{F}_{K^{\perp},\psi}(\delta_{\mathfrak{a}}) = q^{-m-1} d_K^{-1/2} \chi_{\mathfrak{a}'}$ where $\mathfrak{a}'$ is a large compact open. The proposition follows. $\square$

**(5.2.6)** We now explicitly compute the map $\overline{p}'_*$.

**Proposition.** *Let $f$ belong to $\mathscr{S}_0(K)$. Then*

$$((\overline{p}'_K)_* f)(t,\nu) = \frac{d_K^{1/2}}{|\Delta|_F^{1/2} A(\Delta)} \int_F f(\tfrac{1}{2}t + ix) I_2\left(\frac{\Delta - 4\mathbf{d}_K x^2}{\gamma}, \mathbf{d}_K\right) d\mu_F(x)$$

*where $i \in K$ is such that $i^2 = \mathbf{d}_K$, $\gamma = -\beta \mathbf{d}_K$ and $\beta$ is an element of $F^\times$ for which $(\mathbf{d}_K, \beta) = -1$.*

*Proof.* The proof is similar to the one in §4.2.8, so we omit some details. Pick a standard basis for $B'$ so that 1, $i$ is a standard basis for $K$ and $i^2 = \mathbf{d}_K$. Take $\mathfrak{a}$ to be the open set of $K^\perp$ such that $\chi_{\mathfrak{a}}(x_2 j + x_3 k) = \chi_{\mathfrak{p}^{-n}}(x_2)\chi_{\mathfrak{p}^{-m}}(x_3)$ for large integers $n$ and $m$. Our formula for $\pi'_*$ from §5.1.11 then gives

$$q^{-1}(\pi'_*(f \otimes \chi_{\mathfrak{a}}))(t,\nu) = \frac{1}{|\Delta|_F^{1/2} A(\Delta)} \int_{F^2} f(\tfrac{1}{2}t + ix_1)\chi_{\mathfrak{p}^{-n}}(x_2)\chi_{\mathfrak{p}^{-m}}(\sqrt{u}) \frac{1+\eta(u)}{|u|_F^{1/2}} dx_1 dx_2$$

with

$$u = \frac{\Delta/4 - \alpha x_1^2 - \beta x_2^2}{\gamma}.$$

As before, we find that if we omit the $\chi_{\mathfrak{p}^{-n}}(x_2)\chi_{\mathfrak{p}^{-m}}(\sqrt{u})$ from the integrand the result does not change. After having done this, the $x_2$ integral is then $|b|_F^{-1/2} I_2(a,b)$ with

$$a = \frac{\Delta - 4\mathbf{d}_K x_1^2}{4\gamma}, \qquad b = -\frac{\beta}{\gamma} = \mathbf{d}_K^{-1},$$

just as before. We thus find

$$((\overline{p}'_K)_* f)(t,\nu) = \frac{d_K^{1/2}}{|\Delta|_F^{1/2} A(\Delta)} \int_F f(\tfrac{1}{2}t + ix) I_2\left(\frac{\Delta - 4\mathbf{d}_K x^2}{\gamma}, \mathbf{d}_K\right) dx,$$

which is the stated result. □

## 5.3 The cuspidal space $H(X')$ and its $\mathscr{A}$-structure

**(5.3.1)** In §5.3 we introduce the cuspidal space $H(X')$, define an $\mathscr{A}$-module structure on it and compute its structure as an $\mathscr{A}$-module. Although the program is similar to §4.3 and §4.4 we proceed differently, in a more conceptual and less computational manner. (We do perform some computations at the end of the section. These are not needed to establish the main results of the section, but will be used later on.) Here is an overview of the section:

- In §5.3.2 and §5.3.3 we introduce two operations on the space of functions on $X'$: convolution and $f \mapsto f^\vee$.

- In §5.3.4 we define and study a certain "Eisenstein" space of functions on $X'$.
- In §5.3.5 we define the cuspidal space $\mathscr{S}^\circ(X')$ as the orthogonal complement to the Eisenstein space, and prove some basic properties about it. The space $H(X')$ is defined to be the $L^2$-closure of $\mathscr{S}^\circ(X')$.
- In §5.3.6 we define the operators $A_{\psi,\eta}$ which constitute the algebra $\mathscr{A}$. We also define the $T_n$ operators.
- In §§5.3.7–5.3.11 we relate the operators $A_{\psi,\eta}$ to certain convolution operators, culminating in the proof in §5.3.11 that a subspace of $\mathscr{S}^\circ(X')$ is stable under $\mathscr{A}$ if and only if it is stable under convolution by all of $\mathscr{S}_{\mathrm{ns}}(X')$.
- In §5.3.12 we introduce the truncated character functions $\phi_{\pi,k}$ and the spaces $V_\pi$.
- In §5.3.13 we compute $f * \phi_{\pi,k}$, for an arbitrary function $f$.
- In §5.3.14 we determine the structure of the space $\mathscr{S}_{\mathrm{ns}}(X')$ under convolution, in terms of the basis $\phi_{\pi,k}$.
- In §5.3.15 we use the results of §5.3.11 to transfer the results of §5.3.14 to yield the $\mathscr{A}$-structure of $H(X')$. This is the main result of §5.3.
- In §5.3.16 and §5.3.17 we explicitly compute what the $A_{\psi,\eta}$ do to the basis elements $\phi_{\pi,k}$.

**(5.3.2)** For functions $f$ and $g$ on $G'$ define a function $f * g$ on $G'$, called the *convolution* of $f$ and $g$, by

$$(f * g)(x) = \int_{G'} f(y)g(xy^{-1})d\mu_{G'}(y)$$

This integral makes sense so long as one of $f$ or $g$ has compact support. The operation $*$ is associative but not in general commutative. However, if one of $f$ or $g$ is invariant then $f * g = g * f$ does hold. If $f$ and $g$ are both invariant then so is $f * g$ and so the operation $*$ descends to functions on $X$. It is easy to see that we get maps

$$\mathscr{S}_{\mathrm{ns}}(X') \otimes \mathscr{S}_{\mathrm{ns}}(X') \to \mathscr{S}_{\mathrm{ns}}(X'), \qquad \mathscr{C}^\infty(X'_{\mathrm{ns}}) \otimes \mathscr{S}_{\mathrm{ns}}(X') \to \mathscr{C}(X'_{\mathrm{ns}})$$

using $*$. These first is commutative and associative; the second is associative in the obvious sense.

**(5.3.3)** We have a map $X'_{\mathrm{ns}} \to X'_{\mathrm{ns}}$, written $x \mapsto x^{-1}$, gotten by thinking of $X'_{\mathrm{ns}}$ as the set of conjugacy classes in $G'$. For $f \in \mathscr{S}_{\mathrm{ns}}(X')$ we define $f^\vee$ to be the function given by $x \mapsto f(x^{-1})$. It again belongs to $\mathscr{S}_{\mathrm{ns}}(X')$.

**Proposition.** *The map $f \mapsto |\cdot|_F^{-2} f^\vee$ is self-adjoint for $\langle , \rangle_{X'}$ and an isometry for $( , )_{X'}$.*

*Proof.* We have

$$\langle f^\vee, g\rangle_{X'} = \langle (\pi')^* f^\vee, (\pi')^* g\rangle_{B'} = \int_{G'} f(x^{-1}) g(x) d\mu_{B'}(x) = \int_{G'} f(x) g(x^{-1}) |\mathbf{N}\, x|_F^{-4} d\mu_{B'}(x)$$
$$= \langle (\pi')^* f, (\pi')^* (|\cdot|_F^{-4} g^\vee)\rangle_{B'} = \langle f, |\cdot|_F^{-4} g^\vee\rangle_{X'}.$$

Changing $f$ to $|\cdot|_F^2 f$ shows that $f \mapsto |\cdot|_F^{-2} f^\vee$ is self-adjoint for $\langle , \rangle_X$. The isometry statement follows easily from this. ☐

**(5.3.4)** We define $\mathscr{S}^1(X')$ to be the subset of $\mathscr{S}(X')$ consisting of those functions whose restriction to $X'_{\mathrm{ns}}$ factors through the norm $\mathbf{N}$. We write $\mathscr{S}^1_{\mathrm{ns}}(X')$ for the subspace of $\mathscr{S}^1(X')$ consisting of those functions supported on $X'_{\mathrm{ns}}$.

**Proposition.** *We have the following:*

1. *An element $f$ of $\mathscr{S}(X')$ belongs to $\mathscr{S}^1(X')$ if and only if $(\pi')^* f$ is invariant under left (or right) translation by the group $G'_1$ consisting of those elements $x$ of $G'$ with $\mathbf{N}\, x = 1$.*
2. *The spaces $\mathscr{S}^1(X')$ and $\mathscr{S}^1_{\mathrm{ns}}(X)$ are closed under pointwise addition and multiplication.*
3. *The space $\mathscr{S}^1_{\mathrm{ns}}(X')$ is closed under the involution $f \mapsto f^\vee$.*
4. *The space $\mathscr{S}^1_{\mathrm{ns}}(X')$ is closed under convolution by elements of $\mathscr{S}_{\mathrm{ns}}(X')$.*
5. *The space $\mathscr{S}^1(X')$ is closed under the Fourier transform.*

*Proof.* (1), (2) and (3) are straightforward. We now prove (4). Let $f$ belong to $\mathscr{S}^1_{\mathrm{ns}}(X')$ and let $\phi$ belong to $\mathscr{S}_{\mathrm{ns}}(X')$. We then have

$$(\phi * f)(xu) = \int_{G'} f(y) \phi(xuy^{-1}) d\mu_{G'}(y)$$

for $x \in G'$ and $u \in G'_1$. Changing $y$ to $yu$ in the integral and using $\phi(yu) = \phi(y)$ makes the $u$ on the right go away and shows that $\phi * f$ is invariant under translation by $G'_1$ and thus belongs to

$\mathscr{S}^1(X')$. We now prove (5). Let $f$ belong to $\mathscr{S}^1(X')$. We have

$$(\mathscr{F}_\psi f)(xu) = \int_{G'} f(y)\psi_B(xuy)d\mu_{B'}(y)$$

for $x \in B'$ and $u \in G'_1$. Changing $y$ to $u^{-1}y$ and using $f(u^{-1}y) = f(y)$ gives the desired result. □

**(5.3.5)** Let $f \in \mathscr{S}(X')$. We say that $f$ is *cuspidal* if $\langle f, g\rangle_{X'} = 0$ for all $g \in \mathscr{S}^1(X')$. We write $\mathscr{S}^\circ(X')$ for the space of cuspidal functions and $H(X')$ for its closure in $L^2(X')$. If $f$ is cuspidal the its support is contained in $X_{\mathrm{ns}}$ and so $\mathscr{S}^\circ(X') \subset \mathscr{S}_{\mathrm{ns}}(X')$. One easily sees that to check that $f$ is cuspidal it is enough to show $\langle f, g\rangle_{X'} = 0$ for $g \in \mathscr{S}^1_{\mathrm{ns}}(X')$.

**Proposition.** *The space $\mathscr{S}^\circ(X')$ is stable under the following operations.*

1. *The involution $f \mapsto f^\vee$.*

2. *The Fourier transform.*

3. *Pointwise multiplication by elements of $\mathscr{S}^1(X')$.*

4. *Convolution with elements of $\mathscr{S}_{\mathrm{ns}}(X')$.*

*Proof.* These all follow easily from the corresponding properties of the spaces $\mathscr{S}^1(X')$ and $\mathscr{S}^1_{\mathrm{ns}}(X')$ and simple adjointness statements. We prove (1) as an example. Let $f \in \mathscr{S}^\circ(X')$ and let $g \in \mathscr{S}^1_{\mathrm{ns}}(X')$. We then have

$$\langle f^\vee, g\rangle_{X'} = \langle f, |\cdot|_F^{-4} g^\vee\rangle_{X'} = 0$$

since $|\cdot|_F^{-4} g^\vee$ belongs to $\mathscr{S}^1_{\mathrm{ns}}(X)$ and $f$ is cuspidal. Thus $f^\vee$ is cuspidal. □

**(5.3.6)** For a non-trivial additive character $\psi$ of $F$ and a character $\eta$ of $F^\times$ define an operator

$$A_{\psi,\eta} : \mathscr{S}^\circ(X') \to \mathscr{S}^\circ(X'), \qquad A_{\psi,\eta}(f) = -\eta^{-1}\mathscr{F}_{X',\psi}(|\cdot|_F^{-2}\eta^{-1}f^\vee).$$

If $\eta$ is unitary then $A_{\psi,\eta}$ is easily seen to be an isometry for $(,)_{X'}$ and thus extends to an isometry $H(X') \to H(X')$. We will see shortly that the $A_{\psi,\eta}$ commute. We let $\mathscr{A}$ be the polynomial ring over $\mathbb{C}$ generated by the symbols $A_{\psi,\eta}$ with $\eta$ unitary, so that the above definitions give $H(X')$ the structure of an $\mathscr{A}$-module.

We will need a few more operators as well. For an integer $n$ we define an operator

$$T_n : H(X') \to H(X')$$

by letting $T_n(f)$ be the function which is equal to $f$ on the locus where $\mathbf{N}$ has valuation $n$ and 0 off of this set. The operators $T_n$ are mutually orthogonal idempotent operators. We let $\mathscr{T}$ be the polynomial ring over $\mathbb{C}$ in the $T_n$. We have thus given $H(X')$ the structure of a $\mathscr{T}$-module. The actions of $A_{\psi,\eta}$ and $T_n$ do not commute. We write $\mathscr{A} * \mathscr{T}$ for the coproduct of $\mathscr{A}$ and $\mathscr{T}$ in the category of non-commutative algebras, so that $H(X')$ is a module over $\mathscr{A} * \mathscr{T}$.

**(5.3.7)** Let $\phi_{\psi,\eta}$ be the function on $X_{\mathrm{ns}}$ defined by $(t, \nu) \mapsto \psi(t)\eta(\nu)$. A simple computation shows that, for $f \in \mathscr{S}^\circ(X')$, we have

$$\phi_{\psi,\eta} * f = -q^{2m(\psi)+1} A_{\psi,\eta^{-1}}(f).$$

For a compact open set $\Omega$ of $X'$ let $\phi_{\psi,\eta,\Omega}$ be the function which is equal to $\phi_{\psi,\eta}$ on $\Omega$ and 0 off of $\Omega$. We let $\mathscr{K}$ be the set of all compact subsets of $X'$ which are finite unions of the $X'_n$. These are the sets of most importance to us. Any compact subset of $X_{\mathrm{ns}}$ is contained in an element of $\mathscr{K}$. We call a subset of $X_{\mathrm{ns}}$ *bounded* if it is contained in some compact set.

**Proposition.** *Let $f$ belong to $\mathscr{S}^\circ(X')$, let $\psi$ be a non-trivial additive character of $F$ and let $\eta$ be a character of $F^\times$. Then there is a bounded subset $\Omega_0$ of $X'_{\mathrm{ns}}$ such that for any $\Omega \in \mathscr{K}$ containing $\Omega_0$ we have*

$$\phi_{\psi,\eta,\Omega} * f = -q^{2m+1} A_{\psi,\eta^{-1}}(f)$$

*In fact, one may take $\Omega_0$ to be the union of the sets*

$$\mathrm{supp}((T_n f)^\vee) \cdot \mathrm{supp}(A_{\psi,\eta^{-1}}(T_n f)).$$

*Here the product is taken by regarding each factor as a subset of $G'$ via $(\pi')^{-1}$.*

*Proof.* As $f = \sum_{n\in\mathbb{Z}} T_n f$ it suffices to prove the proposition for $T_n f$. In other words, we may assume that $f$ is supported on $X'_n$. Of course, then $f^\vee$ is supported on $X'_{-n}$. Let $\Omega_0 = \varpi_{B'}^{-n} \cdot \mathrm{supp}(A_{\psi,\eta^{-1}} f)$ and let $\Omega \in \mathscr{K}$ contain $\Omega_0$. We then have

$$\begin{aligned}(\phi_{\psi,\eta,\Omega} * f)(x) &= \int_{B'} f^\vee(y)\psi_{B'}(xy)\eta(xy)\chi_\Omega(xy)d^\times y \\ &= \chi_\Omega(\varpi_{B'}^{-n} x) \int_{B'} f^\vee(y)\psi_{B'}(xy)\eta(xy)d^\times y \\ &= -q^{2m+1}\chi_\Omega(\varpi_{B'}^{-n} x)(A_{\psi,\eta^{-1}} f)(x).\end{aligned}$$

If $x$ belongs to $\mathrm{supp}(A_{\psi,\eta^{-1}} f)$ then $\varpi_{B'}^{-n} x$ belongs to $\Omega$ and so $-q^{2m+1}(A_{\psi,\eta} f)(x) = (\phi_{\psi,\eta,\Omega} * f)(x)$.

If $x$ does not belong to the support of $A_{\psi,\eta^{-1}}f$ then both $(\psi_{\psi,\eta,\Omega} * f)(x)$ and $(A_{\psi,\eta^{-1}}f)(x)$ are zero. Thus $-q^{2m+1}(A_{\psi,\eta^{-1}}f)(x) = (\phi_{\psi,\eta,\Omega} * f)(x)$ for all $x$, as was to be shown. □

**(5.3.8)** We now strengthen the previous proposition slightly. We write $F^\vee$ for the set of additive characters on $F^\vee$. It is a topological group and isomorphic to $F$. Recall that the *conductor* of a character $\eta$ of $F^\times$ is defined to be the minimal integer $n$ such that $\chi(U_F^{(n)}) = 1$. If $\chi(U_F) = 1$ then the conductor is defined to be 0.

**Proposition.** *Let $f$ belong to $\mathscr{S}^\circ(X')$, let $S$ be a compact subset of $F^\vee \setminus \{0\}$ and let $S'$ be a set of characters of $F^\times$ of bounded conductor. Then there is a bounded subset $\Omega_0$ of $X'_{\rm ns}$ such that for any $\Omega \in \mathscr{K}$ containing $\Omega_0$ we have*

$$\phi_{\psi,\eta,\Omega} * f = -q^{2m+1} A_{\psi,\eta^{-1}}(f)$$

*for all $\psi \in S$ and all $\eta \in S'$.*

*Proof.* Again, it suffices to treat the case where $f$ is supported on a single coset $\varpi^n U_B$ of $U_B$. Let $\Omega_0$ be the union of the sets $\varpi_B^{-n} \cdot \mathrm{supp}(A_{\psi,\eta^{-1}}f)$ as $\psi$ varies in $S$ and $\nu$ varies in $S'$. So long as this set is bounded, the previous proposition implies the present one. We now show that it is bounded. To begin with, note $\mathrm{supp}(A_{\psi,\eta^{-1}}f) = \mathrm{supp}(\mathscr{F}_{X',\psi}(\eta^{-1}f^\vee))$. If we change $\psi$ to $\psi(ax)$ for some $a \in F^\times$ then the support changes by $a^{-1}$. Thus if we fix a non-trivial character $\psi_0$ of $F$ so that $S$ corresponds to some compact subset $S_1$ of $F^\times$ then

$$\bigcup_{\psi \in S} \mathrm{supp}(A_{\psi,\eta^{-1}}f) \subset S_1 \cdot \mathrm{supp}(A_{\psi_0,\eta^{-1}}f)$$

Now, if we twist $\eta$ be an unramified character then $\eta f$ is just scaled by a constant since $f$ is supported on a single coset of $U_B$. In particular, $\mathrm{supp}(A_{\psi,\eta^{-1}}f)$ is unchanged. Let $S_1'$ be a subset of $S'$ so that every element of $S'$ is an unramified twist of a unique element of $S_1'$. As there are only finitely many characters of a bounded conductor modulo unramified twists, the set $S_1'$ is finite. We now have

$$\Omega_0 = \bigcup_{\psi \in S} \bigcup_{\eta \in S'} \mathrm{supp}(A_{\psi,\eta^{-1}}f) \subset S_1 \cdot \bigcup_{\eta \in S_1'} \mathrm{supp}(A_{\psi_0,\eta^{-1}}f)$$

and so $\Omega_0$ is bounded. □

**(5.3.9)** Let $\mathscr{S}^\dagger(X')$ denote the space of functions spanned by functions of the form $(t,\nu) \mapsto f(t)g(\nu)$ where $f \in \mathscr{S}(F)$ has total integral zero and $g \in \mathscr{S}(F^\times)$. We have $\mathscr{S}^\dagger(X') \subset \mathscr{S}_{\rm ns}(X')$.

**Proposition.** *Let $\phi$ belong to $\mathscr{S}^\dagger(X')$. There then exists a compact subset $S$ of $F^\vee \setminus \{0\}$ and a set $S'$ of characters of $F^\times$ of bounded conductor such that for any bounded set $\Omega_0$ we can find an expression*

$$\phi = \sum_{i=1}^{n} a_i \phi_{\psi_i,\eta_i,\Omega_i}$$

*with $\psi_i \in S$, $\eta_i \in S'$ and $\Omega_i$ an element of $\mathscr{K}$ containing $\Omega_0$.*

We need two lemmas before proving this. We leave these to the reader.

**Lemma.** *Let $f \in \mathscr{S}(F)$. There exists a compact set $S$ of $F^\vee$ such that for any compact subset $A$ of $F$ containing* $\mathrm{supp}(f)$ *we have an expression*

$$f(x) = \chi_A(x) \sum_{\psi \in S} a_\psi \psi(x)$$

*where $a_\psi = 0$ for all but finitely many $\psi$. If $f$ has total integral zero then $S$ can be taken to be a compact subset of $F^\vee \setminus 0$.*

**Lemma.** *Let $g \in \mathscr{S}(F^\times)$. There exists a set $S'$ of characters of $F^\times$ of bounded conductor such that for any compact subset $A'$ of $F^\times$ containing* $\mathrm{supp}(g)$ *we have an expression*

$$g(x) = \chi_{A'}(x) \sum_{\eta \in S'} b_\eta \eta(x)$$

*where $b_\eta = 0$ for all but finitely many $\eta$.*

We now prove the proposition.

*Proof of proposition.* Let $f$ and $g$ be given. Let $S$ and $S'$ be the sets furnished by the previous lemmas. We now show that these sets satisfy the statement of the proposition. Thus let $\Omega_0$ be a given bounded set. Let $\Omega$ be any element of $\mathscr{K}$ containing $\Omega_0$ and the support of the function $(t,\nu) \mapsto f(t)g(\nu)$. Let $A$ be a compact subset of $F$ containing $t(\Omega)$ and let $A'$ be a compact subset of $F^\times$ containing $\nu(\Omega)$. Note that these conditions imply

$$\chi_A(t(x))\chi_\Omega(x) = \chi_\Omega(x), \qquad \chi_{A'}(\nu(x))\chi_\Omega(x) = \chi_\Omega(x).$$

By the lemmas, we have expressions

$$f(x) = \chi_A(x) \sum_{\psi \in S} a_\psi \psi(x), \qquad g(x) = \chi_{A'}(x) \sum_{\eta \in S'} b_\eta \eta(x).$$

We thus have

$$\begin{aligned} f(t(x))g(\nu(x)) &= \chi_\Omega(x) f(t(x))g(\nu(x)) \\ &= \chi_\Omega(x)\chi_A(t(x))\chi_{A'}(\nu(x)) \sum a_\psi b_\eta \psi(t(x))\eta(\nu(x)) \\ &= \sum a_\psi b_\nu \phi_{\psi,\eta,\Omega}(x) \end{aligned}$$

which proves the proposition. □

**(5.3.10)** For a character $\eta$ of $F^\times$ and a compact subset $\Omega$ of $X'_{\mathrm{ns}}$ we let $\phi_{1,\eta,\Omega}$ be the function $x \mapsto \chi_\Omega(x)\eta(\nu(x))$. This is just the function $\phi_{\psi,\eta,\Omega}$ with $\psi$ taken to be the trivial character.

**Proposition.** *The space $\mathscr{S}_{\mathrm{ns}}(X')$ is spanned by $\mathscr{S}^\dagger(X')$ and the $\psi_{1,\eta,\Omega}$ for $\Omega \in \mathscr{K}$.*

*Proof.* It is easy to see that $\mathscr{S}_{\mathrm{ns}}(X')$ is spanned by functions of the form $f(t)g(\nu)$ with $f \in \mathscr{S}(F)$ and $g \in \mathscr{S}(F^\times)$. Let $f$ and $g$ be given. Let $\Omega \in \mathscr{K}$ contain the support of $f(t)g(\nu)$. Let $A'$ be a compact open subset of $F^\times$ containing $\nu(\Omega)$ and write

$$g(x) = \chi_{A'}(x) \sum b_\eta \eta(x)$$

Let $A$ be a non-empty compact open subset of $F$ containing $t(\Omega)$ and let $\alpha \in \mathbb{C}$ be such that $f' = f - \alpha\chi_A$ has total integral zero. We then have

$$f(t)g(\nu) = f'(t)g(\nu) + \alpha\chi_A(t)\chi_{A'}(\nu) \sum b_\eta \eta(\nu)$$

Multiplying each side by $\chi_\Omega$ we find

$$f(t)g(\nu) = f'(t)g(\nu) + \alpha \sum b_\eta \phi_{1,\eta,\Omega}$$

which proves the proposition. □

**(5.3.11)** We can now prove the following important proposition.

**Proposition.** *A subspace $V \subset \mathscr{S}^\circ(X')$ is stable under $\mathscr{A}$ if and only if it is stable under convolution by $\mathscr{S}_{\mathrm{ns}}(X')$.*

*Proof.* Let $V$ be stable under $\mathscr{A}$. As $\mathscr{S}_{\mathrm{ns}}(X')$ is spanned by $\mathscr{S}^\dagger(X')$ and functions of the form $\phi_{1,\eta,\Omega}$

it suffices to show $\mathscr{S}'(X) * V \subset V$ and $\phi_{1,\eta,\Omega} * V \subset V$. Let $f \in V$. We have

$$(\phi_{1,\eta,\Omega} * f)(x) = \int_{G'} f^\vee(y)\eta(xy)\chi_\Omega(xy)d\mu_{G'}(y)$$

For $x$ fixed, the function $y \mapsto \eta(xy)\chi_\Omega(xy)$ factors through the norm (assuming $\Omega \in \mathscr{K}$, as we can). Thus $\phi_{1,\eta,\Omega} * f = 0$. Now let $\phi$ be an element of $\mathscr{S}^\dagger(X')$. Let $S$ and $S'$ be the sets produced by §5.3.9 applied to $\phi$. Let $\Omega_0$ be the set produced by §5.3.8 to $f$, $S$ and $S'$. The conclusion of §5.3.9 gives

$$\phi = \sum_{i=1}^n a_i \phi_{\psi_i,\eta_i,\Omega_i}$$

with $\psi_i \in S$, $\eta_i \in S'$ and $\Omega_i \in \mathscr{K}$ containing $\Omega_0$. The conclusion of §5.3.8 gives $\phi_{\psi_i,\eta_i,\Omega_i} * f = -q^{2m+1}A_{\psi_i,\eta_i^{-1}}(f)$. We thus see that $\phi * f$ belongs to $V$ since $V$ is stable by the $A_{\psi_i,\eta_i}$.

Conversely, say that $V$ is stable under convolution by $\mathscr{S}_{\text{ns}}(X')$. We must show that it is stable by the $A_{\psi,\eta}$. Let $f$ be an element of $V$. By §5.3.7 we have $A_{\psi,\eta}(f) = -q^{-2m-1}(\phi_{\psi,\eta^{-1},\Omega} * f)$ for some choice of $\Omega$. This proves the proposition. □

**(5.3.12)** Let $\pi$ be an element of $\text{Irr}_{G'}$. We let $\chi_\pi : X'_{\text{ns}} \to \mathbb{C}$ be the character of $\pi$ and write $\phi_{\pi,k}$ for the "truncated character," defined to be $\chi_\pi$ on $X_k$ and 0 off of this set. In our notation, $\phi_{\pi,k} = T_k\chi_\pi$. We write $V_{\pi,k}$ for the one dimensional space spanned by $\phi_{\pi,k}$. We let $\widetilde{V}_\pi$ be the space spanned by all of the $\phi_{\pi,k}$ and $V_\pi$ be its closure in $L^2(X')$. Of course, these spaces do not change if $\pi$ is replaced by an unramified twist, and so make sense for $\pi \in \overline{\text{Irr}}_{G'}$. As before, we call $\pi$ *even* if $\xi \otimes \pi = \pi$ and *odd* otherwise. If $\pi$ is even then $\phi_{\pi,k}$ and $V_{\pi,k}$ vanish for $k$ odd. The following is a simple extension of the Peter-Weyl theorem.

**Proposition.** *The $\phi_{\pi,k}$ span $\mathscr{S}_{\text{ns}}(X')$.*

*Proof.* It suffices to show that a conjugation invariant function supported on $G'_k$ lies in the span of the $\phi_{\pi,k}$. Thus let $f$ be such a function. We can then find an non-compact open subgroup $U$ of the center $Z'$ and a function $f'$ on $G'/U$ such that the pull-back of $f'$ to $G'$ agrees with $f$ on $G'_k$. As the group $G'/U$ is compact, we have $f' = \sum a_i\chi_{\pi_i}$ where the $\pi_i$ are irreducible representations of $G'/U$. Of course, the $\pi_i$ can also be regarded as irreducible representations of $G'$. As such, we clearly have $f = \sum a_i\chi_{\pi_i,k}$, which establishes the proposition. □

**(5.3.13)** We need the following result.

**Proposition.** *For $\pi \in \mathrm{Irr}_{G'}$ and $f \in \mathscr{S}_{\mathrm{ns}}(X')$ we have*

$$(f * \phi_{\pi,k})(x) = d_\pi^{-1}\chi_\pi(x)\int_{x\varpi_{B'}^{-k}U_{B'}} f(y)\chi_\pi^\vee(y)d\mu_{G'}(y)$$

*for all $x \in X'_{\mathrm{ns}}$. Here $d_\pi$ is the degree of $\pi$.*

*Proof.* For $x \in G'$ put

$$A(x) = \int_{G'} f(y)\chi_k(xy^{-1})\pi(xy^{-1})d\mu_{G'}(y).$$

Thus $(f * \phi_{\pi,k})(x) = \operatorname{tr} A(x)$. Now, we have

$$A(x) = \pi(x)\int_{x\varpi_{B'}^{-k}U_{B'}} f(y)\pi(y^{-1})d\mu_{G'}(y) = \pi(x)B(x).$$

Since $f$ is invariant under conjugation and $\pi$ is a homomorphism, it follows that $\pi(g)B(x)\pi(g^{-1}) = B(x)$ for any $x \in G'$. Thus, by Schur's lemma, $B(x)$ is a constant. Thus $B(x) = d_\pi^{-1}\operatorname{tr} B(x)$ and so

$$B(x) = \frac{1}{d_\pi}\int_{x\varpi_{B'}^{-k}U_{B'}} f(y)\chi_\pi^\vee(y)d\mu_{G'}(y).$$

Taking the trace of the expression $A(x) = \pi(x)B(x)$ now gives the required formula. □

**(5.3.14)** We now determine the structure of $\mathscr{S}_{\mathrm{ns}}(X')$ as an algebra under convolution.

**Proposition.** *We have the following:*

1. *We have $\widetilde{V}_\pi * \widetilde{V}_{\pi'} = 0$ if $\pi$ is not an unramified twist of $\pi'$.*
2. *If $\pi$ is odd then $\phi_{\pi,i} * \phi_{\pi,j} = \mathrm{Vol}(U_{B'})d_\pi^{-1}\phi_{\pi,i+j}$.*
3. *If $\pi$ is even then $\phi_{\pi,i} * \phi_{\pi,j} = 2\,\mathrm{Vol}(U_{B'})d_\pi^{-1}\phi_{\pi,i+j}$ for $i$ and $j$ even.*
4. *The $\widetilde{V}_\pi$ are precisely the minimal $\mathscr{T}$-stable ideals of $\mathscr{S}_{\mathrm{ns}}(X')$.*

*Proof.* The previous proposition shows that for any $f \in \mathscr{S}_{\mathrm{ns}}(X')$ the function $f * \phi_{\pi,k}$ belongs to $\widetilde{V}_\pi$. If $\pi$ and $\pi'$ are unequal elements of $\overline{\mathrm{Irr}}_{G'}$ then $\widetilde{V}_\pi \cap \widetilde{V}_{\pi'} = 0$ and so we find $\widetilde{V}_\pi * \widetilde{V}_{\pi'} = 0$. This gives (1). Now let $\pi$ be an element of $\mathrm{Irr}_{G'}$. The previous proposition gives

$$(\phi_{\pi,i} * \phi_{\pi,j})(x) = d_\pi^{-1}\chi_\pi(x)\int_{x\varpi_{B'}^{-j}U_{B'}} \phi_{\pi,i}(y)\chi_\pi^\vee(y)d\mu_{G'}(y)$$

If $x$ does not belong to $\varpi_{B'}^{i+j}U_{B'}$ then this is zero. For $x \in \varpi_{B'}^{i+j}U_{B'}$ we get

$$(\phi_{\pi,i} * \phi_{\pi,j})(x) = d_\pi^{-1}\chi_\pi(x) \int_{\varpi_{B'}^i U_{B'}} \chi_\pi(y)\chi_\pi^\vee(y) d\mu_{G'}(y)$$

If $\pi$ is odd then $\pi \otimes \pi^\vee$ contains the trivial representation once and does not contain $\xi$; the above integral is equal to $\mathrm{Vol}(U_{B'})$. If $\pi$ is even then $\pi \otimes \pi^\vee$ contains each of the trivial representation and $\xi$ exactly once; thus, if $i$ and $j$ are even, then the above integral is equal to $2\,\mathrm{Vol}(U_{B'})$. This gives (2) and (3). As for (4), the above shows that the $\widetilde{V}_\pi$ are minimal $\mathscr{T}$-stable ideals. Now let $I$ be some minimal $\mathscr{T}$-stable ideal and let $f \in I$ be some non-zero element. We can then write $f = \sum a_{\pi,k}\phi_{\pi,k}$ where almost all the $a_{\pi,k}$ vanish. Say $a_{\pi_0,k_0}$ is non-zero. By $\mathscr{T}$-stability the element $\sum_\pi a_{\pi,k_0}\phi_{\pi,k_0}$ belongs to $I$. Convolving with $\phi_{\pi_0,0}$, we find that $\phi_{\pi_0,k_0}$ belongs to $I$. It thus follows that $I$ contains $\widetilde{V}_{\pi_0}$ and therefore by minimality is equal to it. □

**(5.3.15)** We now determine the structure of $H(X')$ as an $(\mathscr{A} * \mathscr{T})$-module (compare with §4.4.12).

**Proposition.** *We have the following:*

1. *Each $V_\pi$ is stable under $(\mathscr{A} * \mathscr{T})$ and simple as an $(\mathscr{A} * \mathscr{T})$-module.*
2. *If $\pi$ and $\pi'$ are distinct elements of $\overline{\mathrm{Irr}}_{G'}^\circ$ then $V_\pi$ and $V_{\pi'}$ are not isomorphic as $(\mathscr{A} * \mathscr{T})$-modules.*
3. *We have $H(X') = \bigoplus V_\pi$, the sum taken over $\pi \in \overline{\mathrm{Irr}}_{G'}^\circ$.*

*In particular, $H(X')$ is semi-simple and multiplicity-free as an $(\mathscr{A} * \mathscr{T})$-module.*

*Proof.* It is enough to prove the analogous statements for $\mathscr{S}^\circ(X')$ in place of $H(X')$. The results of §5.3.14 show that $\mathscr{S}^\circ(X')$ is semi-simple and multiplicity free as an $(\mathscr{S}_{\mathrm{ns}}(X') * \mathscr{T})$-module, its simple constituents being the $\widetilde{V}_\pi$. (To see that $\widetilde{V}_\pi$ and $\widetilde{V}_{\pi'}$ are not isomorphic look at their annihilators in $\mathscr{S}_{\mathrm{ns}}(X)$.) The result of §5.3.11 implies that a subspace of $\mathscr{S}^\circ(X')$ is a simple $(\mathscr{A} * \mathscr{T})$-submodule if and only if it is a simple $(\mathscr{S}_{\mathrm{ns}}(X') * \mathscr{T})$-submodule. Now, a module over a ring is semi-simple and multiplicity free if and only if it is the direct sum of its simple submodules. We thus see that $\mathscr{S}^\circ(X')$ is semi-simple and multiplicity free as an $(\mathscr{A} * \mathscr{T})$-module, since its simple $(\mathscr{A} * \mathscr{T})$-submodules are the same as its simple $(\mathscr{S}_{\mathrm{ns}}(X') * \mathscr{T})$-submodules and we know it to be the direct sum of its simple $(\mathscr{S}_{\mathrm{ns}}(X') * \mathscr{T})$-submodules. □

**(5.3.16)** We now compute the Fourier transform in the spanning set $\phi_{\pi,k}$.

**Proposition.** *For* $\pi \in \mathrm{Irr}^\circ$ *we have*

$$\mathscr{F}_{X',\psi}(\phi_{\pi,k}) = -\epsilon(\tfrac{3}{2},\pi,\psi)\phi_{|\cdot|_F^{-2}\pi^\vee,-k-n(\pi)-2m(\psi)}$$

*where*

$$\epsilon(\tfrac{3}{2},\pi,\psi) = -q^{-2m(\psi)-1}d_\pi^{-1}\langle\phi_{\pi,-n(\pi)-2m(\psi)},\psi_{X'}\rangle_{X'}.$$

*Here* $\psi_{X'} = \psi_F \circ \mathrm{tr}$.

*Proof.* Let $n$ be the conductor of $\pi$ and put $n' = n-1$, so that $\pi|_{U_{B'}^{(n')}}$ is trivial but $\pi|_{U_{B'}^{(n'-1)}}$ is non-trivial. Let $\Phi_k$ be the function on $G'$ which is equal to $\pi(x)$ on $G'_n$ and 0 off of this set. Note that $\mathrm{tr}\,\Phi_k = \chi_{\pi,k}$. Put $m = m(\psi)$. We have

$$\begin{aligned}(\mathscr{F}_{B',\psi}\Phi_k)(x) &= q^{-2m-1}\int_{\varpi_{B'}^k U_{B'}} \pi(y)\psi_{B'}(xy)dy \\ &= q^{-2m-2k-1}\sum_{a\in\varpi_{B'}^k U_{B'}/U_{B'}^{(n')}} \pi(a)\int_{U_{B'}^{(n')}}\psi_{B'}(axy)dy \\ &= q^{-2m-2k-1}\sum_{a\in\varpi_{B'}^k U_{B'}/U_{B'}^{(n')}} \pi(a)\psi_{B'}(ax)\int_{\varpi_{B'}^{n'}\mathscr{O}_{B'}}\psi_{B'}(axy)dy \\ &= \delta q^{-2m-2k-2n'-1}\pi(\varpi_{B'}^k)\sum_{a\in U_{B'}/U_{B'}^{(n')}}\pi(a)\psi_{B'}(\varpi_{B'}^k ax)\end{aligned}$$

Here $\delta$ is 1 if $\mathrm{val}\,x \geq -2m-1-n'-k$ and 0 otherwise. Now, say that $\mathrm{val}\,x \geq -2m-n'-k$. The sum on the last line is then equal to

$$\sum_{a\in U_{B'}/U_{B'}^{(n')}}\pi(a)\psi_{B'}(\varpi_{B'}^k ax) = \sum_{a\in U_{B'}/U_{B'}^{(n'-1)}}\ \sum_{b\in U_{B'}^{(n'-1)}/U_{B'}^{(n')}}\pi(ab)\psi_{B'}(\varpi_{B'}^k abx).$$

The quantity $\psi_{B'}(\varpi_{B'}^k abx)$ is in fact independent of $b$, as we can write $b = 1+\varpi_{B'}^{n'-1}u$ and then

$$\psi_{B'}(\varpi_{B'}^k abx) = \psi_{B'}(\varpi_{B'}^k ax)\psi_{B'}(\varpi_{B'}^{k+n'-1}aux) = \psi_{B'}(\varpi_{B'}^k ax)$$

as $\varpi_{B'}^{k+n'-1}aux$ has valuation $k+n'-1+\mathrm{val}\,x \geq -2m-1$. The sum is thus equal to

$$\sum_{a\in U_{B'}/U_{B'}^{(n'-1)}}\pi(a)\psi_{B'}(\varpi_{B'}^k ax)\sum_{b\in U_{B'}^{(n'-1)}/U_{B'}^{(n')}}\pi(b) = 0$$

since $\pi$ is a non-trivial representation of $U_{B'}^{n'-1}$. We have thus shown that $\mathscr{F}_{B',\psi}(\Phi_k)$ vanishes off of the locus in $G'$ where the valuation is equal to $-2m-1-n'-k$ (which is equal to $-2m-n-k$).

Now let $x$ have valuation $-2m-n-k$. Then

$$\begin{aligned}(\mathscr{F}_{B',\psi}\Phi_k)(x) &= q^{-2m-1}\int_{\varpi_{B'}^k U_{B'}} \pi(y)\psi_{B'}(xy)dy \\ &= q^{-2m-1}|\,\mathbf{N}\,x|_F^{-2}\int_{\varpi_{B'}^{-2m-n}U_{B'}} \pi(x^{-1}y)\psi_{B'}(y)dy \\ &= B\cdot(|\cdot|_F^{-2}\pi^\vee)(x)\end{aligned}$$

where

$$B = q^{-2m-1}\int_{\varpi_{B'}^{-2m-n}} \pi(y)\psi_{B'}(y)dy.$$

As we have argued before, $B$ is a scalar. Thus

$$B = d_\pi^{-1}\operatorname{tr} B = q^{-2m-1}d_\pi^{-1}\int_{\varpi_B^{-2m-n}U_B} \chi_\pi(y)\psi_B(y)dy$$

Taking traces of our expression for $\mathscr{F}_{B',\psi}(\Phi_k)$ gives

$$\mathscr{F}_{B',\psi}(\phi_{\pi,k}) = B\cdot\phi_{|\cdot|_F^{-2}\pi^\vee,-2m-n-k},$$

which proves the proposition. (That $\epsilon(\frac{3}{2},\pi,\psi) = -B$ can be taken as a definition of the left side; in fact, this agrees with the usual $\epsilon$-factor.) □

**(5.3.17)** We now compute the operators $A_{\psi,\eta}$ explicitly.

**Proposition.** *For $\pi \in \mathrm{Irr}^\circ$ we have*

$$A_{\psi,\eta}(\phi_{\pi,k}) = \lambda_{\psi,\eta}(\pi)\phi_{\pi,k-n(\eta\pi)-2m(\psi)}$$

*where $\lambda_{\psi,\eta}(\pi) = \epsilon(\frac{3}{2},\eta\pi,\psi)$.*

*Proof.* This is a simple computation using §5.3.16. □

## 5.4 Detecting characters

**(5.4.1)** The results of §5.3 show that the $\mathscr{A}$ and $\mathscr{T}$ structures on $\mathscr{S}^\circ(X')$ are enough to recover the one dimensional spaces $V_{\pi,k}$ spanned by the truncated character functions $\phi_{\pi,k}$. The question we now turn to is: how can one recover the full characters using the $\mathscr{A}$ and $\mathscr{T}$ structures? Concretely, if $\chi$ is a function on $X'$ for which $T_k\chi$ belongs to $V_{\pi,k}$ for each $k$ then how does one determine if $\chi$ is equal to $\chi_\pi$, or perhaps an unramified twist of this? We address these issues in this section. Here is an overview:

- In §5.4.2 we introduce the space of "cuspidal distributions." This space has two key properties: it admits a map from $\mathscr{C}^\infty(X')$ and has an action of $\mathscr{A}$.
- In §5.4.3 we show that the cuspidal distributions coming from cuspidal characters are precisely the eigenvectors of $\mathscr{A}$ acting on the space of cuspidal distributions.
- In §5.4.4 and §5.4.5 we give an alternative approach towards the problem of determining which functions are characters. It is not used in what follows, but helped shape our way of thinking towards the problem, so we decided to include it.
- In §5.4.6 we discuss how the results of this section, and in particular §5.4.5, relate to the local functional equation and local converse theorem.

**(5.4.2)** By a *cuspidal distribution* on $X'$ we mean a linear map $\mathscr{S}^\circ(X') \to \mathbb{C}$. We write $\mathscr{D}^\circ(X')$ for the space of cuspidal distributions. For $f \in \mathscr{D}^\circ(X')$ and $g \in \mathscr{S}^\circ(X')$ we write $\langle f, g\rangle_{X'}$ for the value of $f$ on $g$. We have a linear map $\mathscr{C}^\infty(X') \to \mathscr{D}^\circ(X')$ defined by mapping $f \in \mathscr{C}^\infty(X')$ to the cuspidal distribution defined by

$$\langle f, g\rangle_{X'} = \int_{X'} f(x)g(x)d\mu_{X'}(x).$$

All the operations on cuspidal functions we have considered (*i.e.*, the Fourier transform, the involution $f \mapsto f^\vee$, the operators $A_{\psi,\eta}$, *etc.*) extend to operations on cuspidal distributions, as they all have nice adjointness properties. In particular, the space of cuspidal distributions is a module over $\mathscr{A}$.

**(5.4.3)** We can now finally give a complete characterization of irreducible characters. Note that the only non-trivial structure needed on $X'$ for this characterization is the Fourier transform $\mathscr{F}_{X',\psi}$.

**Proposition.** *Let $\chi$ belong to $\mathscr{C}^\infty(X'_{\mathrm{ns}})$. Then $\chi$ is of the form $\alpha\chi_\pi$ with $\pi \in \mathrm{Irr}^\circ_{G'}$ and $\alpha \in \mathbb{C}$ if and only if it is orthogonal to $\mathscr{S}^1_{\mathrm{ns}}(X')$ and its image in $\mathscr{D}^\circ(X')$ is an eigenvector for $\mathscr{A}$.*

*Proof.* Let $\pi \in \mathrm{Irr}^\circ_{G'}$. It is clear that $\chi_\pi$ is orthogonal to $\mathscr{S}^1_{\mathrm{ns}}(X')$. We now show that $\chi_\pi$, regarded as a cuspidal distribution, is an eigenvector of $\mathscr{A}$. Write $A'_{\psi,\eta}$ for the adjoint of $A_{\psi,\eta}$ Let $\phi$ be a test function in $\mathscr{S}^\circ(X')$ and let $S$ be a finite set of integers such that $\phi$ has its support contained in $\bigcup_{k\in S} X'_{k-2m(\psi)-n(\eta\pi^*)}$ and $A'_{\psi,\eta}\phi$ has its support contained in $\bigcup_{n\in S} X'_n$. We then have

$$\begin{aligned}\langle A_{\psi,\eta}\chi_\pi, \phi\rangle_{X'} = \langle \chi_\pi, A'_{\psi,\eta}\phi\rangle_{X'} &= \sum_{k\in S}\langle \phi_{\pi,k}, A'_{\psi,\eta}\phi\rangle_{X'} = \sum_{k\in S}\langle A_{\psi,\eta}\phi_{\pi,k}, \phi\rangle_{X'} \\ &= \lambda_{\psi,\eta}(\pi)\sum_{k\in S}\langle \phi_{\pi,k-2m(\psi)-n(\eta\pi^*)}, \phi\rangle_{X'} = \lambda_{\psi,\eta}(\pi)\langle\chi_\pi, \phi\rangle_{X'}\end{aligned}$$

and so $A_{\psi,\eta}\chi_\pi = \lambda_{\psi,\eta}(\pi)\chi_\pi$ holds in $\mathscr{D}^\circ(X')$.

Conversely, let $\chi$ be given satisfying the conditions. Let $S$ be a section of $\mathrm{Irr}^\circ_{G'} \to \overline{\mathrm{Irr}}^\circ_{G'}$. We can then write $\chi = \sum a_{\pi,k}\phi_{\pi,k}$ with $\pi$ varying over $S$ and $k$ varying over $\mathbb{Z}$, in a unique manner (subject to the convention that $a_{\pi,k} = 0$ for $\pi$ even and $k$ odd). Now, we have

$$c_{\psi,\eta}\chi = A_{\psi,\eta}\chi = \sum a_{\pi,k}\lambda_{\psi,\eta}(\pi)\phi_{\pi,k-n(\eta\pi)-2m(\psi)}$$

for some scalar $c_{\psi,\eta}$, from which we conclude

$$c_{\psi,\eta}a_{\pi,k} = \lambda_{\psi,\eta}(\pi)a_{\pi,k+n(\eta\pi)+2m(\psi)}.$$

From this we conclude $a_{\pi,k} = \alpha_\pi\beta_\pi^k$. Replacing $\pi$ with an unramified twist, we can assume $\beta_\pi = 1$. We thus find $c_{\psi,\eta} = \lambda_{\psi,\eta}(\pi)$ whenever $\alpha_\pi$ is non-zero. The local converse theorem thus implies that $\alpha_\pi$ is non-zero for at most one $\pi$, which proves the proposition. □

*Remark.* It is possible to take the above proposition further and give conditions that constrain the scalar $\alpha$. For example, $\chi$ is of the form $\pm\chi_\pi$ if and only if the above two conditions are satisfied and additionally

$$\int_{X'_1}(\chi\cdot\chi^\vee)d\mu_X = \mathrm{Vol}(U_{B'})\times\begin{cases}2 & \text{if } \chi \text{ is even}\\ 1 & \text{if } \chi \text{ is odd}\end{cases}$$

where here we say that $\chi$ is *even* if it vanishes on the $X'_n$ with $n$ odd and that $\chi$ is *odd* otherwise. One can even go further than this and nail the sign down: $\chi$ is of the form $\chi_\pi$ if and only if the two conditions of the proposition hold, the above condition holds and furthermore for each quadratic extension $K/F$ the restriction of $\chi$ to $K^\times \subset B'$ is a non-negative integral combination of characters of $K^\times$.

**(5.4.4)** We now give an alternative approach towards the problem of determining which functions are characters. Let $\mathscr{C}^{\mathrm{char}}(X')$ be the subspace of $\mathscr{C}^\infty(X'_{\mathrm{ns}})$ consisting of those functions $f$ such that $p_K^*(f) \in \mathscr{C}^\infty(K^\times)$ is a linear combination of characters of $K^\times$ for all quadratic extensions $K/F$.

This definition has the advantage of being intrinsic to $X'$ in that it does not make reference of $B'$ in any way. Nonetheless, we have:

**Proposition.** *The space* $\mathscr{C}^{\text{char}}(X)$ *has for a basis* $\{\chi_\pi\}$ *with* $\pi \in \text{Irr}_{G'}$.

We need a lemma before giving the proof.

**Lemma.** *Let* $K/F$ *be a quadratic extension and let* $\eta : K^\times \to \mathbb{C}^\times$ *be a character. Let* $f : \mathbb{Z} \to \mathbb{C}$ *by the function given by*

$$f(n) = \int_{\varpi_K^n U_K} \eta(x)|x - \overline{x}|_F^2 dx.$$

*Then* $f$ *is a finite linear combination of functions of the form* $n \mapsto a^n$.

*Proof.* First consider the case where $K/F$. We can then take $\varpi_K = \varpi_F$ and so

$$f(n) = q^{-2n}\eta(\varpi_F)^n \int_{U_K} \eta(x)|x - \overline{x}|_F^2 dx.$$

Therefore $f(n) = ab^n$ where $a$ is the integral and $b = q^{-2}\eta(\varpi_F)$.

Now consider the case where $K/F$ is ramified. The idea is basically the same but slightly more complicated. As $\varpi_K^{2n} U_K = \varpi_F^n U_K$ we find

$$f(2n) = q^{-2n}\eta(\varpi_F)^n \int_{U_K} \eta(x)|x - \overline{x}|_F^2 dx$$

and so $f(2n) = ab^n$ for some $a$ and $b$. Similarly, since $\varpi_K^{2n+1} U_K = \varpi_F^n \varpi_K U_K$ we find

$$f(2n+1) = q^{-2n}\eta(\varpi_F)^n \int_{\varpi_K U_K} \eta(x)|x - \overline{x}|_F^2 dx$$

and so $f(2n+1) = cd^n$ for some $c$ and $d$. We now have

$$f(n) = ab^{n/2}\left(\frac{1 + (-1)^n}{2}\right) + cd^{(n-1)/2}\left(\frac{1 - (-1)^n}{2}\right)$$

which proves the lemma. (The above formula is a special case of the fact that if a function $f : \mathbb{Z} \to \mathbb{C}$ is a linear combination of exponentials on each coset of $m\mathbb{Z}$, for some integer $m$, then so is $f$.) $\square$

We can now prove the proposition.

*Proof of proposition.* Let $\pi$ belong to $\text{Irr}_{G'}$. Then $\chi_\pi|_{K^\times}$ is the character of the semi-simple representation $\pi|_{K^\times}$, for any embedding $K \to B'$. It follows that $\chi_\pi|_{K^\times}$ is a linear combination of characters of $K^\times$. Thus $\chi_\pi$ belongs to $\mathscr{C}^{\text{char}}(X')$.

We now show that the $\chi_\pi$ span $\mathscr{C}^{\mathrm{char}}(X')$. (We take it as a standard fact that the $\chi_\pi$ are linearly independent.) Thus let $f$ belong to $\mathscr{C}^{\mathrm{char}}(X')$. For each integer $k$ we have a unique expression

$$f|_{X'_k} = \sum_{\pi \in \overline{\mathrm{Irr}}} a_{\pi,k} \chi_\pi|_{X'_k}.$$

(Here and in what follows, we fix a section of $\mathrm{Irr} \to \overline{\mathrm{Irr}}$.) The proposition then follows from the following two statements:

1. There exists a finite set $X \subset \overline{\mathrm{Irr}}$ such that $a_{\pi,k} = 0$ for $\pi \notin X$.

2. The function $k \mapsto a_{\pi,k}$ is a linear combination of functions of the form $k \mapsto a^k$.

We now prove these two statements.

First, as $f$ belongs to $\mathscr{C}^{\mathrm{char}}(X')$ it is *uniformly smooth*, that is, there exists an integer $n$ such that $\pi^* f$ factors through $G'/U_{B'}^{(n)}$. It follows easily from this that if $\pi$ is an irreducible representation for which $\pi|_{U_{B'}^{(n)}}$ is non-trivial then $a_{\pi,k} = 0$. We can thus take $X$ to consist of those $\pi$ for which $\pi(U_{B'}^{(n)}) = 1$, *i.e.*, those $\pi$ of conductor $\leq n$. This is clearly a finite set. Thus (1) is established.

We now prove (2). We have

$$a_{\pi,k} = q^{2k} \int_{X'_k} f(x) \chi_\pi^\vee(x) d\mu(x) = \sum_K c_K \int f(x) \chi_\pi^\vee(x) |x - \overline{x}|_F^2 dx$$

(at least for $\pi$ odd; in the even case there should be a $\frac{1}{2}$) where the integral in the sum is over $\varpi_K^k U_K$ if $K$ is ramified and $\varpi_K^{k/2} U_K$ is $k$ is unramified (this is taken to be the empty set if $k$ is odd). Since $f$ and $\chi_\pi$ each belong to $\mathscr{C}^{\mathrm{char}}(X')$, each is a linear combination of characters when restricted to $K^\times$. We thus find that $k \mapsto a_{\pi,k}$ is a linear combination of functions of the form

$$k \mapsto \int_{\varpi_K^k U_K} \eta(x) |x - \overline{x}|_F^2 dx.$$

As shown in the lemma, these functions are linear combinations of functions of the form $k \mapsto a^k$. We have thus established (2) and therefore the proposition. □

**(5.4.5)** We now simply mention the following result, which follows from the earlier results of this section.

**Proposition.** *The image of* $\mathscr{C}^{\mathrm{char}}(X) \to \mathscr{D}^\circ(X)$ *is stable under* $\mathscr{A}$ *and semi-simple and multiplicity free as an* $\mathscr{A}$*-module. Its simple constituents are the one dimensional spaces spanned by the* $\chi_\pi$ *with* $\pi \in \mathrm{Irr}^\circ$. *In fact,* $A_{\psi,\eta}(\chi_\pi) = \lambda_{\psi,\eta}(\pi)\chi_\pi$ *holds in* $\mathscr{D}^\circ(X)$ *for* $\pi \in \mathrm{Irr}^\circ$.

**(5.4.6)** We now discuss how some of the above results are just disguised forms of the local functional equation and the local converse theorem. Let $\pi \in \mathrm{Irr}^{\circ}_{G'}$. For a complex number $s$ and a function $\phi \in \mathscr{S}(X')$ we define the *zeta function* as

$$Z(s, \phi, \pi) = q^{-2m(\psi)-1} \int_{G'} |x|_F^{s-3/2} \phi(x) \chi_\pi(x) dx$$

where $dx$ is the normalized Haar measure on $B'$. (The $q^{-2m(\psi)-1}$ factor out front makes $q^{-2m-1}dx$ the self-dual Haar measure; it will not really be important for us.) The so-called *local functional equation* is the identity

$$Z(1-s, \mathscr{F}_{B',\psi}(\phi), \pi^\vee) = \epsilon(s, \pi, \psi) Z(s, \phi, \pi)$$

for some function $\epsilon(s, \pi, \psi)$ of the form $ab^s$. Note that because $\pi$ is cuspidal the $L$-functions which normally appear on either side are equal to 1. Now, we have the equality

$$Z(s, \phi, \pi) = q^{-2m(\psi)-1} \langle |\cdot|_F^{s-3/2} \chi_\pi, \pi'_*(\phi) \rangle_{X'}.$$

Given this, the local functional equation is exactly equivalent to the fact that $\chi_\pi$ is an eigenvector of $A_{\psi,1}$. Furthermore, the statement that the eigenspace of $\chi_\pi$ is one dimensional, as given in §5.4.5, amounts to the fact that $\pi = \pi'$ if and only if $\lambda_{\psi,\eta}(\pi) = \lambda_{\psi,\eta}(\pi')$ holds for all $\psi$ and $\eta$. This is the converse theorem for $G'$. Notice that our result describing the image of $\mathscr{C}^{\mathrm{char}}(X) \to \mathscr{D}^\circ(X)$ as an $\mathscr{A}$-module thus encapsulates both the local functional equation and the local converse theorem.

*Remark.* Although we used the local converse theorem for $\mathrm{GL}_2$ in our analysis of the split side, its use was not truly necessary. In fact, using our proof of the Jacquet-Langlands correspondence in the next section, we could deduce the local converse theorem from $\mathrm{GL}_2$ from the converse theorem for $G'$, of which we have given a complete proof.

# Chapter 6

# The Jacquet-Langlands correspondence

**(6.1)** In §6, we apply the theory developed in the previous sections to prove the Jacquet-Langlands correspondence. Here is an overview:

- In §6.2 we prove the fundamental identity needed to compare the two sides in the correspondence. This identity is explicit and elementary.
- In §6.3 we use the result of §6.2 to show that the maps $\overline{p}_*$ and $\overline{p}'_*$ agree.
- In §6.4 we use the result of §6.3 to show that $H(X)$ and $H(X')$ are isomorphic as $(\mathscr{A} * \mathscr{T})$-modules, via the natural restriction map.
- Finally, in §6.5 we use §6.4 to deduce the correspondence.

**(6.2)** We begin by proving the fundamental identity needed for our comparison.

**Proposition.** *Let $\alpha$ and $\beta$ be elements of $F^\times$ for which $(\alpha, \beta) = -1$ and put $\gamma = -\alpha\beta$. Then the quantity*

$$I_2(x, \alpha) + I_2(\gamma x, \alpha)$$

*is independent of $x \in F^\times$.*

*Proof.* First consider the case where $\operatorname{val} \alpha$ is even. Necessarily then, $\operatorname{val} \gamma$ is odd (as the residue characteristic of $F$ is odd). We have, for $x \in F^\times$,

$$I_2(x,\alpha) = \begin{cases} 1+q^{-1} & \operatorname{val} x \text{ even} \\ 0 & \operatorname{val} x \text{ odd.} \end{cases}$$

As precisely one of $x$ and $\gamma x$ has even valuation, we see that $I_2(x,\alpha) + I_2(\gamma x, \alpha)$ is always equal to $1 + q^{-1}$.

Now consider the case where $\operatorname{val}\alpha$ is odd. We then have

$$I_2(x,\alpha) = q^{-1/2} \begin{cases} 1+\eta(x) & \operatorname{val} x \text{ even} \\ 1+\eta(-x/\alpha) & \operatorname{val} x \text{ odd.} \end{cases}$$

If $\gamma$ has even valuation, so that $x$ and $\gamma x$ have the same parity of valuation, then

$$I_2(x,\alpha) + I_2(\gamma x,\alpha) = q^{-1/2} \begin{cases} 2+\eta(x)+\eta(\gamma x) & \operatorname{val} x \text{ even} \\ 2+\eta(-x/\alpha)+\eta(-\gamma x/\alpha) & \operatorname{val} x \text{ odd.} \end{cases}$$

As precisely one of $y$ and $\gamma y$ is a square if $y$ has even valuation, we see that the quantity inside the brace is always equal to 1, which proves the result in this case. If $\gamma$ has odd valuation then we find

$$I_2(x,\alpha) + I_2(\gamma x,\alpha) = q^{-1/2} \begin{cases} 2+\eta(-\gamma x/\alpha)+\eta(x) & \operatorname{val} x \text{ even} \\ 2+\eta(\gamma x)+\eta(-x/\alpha) & \operatorname{val} x \text{ odd.} \end{cases}$$

As $-\gamma/\alpha = \beta$ is a non-square of even valuation, the quantity inside the brace always equals 1. This establishes the proposition. □

**(6.3)** We can now compare $\overline{p}_*$ and $\overline{p}'_*$.

**Proposition.** *Let $f$ be an element of $\mathscr{S}_0(\widetilde{X}') = \mathscr{S}_0(\widetilde{X}_{\mathrm{ell}})$. Then $\overline{p}_* f + \overline{p}'_* f = 0$.*

*Proof.* If suffices to show that for $K/F$ a quadratic extension and $f \in \mathscr{S}_0(K)$ we have $(\overline{p}_K)_* f + (\overline{p}'_K)_* f = 0$. By §4.2.8 and §5.2.6 we have

$$((\overline{p}_K)_* f + (\overline{p}'_K)_* f)(t,\nu) = \frac{d_K^{1/2}}{|\Delta|_F^{1/2} A(\Delta)} \int_F f(\tfrac{1}{2}t + ix) J(x) d\mu_F(x)$$

where

$$J(x) = I_2(\Delta - 4\mathbf{d}_K x^2, \mathbf{d}_K) + I_2\left(\frac{\Delta - 4\mathbf{d}_K x^2}{\gamma}, \mathbf{d}_K\right),$$

$\gamma = -\mathbf{d}_K \beta$ and $\beta$ is an element of $F^\times$ for which $(\mathbf{d}_K, \beta) = -1$. By §6.2 we have that $J(x)$ is independent of $x$. As $f$ belongs to $\mathscr{S}_0(K)$ we have that

$$\int_F f(\tfrac{1}{2}t + ix) J(x) d\mu_F(x) = 0,$$

which proves the proposition. □

**(6.4)** We now have our main comparison theorem.

**Proposition.** *Restricting functions on $X_{\mathrm{ell}}$ to $X'$ gives an isomorphism $H(X) \to H(X')$ of $(\mathscr{A} * \mathscr{T})$-modules.*

*Proof.* For a function $f$ on $X_{\mathrm{ell}}$ let $r(f)$ denote its restriction to $X'$. It follows from §4.3.9 and the definition of $H(X')$ that $r : H(X) \to H(X')$ is an isomorphism of Hilbert spaces. It is equally clear that $r$ commutes with the action of $\mathscr{T}$. It remains to check that $r$ commutes with $\mathscr{A}$. To verify this, it suffices to show that for $f \in H(X)$ we have

$$r(\mathscr{F}^{(2)}_{X,\psi}(f)) + \mathscr{F}_{X',\psi}(r(f)) = 0.$$

Recall that we have defined Fourier transforms

$$\mathscr{F}^{(1)}_{X,\psi} : \mathscr{S}(X_{\mathrm{re}}) \to \mathscr{C}^\infty(X_{\mathrm{re}}), \qquad \mathscr{F}_{X',\psi} : \mathscr{S}(X') \to \mathscr{S}(X').$$

By §4.2.7, §5.2.5 and §6.3 we have

$$r(\mathscr{F}^{(1)}_{X,\psi}(f)) + \mathscr{F}_{X',\psi}(r(f)) = 0.$$

As the map $r : \mathscr{S}_{\mathrm{re}}(X) \to \mathscr{S}_{\mathrm{reg}}(X')$ is a linear isomorphism and an $L^2$-isometry and the transform $\mathscr{F}_{X',\psi}$ is an $L^2$-isometry (§5.2.4), it follows that $\mathscr{F}^{(1)}_{X,\psi}$ is an $L^2$-isometry. It therefore extends uniquely to a continuous map

$$\mathscr{F}^{(1)}_{X,\psi} : L^2(X_{\mathrm{ell}}) \to L^2(X_{\mathrm{ell}})$$

since $\mathscr{S}(X_{\mathrm{re}})$ is dense in $L^2(X_{\mathrm{ell}})$. We have previously shown in §4.4.8 that $\mathscr{F}^{(1)}_{X,\psi}$ and $\mathscr{F}^{(2)}_{X,\psi}$ agree on a dense subset of $H(X)$ (which is a closed subspace of $L^2(X_{\mathrm{ell}})$). It follows that $\mathscr{F}^{(1)}_{X,\psi}(f) = \mathscr{F}^{(2)}_{X,\psi}(f)$

for $f \in H(X)$. We thus see that

$$r(\mathscr{F}^{(2)}_{X,\psi}(f)) + \mathscr{F}_{X',\psi}(r(f)) = 0$$

for $f$ in $\mathscr{S}_{\mathrm{re}}(X) \cap H(X)$. As this space is dense in $H(X)$ and $r$, $\mathscr{F}^{(2)}_{X,\psi}$ and $\mathscr{F}_{X',\psi}$ are continuous, it follows that the above equation holds for all $f \in H(X)$, which proves the proposition. □

**(6.5)** We can now prove the Jacquet-Langlands correspondence.

**Theorem.** *If $\pi$ is a cuspidal representation of $G$ then there exists a unique cuspidal representation $\pi'$ of $G'$ such that $\chi_\pi|_{X_{\mathrm{re}}} = -\chi_{\pi'}$. Every cuspidal representation $\pi'$ of $G'$ arises from some $\pi$. Furthermore if $\pi$ and $\pi'$ are thusly related then $d_\pi = C \cdot d_{\pi'}$ for some absolute constant $C$ and $\epsilon(s, \eta\pi, \psi) = \epsilon(s, \eta\pi', \psi)$ for all $\eta$ and $\psi$.*

*Proof.* As $H(X) \to H(X')$ is an isomorphism of $(\mathscr{A} * \mathscr{T})$-modules, it follows that for $\pi \in \overline{\mathrm{Irr}}^\circ_G$ and $k \in \mathbb{Z}$ there exists $\pi' \in \overline{\mathrm{Irr}}^\circ_{G'}$, unique up to unramified twist, such that $V_{\pi,k} = V_{\pi',k}$. Note that this implies that $\phi_{\pi,k}$ belongs to $\mathscr{S}(X_{\mathrm{ell}})$ for $\pi \in \mathrm{Irr}^\circ_G$. Using the same reasoning as in §5.4.3, one now finds that for $\pi \in \mathrm{Irr}^\circ_G$ we have $A_{\psi,\eta}(\chi_\pi) = \lambda_{\psi,\eta}(\pi)\chi_\pi$, where here we regard $\chi_\pi$ as a cuspidal distribution. As $\chi_\pi$ is orthogonal to $\mathscr{S}^1(X')$, the results of §5.4.3 imply that $\chi_\pi|_{X'} = \alpha\chi_{\pi'}$ for some $\pi' \in \mathrm{Irr}^\circ_{G'}$. Of course, we then have $\lambda_{\psi,\eta}(\pi) = \lambda_{\psi,\eta}(\pi')$ for all $\psi$ and $\eta$, which shows $\epsilon(\frac{3}{2}, \eta\pi, \psi) = \epsilon(\frac{3}{2}, \eta\pi', \psi)$. The action of $A_{\psi,\eta}$ on $V_\pi = V_{\pi'}$ (*c.f.* §4.4.11 and §5.3.17) shows that $n(\eta\pi) = n(\eta\pi')$. We therefore have $\epsilon(s, \eta\pi, \psi) = \epsilon(s, \eta\pi', \psi)$ for all $\eta$ and $\psi$. The equality $\langle \phi_{\pi,0}, \phi^\vee_{\pi,0} \rangle_X = \alpha^2 \langle \phi_{\pi',0}, \phi^\vee_{\pi',0} \rangle_{X'}$ together with the computation of each side now gives $\alpha = \pm 1$. The identities of §4.4.4 and §5.3.16, together with the positivity of $d_\pi$ and $d_{\pi'}$ now give $\alpha = -1$. We can now again apply §4.4.4 and §5.3.16 to conclude the statement about formal degrees agreeing up to an absolute constant (in fact, the constant is $\frac{1}{2}q^2/(q+1)$).

We have now proved everything except for the statement that every cuspidal representation of $G'$ arises from a cuspidal representation of $G$. To see this, observe that because $H(X) \to H(X')$ is an isomorphism of $(\mathscr{A} * \mathscr{T})$-modules, the map $r : \mathrm{Irr}^\circ_G \to \mathrm{Irr}^\circ_{G'}$ demonstrated above induces an isomorphism $\overline{r} : \overline{\mathrm{Irr}}^\circ_G \to \overline{\mathrm{Irr}}^\circ_{G'}$ (this follows from §4.4.12 and §5.3.15). Because $r$ behaves well with respect to twists and $\overline{r}$ is an isomorphism it follows that $r$ is an isomorphism. This proves the theorem. □

*Remark.* By normalizing our Haar measures differently, the constant $C$ in the above theorem can be made to equal 1.

# Chapter 7

# Future directions

**(7.1)** We now give some discussion about how the above results might be adapted to $\mathrm{GL}_n$ with $n > 2$. We begin by giving some notation and basic definitions. These override previous notations and definitions.

- Fix an integer $n > 1$.
- Let $X$ be the space of monic degree $n$ polynomials over $F$ which are either irreducible or $n$th powers.
- Let $X_{\mathrm{reg}}$ be the subspace of $X$ consisting of those polynomials which are irreducible.
- Let $d\mu_X$ be the measure on $X$ analogous to the one we have used above. We give a more precise definition below.

Let $B$ be a central simple $F$-algebra of rank $n^2$.

- We call $x \in B$ *elliptic* if its characteristic polynomial belongs to $X$. We write $B_{\mathrm{ell}}$ for the set of elliptic elements of $B$.
- We call $x \in B$ *regular elliptic* if its characteristic polynomial belongs to $X_{\mathrm{reg}}$. We write $B_{\mathrm{re}}$ for the set of regular elliptic elements of $B$.
- We let $\pi : B_{\mathrm{ell}} \to X$ be the characteristic polynomial map.
- We let $d\mu_B$ be the Haar measure on $B$ giving volume 1 to any maximal order.

- For a function $f : B_{\text{ell}} \to \mathbb{C}$ we let $\pi_*(f)$ be the function on $X$ given by the Radon-Nikodym derivative of $\pi_*(f d\mu_B)$ with respect to $d\mu_X$, wherever this exists. If $f \in \mathscr{S}(B_{\text{ell}})$ then $\pi_*(f)$ is well-defined and smooth on $X_{\text{reg}}$.

- For an additive character $\psi$ of $F$ we let $\mathscr{F}_{B,\psi}$ be the Fourier transform on $B$ with respect to $\psi$, normalized so that it is an $L^2$-isometry.

- Write $B = M_k(D)$ where $D$ is a central simple division algebra over $F$. We define $\epsilon_B$ to be $(-1)^{n-k}$.

If $B$ is a central simple division algebra over $F$ of rank $n^2$ then $B = B_{\text{ell}}$ and $\pi : B \to X$ is proper. The measure $d\mu_X$ can be taken to be $\pi_*(d\mu_B)$.

**(7.2)** Consider the following statement:

**Statement** (FT$_n$)**.** *Let $\psi$ be a non-trivial additive character of $F$. Then there exists a map*

$$\mathscr{F}_{X,\psi} : L^2(X) \to L^2(X)$$

*which is self-adjoint, an isometry, preserves $\mathscr{S}(X)$ and has the following property. Let $B$ be a central simple $F$-algebra of rank $n^2$. Let $f$ be a Schwartz function on $B$ which either has regular elliptic support or is a linear combination of truncated matrix coefficients of essentially square integrable representations. Then*

$$\mathscr{F}_{X,\psi}(\pi_*(f|_{B_{\text{ell}}})) = \epsilon_B \pi_*((\mathscr{F}_{B,\psi} f)|_{B_{\text{ell}}}).$$

Note that the map $\mathscr{F}_{X,\psi}$, if it exists, is determined uniquely by the final condition imposed on it. (This uses the facts that the map $\pi_* : \mathscr{S}(B_{\text{re}}) \to \mathscr{S}(X_{\text{reg}})$ is surjective and that $\mathscr{S}(X_{\text{reg}})$ is dense in $L^2(X)$.) By "truncated matrix coefficients" we mean functions like the $\phi_{\pi,v,v^*,n}$ that we used.

**(7.3)** For a totally disconnected locally compact group $G$ let $\text{Irr}_{\text{si}}(G)$ denote the set of irreducible admissible representations of $G$ whose matrix coefficients are square integrable. Consider the following statement:

**Statement** (JL$_n$)**.** *Let $B$ and $B'$ be central simple $F$-algebras of rank $n^2$. Then there is a bijection*

$$\text{Irr}_{\text{si}}(B^\times) \to \text{Irr}_{\text{si}}((B')^\times)$$

*characterized by the following property: if $\pi$ corresponds to $\pi'$ then $\epsilon_B \chi_\pi(x) = \epsilon_{B'} \chi_{\pi'}(x')$ whenever*

*$x \in B$ and $x' \in B'$ are regular elliptic elements with the same characteristic polynomial. The bijection also preserves $\epsilon$-factors and $L$-functions.*

**(7.4)** We can now discuss the manner in which our work extends to $\mathrm{GL}_n$. To begin with, our methods show

$$\mathrm{JL}_n \implies \mathrm{FT}_n \quad \text{for all } n.$$

This can be seen using explicit computations of the Fourier transforms of matrix elements of square integrable representations, such as those in §4.4.2. (Of course, there are a lot of details to fill in, but we believe that no new ideas are needed in addition to our work.) In the other direction, our methods show

$$\mathrm{FT}_n \implies \mathrm{JL}_n \quad \text{for } n = 2, 3.$$

This can be proved by considering the structure of the cuspidal space $H(X)$ as a module over $\mathscr{A} * \mathscr{T}$. We carried this out for $n = 2$; for $n = 3$ the same argument works. The essential feature which fails for $n > 3$ is the form of the local converse theorem which we use. The entirety of this thesis was devoted to proving $\mathrm{FT}_2$ and the above implication for $n = 2$, which gave us a proof of $\mathrm{JL}_2$.

**(7.5)** We proved $\mathrm{FT}_2$ by attaching a Fourier transform on $X$ to each rank 4 central simple algebra over $F$ and then verifying that all of these Fourier transforms agreed. To prove that they agreed, we factored them into something intrinsic to $X$ followed by an operator $\overline{p}$ and then showed that the $\overline{p}$'s agreed. We feel that this approach should work, in theory, to prove $\mathrm{FT}_n$. The problem lies in the comparison of the $\overline{p}$ operators. We succeeded in our situation because we could explicitly compute the $\overline{p}$'s and then just look at the resulting formulas to see that they agreed. For $n > 2$ such an explicit computation does not seem feasible, at least not by our methods, so a new idea is needed. Nonetheless, comparing these $\overline{p}$ operators amounts to showing that two rather elementary integrals agree. If one could establish this for $n = 3$ then one would have a purely local proof of $\mathrm{JL}_3$.

**(7.6)** We should remark that while we have put some thought into the claims in §7.4 and §7.5, we have not worked carefully through the details, so they should be taken with a grain of salt.

**(7.7)** As mentioned in the introduction, the statement $\mathrm{JL}_n$ has been proven (by global means). Thus the statement $\mathrm{FT}_n$ is true. We find the Fourier transform $\mathscr{F}_{X,\psi}$ to be very intriguing: it is a natural transform on the space functions on the space of irreducible monic polynomials. There are a number of questions we have about it:

- To compute $\mathscr{F}_{X,\psi}(f)$ one has to pick a central simple algebra $B$ and choose a lift of $f$ to a function on $B$. Is there a more natural description of $\mathscr{F}_{X,\psi}$, one that does not involve any choices? In particular, can one describe $\mathscr{F}_{X,\psi}$ without mentioning central simple algebras?
- Is there a natural way to extend $\mathscr{F}_{X,\psi}$ to the space of functions on the space of *all* monic polynomials?
- We feel that the principle of functoriality should imply, at least on a philosophical level, that $\mathscr{F}_{X,\psi}$ interacts nicely with certain operations on polynomials. Can this be made precise?

**(7.8)** As mentioned, our methods only show $\mathrm{FT}_n$ implies $\mathrm{JL}_n$ for $n = 2, 3$. We have one idea about how our approach could be modified to obtain $\mathrm{JL}_n$ for larger $n$. In the case $n = 2$ we defined a convolution operation on the space of Schwartz functions on $X$ with regular elliptic support by pushing forward the multiplicative convolution operation on the non-split quaternion algebra. The Fourier transform on $X$ could be recovered from this structure by convolving with a specific function. For general $n$, if one could define such convolutions on $X$ using any central simple algebra and compare the resulting operations then one should be able to deduce $\mathrm{JL}_n$.

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