UNIFORMIZATION OF DRINFELD MODULAR CURVES

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Abstract. These are notes from a talk in the Arithmetic Geometry Learning Seminar on Drinfeld modules during the fall of 2017. In this talk, we discuss the uniformization of the Drinfeld modular curves (of fixed rank, with level structure), following [Dri74, §6]. In addition, we introduce the relevant rigid-analytic background. The exposition draws heavily on [DH87, Gek86].

1. Uniformization of Complex Modular Curves

If $N \in \mathbf{Z}_{>0}$, consider the functor $F_N\colon (\mathrm{Sch}/\mathbf{Z}[\frac{1}{N}])^{\mathrm{op}} \to (\mathrm{Sets})$ given by

$$S \mapsto \Big\{\text{elliptic curves } E \text{ over } S\text{, equipped with a full level-}N\text{ structure}\Big\}\Big/ \simeq,$$

where a full level-N structure on an elliptic curve E over S is the choice of an isomorphism $(\mathbf{Z}/N\mathbf{Z})^2 \xrightarrow{\simeq} E[N]$ of S-group schemes.

Theorem 1.1. *If $N \geq 3$, the functor F_N is representable by a smooth affine $\mathbf{Z}[\frac{1}{N}]$-scheme $Y(N)$ of finite type.*

Taking the $\mathbf{C}$-points of the moduli scheme $Y(N)$ gives a complex manifold $\mathcal{Y}(N) \coloneqq Y(N)(\mathbf{C})$ called the *modular curve of level* N over $\mathbf{C}$, whose points parametrize elliptic curves over $\mathbf{C}$ together with an isomorphism (of abelian groups) between the N-torsion of the elliptic curve and $(\mathbf{Z}/N\mathbf{Z})^2$.

Let $\mathbf{H} \coloneqq \{\tau \in \mathbf{C}\colon \mathrm{im}(\tau) > 0\}$ be the complex upper half-plane. Recall that $\mathrm{SL}(2, \mathbf{Z})$ acts on $\mathbf{H}$ (on the left) by fractional linear transformations. We will be particularly interested in the action of the principal congruence subgroup

$$\Gamma(N) \coloneqq \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z})\colon \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N\right\}$$

of level N; equivalently, $\Gamma(N)$ is the kernel of the "reduction mod N" map $\mathrm{SL}(2, \mathbf{Z}) \to \mathrm{SL}(2, \mathbf{Z}/N\mathbf{Z})$. The action of $\Gamma(N)$ on $\mathbf{H}$ is discrete (equivalently, properly discontinuous), so the quotient $\Gamma(N)\backslash\mathbf{H}$ exists as a complex-analytic space.

For any $\tau \in \mathbf{H}$, the lattice $\Lambda_\tau \coloneqq \mathbf{Z} + \mathbf{Z}\tau$ gives rise to an elliptic curve $E_\tau \coloneqq \mathbf{C}/\Lambda_\tau$ over $\mathbf{C}$, along with a full level-N structure $(\mathbf{Z}/n\mathbf{Z})^2 \xrightarrow{\simeq} E_\tau[N]$, sending the two generators of $(\mathbf{Z}/N\mathbf{Z})^2$ to $\frac{1}{N}, \frac{\tau}{N} \in E_\tau[N]$. This construction gives rise a surjective, $\Gamma(N)$-invariant, holomorphic map

$$\mathbf{H} \longrightarrow \mathcal{Y}(N)$$

that induces an isomorphism

$$\Gamma(N)\backslash\mathbf{H} \xrightarrow{\simeq} \mathcal{Y}(N)$$

of complex-analytic spaces. The realization of $\mathcal{Y}(N)$ as the quotient of the complex manifold $\mathbf{H}$ by the discrete action of the group $\Gamma(N)$ is often referred to as the (complex-analytic) uniformization of the modular curve of level N.

The goal of today's lecture is to explain the analogue of the above uniformization theorem for the moduli spaces of Drinfeld modules of fixed rank and with some level structure. To do so, we must first discuss the correct analytic framework.

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2. Rigid-Analytic Geometry

This section provides a brief overview of the basic definitions in Tate's theory of rigid-analytic geometry. For careful introductory treatments, see [Bos14, Con08].

Let K be a complete (nontrivially-valued) non-Archimedean field with norm $|\cdot|$. The fundamental algebraic object in the theory is the Tate algebra $K\{T_1, \dots, T_n\}$ over K, which is the Banach K-algebra defined by

$$K\{T_1, \dots, T_n\} \coloneqq \left\{ f = \sum_{\mu \geq 0} a_\mu T^\mu \in K[[T_1, \dots, T_n]] \colon a_\mu \in K, |a_\mu| \to 0 \text{ as } |\mu| \to +\infty \right\}.$$

Intuitively, $K\{T_1, \dots, T_n\}$ is the algebra of analytic functions on the closed unit ball over K. More generally, the basic algebraic building blocks of Tate's theory are the Banach K-algebras isomorphic to a quotient of a Tate algebra by an ideal; these are called *K-affinoid algebras*.

Just as in the theory of schemes, the basic algebraic objects give rise to the basic spaces: to a K-affinoid algebra, we construct a *K-affinoid space* $\mathrm{Sp}(A)$. In order to call $\mathrm{Sp}(A)$ a 'space', we must specify 3 pieces of data: the underlying set, the topology, and the structure sheaf.

The underlying set of $\mathrm{Sp}(A)$ is the set $\mathrm{Max}(A)$ of maximal ideals of A. One can check that any K-algebra map $\phi\colon A \to B$ induces a well-defined pullback map $\phi^*\colon \mathrm{Max}(B) \to \mathrm{Max}(A)$. Moreover, $\mathrm{Sp}(A)$ comes with a distinguished collection of subsets, known as the *affinoid subdomains* of $\mathrm{Sp}(A)$: every affinoid subdomain is (a finite union of subsets) of the form

$$\{x \in \mathrm{Max}(A) \colon |f_1(x)| \leq |g(x)|, \dots, |f_m(x)| \leq |g(x)|\}$$

for $f_1, \dots, f_m, g \in A$ that generate the unit ideal. Here, if $f \in A$ and $x \in \mathrm{Max}(A)$, then $f(x)$ denotes the image of f in the residue field of A at x (the residue field is a finite extension of K, and hence $|\cdot|$ extends in a unique way).

The set $\mathrm{Max}(A)$ is equipped with a (strong) Grothendieck topology given by the following data:

- a subset $U \subseteq \mathrm{Max}(A)$ is an *admissible open* if there is a (set-theoretic) cover $\{U_i\}_{i \in I}$ of U by affinoid subdomains such that for any map $\phi\colon A \to B$ of K-affinoid algebras such $\phi^*(\mathrm{Max}(B)) \subseteq U$, then finitely-many of the U_i's cover $\phi^*(\mathrm{Max}(B))$;
- a collection $\{V_j\}_{j \in J}$ of admissible opens is an *admissible cover* of $V = \bigcup_{j \in J} V_j$ if for any map $\phi\colon A \to B$ of K-affinoid algebras such $\phi^*(\mathrm{Max}(B)) \subseteq V$, the cover $\{(\phi^*)^{-1}(V_j)\}_{j \in J}$ of $\mathrm{Max}(B)$ has a finite refinement by affinoid subdomains of $\mathrm{Max}(B)$.

Furthermore, there is a natural structure sheaf on this Grothendieck topology (with A as the algebra of global sections), which we will not define here.

More generally, a *rigid space* over K is a locally G-ringed space with an admissible covering by K-affinoid spaces; that is, a rigid space over K is obtained by gluing together a collection of K-affinoid spaces.

Example 2.1.

(1) The closed unit polydisc is $\mathrm{Sp}(K\{T_1, \dots, T_n\})$, and its underlying set corresponds to $\mathrm{Gal}(\overline{K}/K)$-orbits of n-tuples $(x_1, \dots, x_n) \in \overline{K}^n$ such that $|x_1|, \dots, |x_n| \leq 1$.

(2) Pick $c \in K$ with $|c| > 1$ and let $A_i = K\{c^{-i}T_1, \dots, c^{-i}T_n\}$ be the K-affinoid algebra of power series convergent on a ball of radius $|c|$. There is a sequence of inclusions

$$\mathrm{Sp}(A_0) \hookrightarrow \mathrm{Sp}(A_1) \hookrightarrow \dots \hookrightarrow \mathrm{Sp}(A_n) \hookrightarrow \dots$$

and these glue to give the rigid affine n-space, denoted $\mathbf{A}_K^{n,\mathrm{rig}}$ (said differently, we have constructed affine space by expressing it as a nested union of closed balls of increasing radii).

In fact, for any scheme X locally of finite type over K, there is an associated rigid space X^{rig}, called the *rigid analytification* of X, whose underlying point set is the set of closed points of X. The operation $X \mapsto X^{\mathrm{rig}}$ is functorial, and satisfies the GAGA theorems that exist for complex analytification.

In the sequel, we will be discussing the quotients of rigid spaces by certain group actions, so the following existence result will be crucial.

Proposition 2.2. [Dri74, §6.2] *If X is a separable rigid space and Γ is a group acting discretely on X, then the quotient $\Gamma\backslash X$ exists as a rigid space.*

The proof proceeds as one might expect: choose an affinoid cover of X such that Γ acts by a finite subgroup on each member of the cover, and hence one can form the algebra of invariants to form the quotient locally, and then glue.

3. Drinfeld Modular Curves

Let q be a power of a prime p. Let X be a geometrically irreducible, smooth, projective curve over $\mathbf{F}_q$, and let K be the function field of X. Fix a place ∞ of K (that is, a closed point of X), and let $|\cdot|_\infty$ be the corresponding norm on K, normalized so that $|\pi|_\infty = q^{-1}$, where π is any uniformizing parameter at ∞. Let $A = H^0(X\backslash\{\infty\}, \mathcal{O}_X)$ and let $I \subseteq A$ be an ideal such that $\#\operatorname{Spec}(A/I) > 1$. Let K_∞ be the completion of K with respect to the norm $|\cdot|_\infty$, and let C be the completion of the algebraic closure of K_∞.

Consider the functor $\mathrm{DM}_I^r \colon (\mathrm{Sch}\,/A)^{\mathrm{op}} \to (\mathrm{Sets})$ given by

$$S \mapsto \{\text{Drinfeld modules over } S \text{ of rank } r \text{ with level } I\text{-structure}\}\,/\simeq .$$

The goal of the previous lecture was to prove the following representability theorem of Drinfeld:

Theorem 3.1. [Dri74, Proposition 5.3] *The functor DM_I^r is represented by an affine A-scheme M_I^r of finite type.*

When viewed as a scheme over $\mathbf{F}_q$, M_I^r is smooth of dimension r; however, the structure map $M_I^r \to \operatorname{Spec}(A)$ is only smooth away from $\operatorname{Supp}(I)$ in general.

The *Drinfeld modular curve* (of rank r and level I) is the rigid analytification

$$\mathcal{M}_I^r \coloneqq (M_I^r \otimes_A K_\infty)^{\mathrm{rig}}$$

of the moduli scheme. The C-points of the rigid space $\mathcal{M}_I^r$ classify isomorphism classes of Drinfeld modules over C of rank r with level-I structure. In [Dri74, §3], we saw an alternate classification of such objects in terms of r-lattices[1] in C with some level structure: more precisely, there is an equivalence of categories

$$\left\{\begin{matrix}\text{homothety classes } r\text{-lattices } \Lambda \text{ in } C \text{ with} \\ A/I\text{-module isomorphisms } (A/I)^r \simeq \Lambda/I\Lambda\end{matrix}\right\} \xrightarrow{\simeq} \left\{\begin{matrix}\text{isomorphism classes of Drinfeld modules} \\ \text{over } C \text{ of rank } r \text{ with level-}I \text{ structure}\end{matrix}\right\} = \mathcal{M}_I^r(C),$$

where an r-lattice $\Lambda \subseteq C$ is sent to the Drinfeld module ϕ^Λ (as defined in [Gek86, Chapter I, 2.3])

The goal of this lecture is to produce a rigid-analytic space $\widetilde{\Omega}^r$ over K_∞, equipped a discrete group action Γ_I on $\tilde{\Omega}^r$, and a surjective Γ_I-invariant morphism $\widetilde{\Omega}^r \to \mathcal{M}_I^r$ such that the induced map

$$\Gamma_I\backslash\widetilde{\Omega}^r \xrightarrow{\simeq} \mathcal{M}_I^r \tag{3.1}$$

is an isomorphism. The space $\widetilde{\Omega}^r$ will (in a certain sense) parametrize a large class of lattices with some level structure and the Γ_I-action will identify equivalent level structures; thus, the isomorphism (3.1) aims to generalize the above equivalence of categories.

[1] If $r \in \mathbf{Z}_{>0}$, an *r-lattice* in C is a finitely-generated projective rank-r A-submodule of C that is a discrete subset of C, i.e. it intersects every ball of finite radius in C in only finitely-many points.

4. Drinfeld Upper Half Space

In this section, let K be a local non-Archimedean field with valuation ring R, and uniformizer π. Let $|\cdot|$ be a norm on K and let $q^{-1} = |\pi|$.

For $r \geq 2$, consider the rigid projective space $\mathbf{P}_K^{r-1,\mathrm{rig}}$ with homogeneous coordinates $[z_1\colon \ldots : z_r]$. The underlying point set of $\mathbf{P}_K^{r-1,\mathrm{rig}}$ consists of $\mathrm{Gal}(\overline{K}/K)$-orbits of points $\mathbf{P}^{r-1}(\overline{K})$, i.e. the closed points of the K-scheme $\mathbf{P}_K^{r-1}$. Let Ω^r be the set-theoretic complement of all K-rational hyperplanes in $\mathbf{P}_K^{r-1,\mathrm{rig}}$. The goal is to show that Ω^r is naturally a rigid-analytic space over K (by showing that Ω^r is an admissible open subset of $\mathbf{P}_K^{r-1,\mathrm{rig}}$). The rigid space Ω^r is known as the *Drinfeld upper half space* over K.

In order to prove that Ω^r is an admissible open in $\mathbf{P}_K^{r-1,\mathrm{rig}}$, we must first relate Ω^r to the (geometric realization of the) Bruhat–Tits building for the group $\mathrm{GL}(r,K)$ (with the key relationship coming from the fact that Ω^r is invariant for the usual action of $\mathrm{GL}(r,K)$ on $\mathbf{P}_K^{r-1,\mathrm{rig}}$). For a complete exposition, see [DH87, §3].

4.1. **The Bruhat–Tits Building and the Building Map.** Given a R-lattice L in the vector space K^r, let $\{L\} \coloneqq \{cL\colon c \in K^*\}$ denote the homothety class of L. The set S^r of homothety classes of R-lattices in K^r can be realized as the vertex set of a simplicial complex $\mathcal{B}(r,K)$, known as the *Bruhat–Tits building* of the group $\mathrm{PGL}(r,K)$. More precisely, define $\mathcal{B}(r,K)$ to be the simplicial complex of dimension $r-1$ given by the following data:

(1) the 0-simplices correspond to the points of S^d;
(2) the homothety classes $\{L_0\}, \{L_1\}, \ldots, \{L_n\}$ span an n-simplex if there are representatives $L_i \in \{L_i\}$ such that

$$\pi L_0 \subsetneq L_n \subsetneq \ldots \subsetneq L_1 \subsetneq L_0.$$

The *geometric realization* of $\mathcal{B}(r,K)$ is the subset $S^r(\mathbf{R}) \subseteq \prod_{v\in\mathcal{B}(r,K)}[0,1]$ consisting of those tuples (t_v) such that $\sum_v t_v = 1$ and the subset $\{v\colon t_v \neq 0\}$ determines a simplex of $\mathcal{B}(r,K)$.

Each n-simplex $\sigma = \{v_0,\ldots,v_n\}$ of $\mathcal{B}(r,K)$ has its geometric realization $|\sigma|$ living naturally in $(\mathbf{R}_{\geq 0})^{n+1}$, and we equip it with the induced topology. The geometric realization $S^d(\mathbf{R})$ is equipped with the inductive limit topology; that is, a subset $U \subseteq S^r(\mathbf{R})$ is open iff $U \cap |\sigma|$ is open for all simplices σ. There are two distinguished subsets of $S^r(\mathbf{R})$, namely the points $S^r(\mathbf{Z})$ with integer coordinates (these are precisely the 0-simplices of $\mathcal{B}(r,K)$), and the dense subset $S^r(\mathbf{Q})$ of points with rational coordinates.

The geometric realization $S^d(\mathbf{R})$ admits an alternate description in terms of a space of norms, which we discuss now.

If V is a finite-dimensional K-vector space, a *norm* on V is a function $\alpha\colon V \to \mathbf{R}_{\geq 0}$ satisfying

(1) $\alpha(x) = 0$ iff $x = 0$;
(2) $\alpha(ax) = |a|\alpha(x)$ for $x \in V$ and $a \in K$;
(3) $\alpha(x+y) \leq \max\{\alpha(x), \alpha(y)\}$.

A norm α on V is *integral* if $\alpha(V) = |K| = \{0\} \cup q^{\mathbf{Z}}$, and it is *rational* if $\alpha(V) \subseteq \{0\} \cup q^{\mathbf{Q}}$. Moreover, if α is a norm on V and $t > 0$, then the dilation $t\alpha$ is still a norm on V. Denote by $N(V)$ the set of dilation classes of norms on V. A dilation class is integral or rational if it contains an integral or rational norm.

Example 4.1. Given a lattice L in V, one can construct a norm α_L on V as follows: if $x \in V$, set

$$\alpha_L(x) \coloneqq \inf\{|a|^{-1}\colon a \in K, ax \in L\}.$$

Notice that this construction behaves well under homothety of the lattice: if $c \in K^*$, then $\alpha_{cL} = |c|^{-1}\alpha_L$. In addition, given an R-basis $x_1,\ldots,x_r \in L$ of L, one can show that

$$\alpha_L(a_1x_1 + \ldots + a_rx_r) = \max\{|a_1|,\ldots,|a_r|\},$$

In particular, the unit ball of α_L is precisely L.

The construction in Example 4.1 provides the connection between the Bruhat–Tits building and the space of norms: if $\sigma = \{v_0, \dots, v_n\}$ is an n-simplex of $\mathcal{B}(r, K)$ and $t = (t_0, \dots, t_n) \in |\sigma|$, pick representatives L_i of v_i such that $\pi L_0 \subsetneq Ln \subsetneq \dots \subsetneq L_1 \subsetneq L_0$ and $t_n > 0$, and set

$$\theta(t) := \max_{i=0,\dots,n} q^{t_i + \dots + t_n} \alpha_{L_i}.$$

Miraculously, $\theta(t)$ is norm on K^r, and different choices of representatives of the v_i's yield dilation-equivalent norms. In fact, much more is true: θ defines a bijective map $S^r(\mathbf{R}) \xrightarrow{\simeq} N(K^r)$, sending $S^r(\mathbf{Z})$ and $S^r(\mathbf{Q})$ onto the classes of integral and rational norms, respectively.

There is a function ρ on the space of norms, defined as follows: for two norms α, β on V, set

$$\rho(\alpha, \beta) := \log_q \left(\sup_{x \in V \setminus \{0\}} \frac{\alpha(x)}{\beta(x)} \right) + \log_q \left(\sup_{x \in V \setminus \{0\}} \frac{\beta(x)}{\alpha(x)} \right).$$

It is clear that $\rho(\alpha, \beta)$ is invariant if α and β are dilated, so it depends only on the dilation classes $\{\alpha\}$ and $\{\beta\}$. Even better, one can show that the function $\{\alpha\} \times \{\beta\} \mapsto \rho(\{\alpha\}, \{\beta\})$ defines a metric on the set $N(V)$, and ρ is invariant under the $\mathrm{GL}(V)$-action on $N(V)$ (given by precomposition).

Example 4.2. If L is a lattice in V and α is a norm on V, then one can check that

$$\rho(\alpha, \alpha_L) = \log_q \left(\sup_{x \in L \setminus \pi L} \alpha(x) \right) - \log_q \left(\inf_{x \in L \setminus \pi L} \alpha(x) \right).$$

Consider the *building map* $\lambda \colon \Omega^r \to N(K^r)$, which sends a point $[z_1 \colon \dots \colon z_r] \in \Omega^r$ to the (dilation class of) the norm

$$(a_1, \dots, a_r) \mapsto |a_1 z_1 + \dots + a_r z_r|.$$

Note that this is a norm (as opposed to a seminorm) precisely because the z_i's have no K-linear dependence. Furthermore, the definition of λ appears to depend on a choice of representative of $[z_1 \colon \dots \colon z_r]$, but different choices yield dilation-equivalent norms, so λ is well-defined.

The norm on K is discretely-valued, so $|C^*| = q^{\mathbf{Q}}$; in particular, the image of λ lands in the subset of rational classes of norms. Moreover, viewing $\mathrm{GL}(r, K)$ as acting on $N(K^r)$ by precomposition, it is easy to see that λ is $\mathrm{GL}(r, K)$-equivariant.

4.2. Rigid Structure on the Drinfeld Upper Half Space.

Theorem 4.3. [Dri74, Proposition 6.1]

(1) If $\alpha_1, \dots, \alpha_k \in S^r(\mathbf{Z})$ and $c \in \mathbf{Q}$, then

$$X_c := \left\{ z \in \Omega^r \colon \sum_{i=1}^{k} \rho(\alpha_i, \lambda(z)) \le c \right\}$$

is an affinoid subdomain of $\mathbf{P}_K^{r-1,rig}$.

(2) The subset Ω^r is an admissible open subset $\mathbf{P}_K^{r-1,rig}$; in particular, Ω^r naturally admits the structure of a rigid-analytic space over K.

We will only prove Theorem 4.3(1) in the case $k = 1$, but this is sufficient to prove (2).

Proof. To prove (1) when $k = 1$, we may assume that $\alpha_1 = \alpha_\Lambda$, where $\Lambda = R^r$ is the standard lattice in K^r (indeed, any other lattice can be obtained from Λ from the $\mathrm{GL}(r, K)$-action). Then,

$$\rho(\alpha_\Lambda, \lambda(z)) = \log_q \left(\sup_{a \in \Lambda \setminus \pi\Lambda} |a_1 z_1 + \dots + a_r z_r| \right) - \log_q \left(\inf_{a \in \Lambda \setminus \pi\Lambda} |b_1 z_1 + \dots + b_r z_r| \right),$$

so $\rho(\alpha_\Lambda, \lambda(z)) \le c$ iff for any $a, b \in \Lambda\backslash\pi\Lambda$, we have the inequality

$$\frac{|a_1 z_1 + \ldots + a_r z_r|}{|b_1 z_1 + \ldots + b_r z_r|} \le q^c. \tag{4.1}$$

By passing to a finite q^{-n}-net in $\Lambda\backslash\pi\Lambda$ for $n \gg 0$, it suffices to check (4.1) for finitely-many pairs of a, b in $\Lambda\backslash\pi\Lambda$. For simplicity, assume that we must check it only for a single pair a, b.

Let $W \subseteq \mathbf{P}_K^{r-1,\mathrm{rig}}$ be the locus where $b_1 z_1 + \ldots + b_r z_r \neq 0$. As W is Zariski-open, W is an admissible open subset of $\mathbf{P}_K^{r-1,\mathrm{rig}}$. The function $z \mapsto \frac{a_1 z_1 + \ldots + a_r z_r}{b_1 z_1 + \ldots + b_r z_r}$ is analytic on W, so the subset of $z \in W$ where (4.1) holds is an affinoid subdomain of W, and hence of $\mathbf{P}_K^{r-1,\mathrm{rig}}$.

For (2), it suffices to show that for any morphism $\varphi\colon \mathrm{Sp}(B) \to \mathbf{P}_K^{r-1,\mathrm{rig}}$ such that $\varphi(\mathrm{Sp}(B)) \subseteq \Omega^r$, there is $c \in \mathbf{Q}$ such that $\varphi(\mathrm{Sp}(B)) \subseteq X_c$. The image of φ lands in some polydisc, and so by homogenizing and rescaling we may write φ as

$$x \mapsto [b_1(x)\colon \ldots b_r(x)]$$

for some functions $b_i \in B$ such that $|b_i(y)| \le 1$ for all $y \in \mathrm{Sp}(B)$. We claim that there exists $c \in \mathbf{Q}$ such that $\rho(\alpha_\Lambda, \lambda(\varphi(x))) \le c$ for all $x \in \mathrm{Sp}(B)$, from which it follows that $\varphi(\mathrm{Sp}(B)) \subseteq X_c$.

Define the function $\psi\colon K^r \to \mathbf{R}_{\ge 0}$ by the formula

$$\psi(a_1, \ldots, a_r) := \inf_{x \in \mathrm{Sp}(B)} \left| \sum_{i=1}^r a_i b_i(x) \right|,$$

and notice that $\psi(a) = 0$ iff $a = 0$ (if not, one could use the completeness of B to construct a point $x \in \mathrm{Sp}(B)$ where φ is not defined); in particular, $\epsilon := \inf_{a \in \Lambda\backslash\pi\Lambda} \psi(a) > 0$. Furthermore, by our choice of b_i's, the ultrametric inequality gives that

$$\sup_{a \in \Lambda\backslash\pi\Lambda} \left| \sum_{i=1}^r a_i b_i(x) \right| \le \sup_{a \in \Lambda\backslash\pi\Lambda} \max_{i=1,\ldots,r} |a_i||b_i(x)| \le 1.$$

It follows that

$$\begin{aligned}\sup_{x \in \mathrm{Sp}(B)} \rho(\alpha_\Lambda, \lambda(\varphi(x))) &= \sup_{x \in \mathrm{Sp}(B)} \log_q \left(\sup_{a \in \Lambda\backslash\pi\Lambda} \left| \sum_{i=1}^r a_i b_i(x) \right| \right) - \log_q \left(\inf_{a \in \Lambda\backslash\pi\Lambda} \inf_{x \in \mathrm{Sp}(B)} \left| \sum_{i=1}^r a_i b_i(x) \right| \right) \\ &\le \log_q(1) - \log_q(\epsilon).\end{aligned}$$

Thus, if $c \in \mathbf{Q}$ is larger than $-\log_q(\epsilon)$, then we have $\varphi(\mathrm{Sp}(B)) \subseteq X_c$, as required. □

5. Uniformization of the Drinfeld Modular Curve

We revert now to the notation of §3. Let $\mathbf{A}_K^{(\infty)}$ be the ring of adèles of K without the component ∞, and let $\mathcal{O}_K^{(\infty)}$ be the subring of integral elements. Given a matrix $g = (g_v)_{v \neq \infty} \in \mathrm{GL}(r, \mathbf{A}_K^{(\infty)})$, the subset

$$\Lambda_g := \bigcap_{v \neq \infty} (g_v \cdot \mathcal{O}_v^r \cap K^r)$$

of K^r is an A-lattice (and comes equipped with a distinguished basis for this lattice); see [BS97, Lemma 1.4.6]. Set

$$\mathrm{GL}(r, \mathcal{O}_{K,I}^{(\infty)}) := \ker\left(\mathrm{GL}(r, \mathcal{O}_I^{(\infty)}) \to \mathrm{GL}(r, A/I)\right).$$

In [Dri74], Drinfeld writes U_I for $\mathrm{GL}(r, \mathcal{O}_{K,I}^{(\infty)})$. Two matrices $g, g' \in \mathrm{GL}(r, \mathbf{A}_K^{(\infty)})$ give the same A-lattice with level-I structure if they differ by the $\mathrm{GL}(r, \mathcal{O}_{K,I}^{(\infty)})$-action.

Let $\mathrm{Mon}_K(K^r, C)$ denote the set of injective K-linear maps $K^r \to C$. Given an A-lattice Λ in K^r and a map $j \in \mathrm{Mon}_K(K^r, C)$, the subset $j(\Lambda)$ is an r-lattice in C. Thus, there is a well-defined map

$$\mathrm{GL}(r, \mathbf{A}_K^{(\infty)}) \times \mathrm{Mon}_K(K^r, C)/C^* \longrightarrow \mathcal{M}_I^r(C), \tag{5.1}$$

where we think of $\mathcal{M}_I^r(C)$ as homothety classes of r-lattices in C with level-I structure. In fact, there is a $\mathrm{GL}(r,K)$-action on the product $\mathrm{GL}(r,\mathbf{A}_K^{(\infty)}) \times \mathrm{Mon}_K(K^r,C)$ given by

$$g \cdot ((g_v)_{v\neq\infty}, j) \coloneqq \left((g \cdot g_v)_{v\neq\infty}, j \circ g^{-1}\right),$$

for $g \in \mathrm{GL}(r,K)$, and we have $j(\Lambda_h) \simeq j'(\Lambda_{h'})$ precisely when (h,j) and (h',j') are related by the $\mathrm{GL}(r,K)$-action. In particular, (5.1) factors through the quotient by $\mathrm{GL}(r,K)$ to give a map

$$\mathrm{GL}(r,K)\backslash\left(\mathrm{GL}(r,\mathbf{A}_K^{(\infty)}) \times \mathrm{Mon}_K(K^r,C)/C^*\right) \longrightarrow \mathcal{M}_I^r(C) \tag{5.2}$$

Now, there is a bijection $\Omega^r(C) \xrightarrow{\simeq} \mathrm{Mon}_K(K^r,C)/C^*$ given by

$$[z_1 \colon \ldots \colon z_r] \mapsto ((a_1,\ldots,a_r) \mapsto a_1z_1 + \ldots + a_rz_r),$$

and the inverse sends a map $j \in \mathrm{Mon}_K(K^r,C)$ to the point $[j(e_1)\colon \ldots \colon j(e_r)]$, where $e_1,\ldots,e_r$ are the standard basis vectors of K^r. Thus, we can rewrite (5.2) as

$$\mathrm{GL}(r,K)\backslash\left(\mathrm{GL}(r,\mathbf{A}_K^{(\infty)}) \times \Omega^r(C)\right) \longrightarrow \mathcal{M}_I^r(C). \tag{5.3}$$

The same story holds if one replaces C with $\overline{K_\infty}$. Consider the rigid space

$$\widetilde{\Omega}^r \coloneqq \mathrm{GL}(r,K)\backslash\left(\mathrm{GL}(r,\mathbf{A}_K^{(\infty)}) \times \Omega^r\right).$$

The map (5.3) defines a $\mathrm{GL}(r,\mathcal{O}_{K,I}^{(\infty)})$-invariant morphism $\widetilde{\Omega}^r \longrightarrow \mathcal{M}_I^r$ of rigid spaces, and hence it descends to a morphism

$$\phi_I \colon \mathrm{GL}(r,\mathcal{O}_{K,I}^{(\infty)})\backslash\widetilde{\Omega}^r \longrightarrow \mathcal{M}_I^r$$

of rigid spaces from the quotient.

Theorem 5.1. [Dri74, Proposition 6.6] *The morphism ϕ_I is an isomorphism of rigid spaces.*

The calculations in this section show that ϕ_I is a bijection, and one can show that ϕ_I induces an isomorphism on all completed local rings; in particular, ϕ_I is a local isomorphism. Finally, Drinfeld shows that ϕ_I is a quasi-compact morphism (this is the difficult part), from which it follows that ϕ_I is an isomorphism.

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