18.904

2/25/11: LIFTING PROPERTIES I

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In order to deduce the Galois correspondence on Monday, we will need to prove some technical results about covering spaces. These are the so-called lifting properties, and we will prove them today. For this talk, let $p : \tilde{X} \to X$ be a covering space.

Definition 0.1. *Let $f : Y \to X$ be a map. Then, a map $\tilde{f} : Y \to X$ is said to be a **lift** of f if $p \circ \tilde{f} = f$. In particular, if $F : Y \times I \to X$ be a homotopy, a homotopy $\tilde{F} : Y \times I \to \tilde{X}$ is said to be a **lift** of F if $p \circ \tilde{f} = f$.*

Proposition 0.2 (The homotopy lifting property)**.** *Given a homotopy $F : Y \times I \to X$, and a lift $\tilde{f}_0$ of the map $f_0 = F|_{Y\times\{0\}}$, there is a unique lift $\tilde{F} : Y \times I \to X$ of F such that $\tilde{F}|_{Y\times\{0\}} = \tilde{f}_0$.*

Proof. Pick an open cover U_α of X such that $p^{-1}(U_\alpha)$ can be decomposed into a disjoint union of open sets each of which is homeomorphic to U_α under p.

Step 1. Let $y \in Y$ be given. We begin by constructing a lift $\tilde{F} : V \times I \to \tilde{X}$, for some neighborhood V of y. For each $t \in I$, we may pick some neighborhood V_t and some open interval I_t containing t such that $F(V_t \times I_t) \subset U_\alpha$. Cover I by finitely many I_t, and let V be the intersection of the corresponding V_t. Then, we may choose a finite partition $0 = t_0 < t_1 < \ldots < t_n = 1$ such that $F(V \times [t_i, t_{i+1}]) \subset U_{\alpha_i}$ for some index α_i.

We now construct a lift $\tilde{F} : V \times [0, t_i] \to \tilde{X}$ by induction on i. The base case $i = 0$ is given by $\tilde{f}_0$. Suppose $\tilde{F} : V \times [0, t_i] \to \tilde{X}$ has been constructed. As $F(V \times [t_i, t_{i+1}]) \subset U_{\alpha_i}$, we may choose some set $\tilde{U}_{\alpha_i} \in \tilde{X}$ containing $\tilde{F}(y, t_i)$ such that $\tilde{U}_{\alpha_i}$ is homeomorphic to U_{α_i} under p. Shrinking V if needed, by continuity we may assume that $\tilde{F}(V \times \{t_i\}) \subset \tilde{U}_{\alpha_i}$. We may then define $\tilde{F}$ on the set $V \times [t_i, t_{i+1}]$ to be $p^{-1}F$, where p^{-1} denotes the homeomorphism $p^{-1} : U_{\alpha_i} \to \tilde{U}_{\alpha_i}$. By the pasting lemma, the resulting function $\tilde{F} : V \times [0, t_{i+1}] \to \tilde{X}$ is continuous. This completes the induction, furnishing a map $\tilde{F} : V \times I \to \tilde{X}$ that lifts $F|_{V\times I}$.

Step 2. We prove uniqueness in the case where Y is a single point y. Let $\tilde{F}, \tilde{F}'$ be two lifts of $F : \{y\} \times I \to X$ for which $\tilde{F}(y, 0) = \tilde{F}'(y, 0)$; once again, we may choose a finite partition $0 = t_0 < t_1 < \ldots < t_n = 1$ such that $F(\{y\} \times [t_i, t_{i+1}]) \subset U_{\alpha_i}$ for some index α_i. We claim that $\tilde{F} = \tilde{F}'$ on $[0, t_i]$ for all i; once again we proceed by induction. The base case $i = 0$ follows by assumption. Suppose $\tilde{F} = \tilde{F}'$ on $[0, t_i]$; as $\tilde{F}(\{y\} \times [t_i, t_{i+1}]), \tilde{F}'(\{y\} \times [t_i, t_{i+1}])$ are connected and $\tilde{F}(y, t_i) = \tilde{F}'(y, t_i)$, both must lie in the same open set $\tilde{U}_{\alpha_i} \in \tilde{X}$ that is homeomorphic to U_{α_i} under p. But, as $p|_{\tilde{U}_{\alpha_i}}$ is injective and $p\tilde{F} = F = p\tilde{F}'$, this implies that $\tilde{F} = \tilde{F}'$ on $\{y\} \times [t_i, t_{i+1}]$, completing the induction.

Step 3. We now prove the theorem. First, we show uniqueness: if $\tilde{F} : Y \times I \to \tilde{X}$ is a lift of F, then $\tilde{F}|_{\{y\}\times I}$ is a lift of $F|_{\{y\}\times I}$, so by Step 2, $\tilde{F}$ is unique. Furthermore, given two lifts $\tilde{F} : V \times I \to \tilde{X}, \tilde{F}' : V \times I \to \tilde{X}$ constructed in Step 1, by Step 2, $\tilde{F}$ and $\tilde{F}'$ must agree on $V \cap V'$.

Therefore, by pasting together lifts $\tilde{F} : V \times I \to \tilde{X}$ for each point $y \in Y$, one obtains a well-defined lift $\tilde{F} : Y \times I \to \tilde{X}$, completing the proof of the proposition. □

Corollary 0.3 (The path lifting property)**.** *Let $f : I \to X$ be a path such that $f(0) = x_0$. Given a point $\tilde{x}_0 \in p^{-1}(x_0)$, there exists a unique lift $\tilde{f} : I \to X$ of f such that $f(0) = \tilde{x}_0$. In particular, every lift of a constant path is constant.*

Corollary 0.4 (The path homotopy lifting property)**.** *Let f_t be a path homotopy in X. Given a lift $\tilde{f}_0$ of f_0, there exists a unique lift $\tilde{f}_t$ in $\tilde{X}$ of f_t; this lift $\tilde{f}_t$ is also a path homotopy.*

As an application of the path lifting property, we have the following proposition. This is the result JJ assumed at the conclusion of Wednesday's lecture.

Proposition 0.5. *The map $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective.*

Proof. Suppose $[\tilde{f}_0] \in \ker p_*$; then there exists a homotopy f_t between $p\tilde{f}_0$ and the trivial loop e_{x_0}. Then, by the path homotopy lifting property, there exists a lift $\tilde{f}_t$ of f_t between $\tilde{f}_0$ and $e_{\tilde{x}_0}$. Therefore, $[\tilde{f}_0] = [e_{\tilde{x}_0}]$, so $\ker p_*$ is trivial as desired. □