18.904 Lecture Notes - 2/28/11

Danny Bulmash

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Outline

1. Action of π_1 on the universal cover
2. Covers corresponding to subgroups of π_1
3. Galois covers

Action of π_1 on the Universal Cover

Recall that a group action of a group G on a set X is a map $G \times X \to X$ (which we will denote by juxataposition) such that $(gh)x = g(hx)$ and $ex = x$ where e is the identity element of g.

Recall that the universal cover $(\tilde{X}, \tilde{x_0})$ of a base-pointed space (X, x_0) can be thought of as a set of homotopy classes of paths from x_0. There is a natural way to define a group action of π_1 on $\tilde{X}$: just concatenate a loop in X onto the path modulo homotopy.

More fomally, we define an action $\pi_1(X, x_0) \times (\tilde{X}, \tilde{x_0}) \to (\tilde{X}, \tilde{x_0})$ by $[f] \times [g] \mapsto [fg]$. It is easy to see that this is well-defined and is a group action. The continuity of the map created by the action of each element is left as an exercise.

Covers Corresponding to Subgroups of π_1

We now prove the main result of this talk. This is part of the Galois correspondence, which says that there is a bijective correspondence between connected pointed covers of a space (up to isomorphism) and subgroups of its fundamental group. This portion gives us surjectivity, i.e. it gives us a cover for every subgroup of the fundamental group.

Proposition: Let X be path-connected, locally path-connected, and semi-locally simply connected. Then given a subgroup H of $\pi_1(X, x_0)$, there exists a cover $p : Y \to X$ such that $p_*(\pi_1(Y, y_0)) = H$ for some basepoint $y \in y_0$.

Proof:
Step 1: Construct the cover.

Define a relation $\sim$ on the universal cover $\tilde{X}$ (with paths base-pointed at x_0) by saying $[\gamma] \sim [\gamma']$ if $\gamma(1) = \gamma'(1)$, i.e. the paths are between the same points, and $[\gamma\bar{\gamma'}] \in H$. We claim that this is an equivalence relation.

1. Reflexivity: H contains the identity element (the class of the constant loop), so $[\gamma\bar{\gamma}] \in H$
2. Symmetry: H is closed under inverses, so $[\gamma\bar{\gamma'}] \in H$ implies $[\gamma\bar{\gamma'}]^{-1} = [\gamma'\bar{\gamma}] \in H$
3. Transitivity: Suppose $[\gamma] \sim [\gamma']$ and $[\gamma] \sim [\eta]$. Then $[\gamma\bar{\gamma'}], [\gamma\bar{\eta}] \in H$, so by closure under inverses $[\gamma'\bar{\gamma}] \in H$ as well. Thus from closure under multiplication $[\gamma'\bar{\gamma}][\gamma\bar{\eta}] = [\gamma'\bar{\eta}] \in H$.

Let Y be the quotient space obtained from $\tilde{X}$ by identifying elements which are equivalent under $\sim$. We claim that defining $p : Y \to X$ by sending the equivalence class of $[\gamma]$ to $\gamma(1)$ gives a covering space. p is well-defined because equivalent homotopy classes come from paths with the same endpoint.

Step 2: Show that we have a cover.

To show that p and Y define a covering space, we show that the inverse image of a basis element U of X (as defined in the construction of the universal cover) is a disjoint union of open sets in Y. $p^{-1}(U)$ is by definition the set of equivalence classes of homotopy classes of paths ending in U. Examine the inverse image of $p^{-1}(U)$ under the quotient map from $\tilde{X}$ to Y.

We claim that this inverse image is the union of all $U_{[\gamma]}$ where γ is a path ending in U. Certainly all such $U_{[\gamma]}$ map into U under the quotient map and p, as endpoints are preserved. Moreover, $[\gamma] \in U_{[\gamma]}$, so if we pick any homotopy class of the equivalence class of paths mapping into U, then it will be in the U for that homotopy class.

If two paths γ and γ' have the same endpoint, then for any other path η we have $[\gamma] \sim [\gamma']$ if and only if $[\gamma\eta] \sim [\gamma'\eta]$, so if any two points in neighborhoods $U_{[\gamma]}$ and $U_{[\gamma']}$ are are identified, then the whole neighborhoods are identified. This gives us the breakdown into disjoint open sets - the image of each neighborhood under the quotient map gives us one open set, and we throw out "duplicate" neighborhoods.

Step 3: Show that the cover has the desired property.

Let y_0 be the equivalence class of $[c]$ where c is the constant loop. We claim that $p_*(\pi_1(Y, y_0)) = H$.

Lemma: $p_*(\pi_1(Y, y_0))$ is the set of homotopy classes of loops in X based at x_0 whose lifts to Y starting at $[c]$ are loops.

Proof of lemma: Suppose that a loop γ in X lifts to a loop in Y starting at $[c]$. Then certainly $[\gamma]$ is in $p_*(\pi_1(Y, y_0))$ because p applied to this loop is γ. Now suppose that $[\gamma]$ is in $p_*(\pi_1(Y, y_0))$. Then there exists a loop η in Y for which $p_*([\eta]) = [p\eta] = [\gamma]$. Define $p\eta = \gamma'$, which must be a loop in X. Then γ is homotopic to γ', which has a lift (namely η) to Y, so by the homotopy lifting property γ has a lift to Y too.

Let γ be a loop in X at x_0. Then we can create a lift to Y by sending t to the equivalence class of $[\gamma_t]$, where $\gamma_t(s) = \gamma(ts)$, that is, γ_t is γ but truncated at $\gamma(t)$. This lift starts at y_0, so by the unique lifting property this is the only such lift. This lift is a loop iff $[\gamma] \sim [c]$, that is, $[\gamma\bar{c}] = [\gamma] \in H$. $\square$

Galois Covers

Normal subgroups of a group are interesting - are the covers that we get from them (by what we just did) also interesting? The answer is yes. We begin by defining the interesting property:

Definition: A cover $p : Y \to X$ is **Galois** (or **normal**) if for each $x \in X$ and each pair of points y, y' in $p^{-1}(x)$, there exists an isomorphism $Y \to Y$ which sends y to y'.

Recall Andrew's example of a Galois covering - cover S^1 (in the complex plane) by itself with the map $z \mapsto z^n$. Then we can send $z \mapsto \xi z$ where ξ is an $n$the root of unity - the collection of these maps gives us all the isomorphisms that we need. We now relate these "normal" covers to normal subgroups.

Proposition: The covering space constructed in our earlier proposition is Galois iff H is a normal subgroup.

Proof: Umut will show that changing the base point of the fundamental group of Y to another point in $p^{-1}(x)$ corresponds to conjugating $H = p_*(\pi_1(Y, y_0)$ (a subgroup of the fundamental group of X) by a loop $[\gamma]$, where the lift of $[\gamma]$ is a path from the old base point to the new base point. H is closed under conjugation, i.e. normal, if and only if the conjugated group is equal to the new group, i.e. $p_*(\pi_1(Y, y_0) = p_*(\pi_1(Y, y_1)$ for all $y_1 \in p^{-1}(x)$. But the lifting criterion tells us that this condition is equivalent to there being an isomorphism of Y with itself which takes y_0 to y_1, which is all we needed. $\square$