Exercises from February 14th

18.904 Spring 2011

These are exercises from John's and Noah's talks. Solutions are on the following page.

Exercise 1 (John)**.** Consider the embedding $\phi : S^1 \vee S^1 \to (S^1)^2$ defined such that the point x of the first S^1 maps to $(x, 1)$ and x of the second S^1 maps to $(1, x)$. Describe the pushforward ϕ_* (a homomorphism from the free group on two generators to $\mathbb{Z}^2$).

Exercise 2 (John)**.** Let $\phi : X \to Y$ be a continuous map of topological spaces, and call $\phi_{x_0} : (X, x_0) \to (Y, \phi(x_0))$ the corresponding map of pointed topological spaces. Recall that a path $f : I \to X$ from $f(0) = x_0$ to $f(1) = x_1$ induces an isomorphism which I call $f_\sharp : \pi_1(X, x_0) \to \pi_1(X, x_1)$. Prove that

$$(\phi_{x_1})_* \circ f_\sharp = (\phi \circ f)_\sharp \circ (\phi_{x_0})_*$$

for all such ϕ and f.

Exercise 3 (Noah)**.** In class it was stated that the spaces $\mathbf{R}^3 - S^2$ and $S^2 \coprod \{0\}$ are of the same homotopy type. Prove this by finding a homotopy equivalence between the two spaces.

Exercise 4 (Noah, 58.2 of Munkres)**.** For each of the following subsets of $\mathbf{R}^2$, determine if the fundamental group is 1, $\mathbf{Z}$, or the free abelian group with two generators:

(1) $\{x \mid ||x|| > 1\}$
(2) $S^1 \cup (\mathbf{R}_+ \times 0)$
(3) $S^1 \cup (\mathbf{R}_+ \times \mathbf{R})$
(4) $S^1 \cup (\mathbf{R} \times 0)$
(5) $\mathbf{R}^2 - (\mathbf{R}_+ \times 0)$

Solution to Exercise 1: The fundamental group $\pi_1(S^1 \vee S^1)$ is generated by (the homotopy classes of) a loop f around the first S^1 and a loop g around the second S^1 with no other relations. We may see that $\phi_*[f]$ is a loop around S^1 in the first coordinate of $(S^1)^2$ and $\phi_*[g]$ around the second coordinate of $(S^1)^2$. These are the generators of $\pi_1((S^1)^2)$, and have no other relations except commutativity (i.e., they generate a free abelian group). Thus ϕ_* is just the abelianization which maps the generators of the free group on two generators to the generators of the free *abelian* group on two generators.

Solution to Exercise 2: Both $(\phi_{x_1})_* \circ f_\sharp$ and $(\phi \circ f)_\sharp \circ (\phi_{x_0})_*$ are homomorphisms from $\pi_1(X, x_0)$ to $\pi_1(Y, \phi(x_1))$. Let $g : S^1 \to (X, x_0)$ be a loop based at x_0 so that $[g]$ represents an arbitrary element in the domain of the two homomorphisms.

$$(\phi_{x_1})_* \circ f_\sharp[g] = (\phi_{x_1})_*[f^{-1} \cdot g \cdot f] = [\phi \circ (f^{-1} \cdot g \cdot f)]$$

where f^{-1} represents the corresponding path from x_1 to x_0. On the other hand,

$$(\phi \circ f)_\sharp \circ (\phi_{x_0})_*[g] = (\phi \circ f)_\sharp[\phi_{x_0} \circ g] = [(\phi \circ f)^{-1} \cdot (\phi \circ g) \cdot (\phi \circ f)]$$

We need to show that these two expressions inside the brackets are homotopic. In fact, it is more generally true that $\phi \circ (f \cdot g) = (\phi \circ f) \cdot (\phi \circ g)$ whenever f and g can be composed ($f(1) = g(0)$). This is because this expression is $\phi(f(2t))$ on $0 \le t \le \frac{1}{2}$ and $\phi(g(2t-1))$ on $\frac{1}{2} \le t \le 1$. Therefore the two expressions act the same on any $[g]$, and so

$$(\phi_{x_1})_* \circ f_\sharp = (\phi \circ f)_\sharp \circ (\phi_{x_0})_*$$

Solution to Exercise 3: Let $X = \mathbf{R}^3 - S^2, Y = S^2 \coprod \{0\}$, and define the map $\phi : X \to Y$ as follows:

$$\phi(x) = \begin{cases} 0 & : |x| < 1 \\ x \| x| & : |x| > 1 \end{cases}$$

We claim that ϕ is a homotopy equivalence. Let $\psi : Y \to X$ be as follows

$$\psi(x) = \begin{cases} 0 & : x = 0 \\ 2x & : |x| = 1 \end{cases}$$

It is straightforward to verify that ϕ, ψ are continuous. Also, $\phi\psi = \mathrm{Id}_Y$, so trivially $\phi\psi \simeq \mathrm{Id}_Y$. Then, the family of maps $f_t, t \in I$ given by

$$f_t(x) = \begin{cases} tx & : |x| < 1 \\ \frac{2}{|x|} + (1 - \frac{2}{|x|})t & : |x| > 1 \end{cases}$$

is a homotopy between $\phi\psi$ and Id_X so that $\phi\psi \simeq \mathrm{Id}_X$. This completes the proof. Note that we have in fact proved the stronger statement that the base-pointed spaces $(X, 0), (Y, 0)$ are homotopy equivalent.

Solution to Exercise 4: $1)\mathbf{Z}, 2)\mathbf{Z}, 3)\mathbf{Z}, 4)$ free abelian group with two generators, $5)1$.