Exercises from February 11th

18.904 Spring 2011

These are exercises from Marcel's and Danny's talks. Solutions are on the following page.

Exercise 1 (Marcel)**.** Prove that any convex subset X of $\mathbf{R}^n$ is contractible.

Exercise 2 (Marcel, adapted from Ch. 2 Ex. 2 of Hatcher)**.** Construct an explicit homotopy between $\mathbf{R}^n - \{0\}$ and the subset S^{n-1} which is constant on S^{n-1}. This is an example of a "deformation retraction," a special case of homotopy. Use this to prove that $\mathbf{R}^n - \{0\}$ is not contractible.

Exercise 3 (Danny, adapted from Hatcher Ex. 1.1.14)**.** Let X and Y be spaces containing the points x_0 and y_0 respectively. Find (with proof) an explicit isomorphism between $\pi_1(X \times Y, (x_0, y_0))$ and $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Exercise 4 (Danny, adapted from Munkres 60.1)**.** Find the fundamental group of the "solid torus" consisting of the torus plus its interior (as in a solid donut).

Solution to Exercise 1: We show that X is contractible by providing a homotopy between X and a point x_0 in X. We claim that the map $f : I \times X \to X$ defined by $f_t(x) = tx_0 + (1-t)x$ is such a homotopy. We see that $f_0(x) = x$ and $f_1(x) = x_0$. This map is continuous because it is a linear addition of vectors. Finally, this map is well defined because it defines a straight line between x and x_0, which by our assumption of X being convex, is contained entirely in X.

Solution to Exercise 2: The map $f_t(x) = x/|x|^t$ works. $f_0(x) = x$ and $f_1(x) = x/|x|$, which normalizes the length of every vector to 1 so that it is in S^{n-1}. For x in S^{n-1}, $|x| = 1$ so $f_t(x) = x$. This map is continuous because it is a composition of algebraic functions. It is well defined because $|x|$ is never 0. Because $\mathbf{R}^n - \{0\}$ is homotopic to the space S^{n-1}, which is not contractible, the space $\mathbf{R}^n - \{0\}$ is not contractible.

Solution to Exercise 3: Let f be a loop in $X \times Y$ based at (x_0, y_0) and p_i be the projection map from $X \times Y$ on the ith coordinate.

We claim that $\phi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ which defined by $[f] \mapsto ([p_1 \circ f], [p_2 \circ f])$ is the desired isomorphism. We need to show:

(1) ϕ is well-defined. For this we need to show that if $\tilde{f}$ is another loop with the same base point as f for which $f \simeq \tilde{f}$, then $p_i \circ f \simeq p_i \circ \tilde{f}$ for each i. First notice that since $f(0) = f(1) = \tilde{f}(0) = \tilde{f}(1) = (x_0, y_0)$, then certainly $p_i \circ f(0) = p_i \circ \tilde{f}(0) = x_0$ or y_0 (depending on i), and the same holds for $f(1)$. Moreover, projections are continuous and compositions of continuous maps are continuous, so the projections are continuous. This tells us that $p_i \circ f$ and $p_i \circ \tilde{f}$ are loops in X and Y based at x_0 or y_0 (depending on i).

Now let $F(t, s)$ be a homotopy with $F(0, s) = f(s)$ and $F(1, s) = \tilde{f}(s)$. $\pi_i \circ F$ is continuous since F is, and we have both $\pi_i \circ F(0, s) = \pi_i \circ f(s)$ and $\pi_i \circ F(1, s) = \pi_i \circ (\tilde{f}(s))$ by construction. Thus $\pi_i \circ F$ is a homotopy from $\pi_i \circ f$ to $\pi_i \circ \tilde{f}$.

(2) ϕ^{-1} is well-defined. For this we need to show that if $g \simeq \tilde{g}$ are loops in X and $h \simeq \tilde{h}$ are loops in Y, then $(g, h) \simeq (\tilde{g}, \tilde{h})$. It is an elementary fact of point-set topology that if $g : Z \to X$ and $h : Z \to Y$ are continuous, then the map $f : Z \to X \times Y$ defined by $f(z) = (g(z), h(z))$ is continuous. This allows us to reverse the logic in our previous proof to get the desired homotopy (treating the homotopies from g to $\tilde{g}$ and h to $\tilde{h}$ as component functions of a new homotopy).

(3) ϕ is a homomorphism. For this we have:

$$\begin{aligned}
\phi([f] \cdot [g]) &= \phi([fg]) \\
&= ([p_1 \circ (fg)], [p_2 \circ (fg)] \\
&= ([p_1 \circ f][p_1 \circ g], [p_2 \circ f][p_2 \circ g]) \\
&= ([p_1 \circ f], [p_2 \circ f])([p_1 \circ g], [p_2 \circ g]) = \phi([f])\phi([g])
\end{aligned}$$

Solution to Exercise 4: We can write the solid torus as $S^1 \times D^2$. Hence the solid torus has fundamental group isomorphic to $\pi_1(\mathbb{Z}) \times \pi_1(1) \approx \mathbb{Z}$.