Problem Set 1

18.904 Spring 2011

Instructions. Write-up solutions in Latex, print them out and hand them in at the beginning of class on Tuesday, February 22nd. See the website for additional instructions.

Problem 1. Let $n \geq 1$ be an integer. Let $\mathbf{CP}^n$ denote the set of all lines in $\mathbf{C}^{n+1}$ passing through the origin. There is a natural map $\pi : \mathbf{C}^{n+1} \setminus \{0\} \to \mathbf{CP}^n$ taking a point to the line it spans. We give $\mathbf{CP}^n$ the quotient topology, so that a set U in $\mathbf{CP}^n$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbf{C}^{n+1}$. Let $U_i \subset \mathbf{CP}^n$ denote the set of points of the form $\pi(x_0, \ldots, x_n)$ where $x_i \neq 0$.

(a) Show that the U_i form an open cover of $\mathbf{CP}^n$.
(b) Show that an intersection of $k+1$ distinct elements of $\{U_0, \ldots, U_n\}$ is homeomorphic to $(\mathbf{C}^\times)^k \times \mathbf{C}^{n-k}$, for $0 \leq k \leq n$. (In particular, each U_i is homeomorphic to $\mathbf{C}^n$.)
(c) Prove the following lemma. Let X be a topological space and let $\mathscr{U}$ be a finite open cover of X. Suppose that each element of $\mathscr{U}$ is simply connected and any intersection of elements of $\mathscr{U}$ is non-empty and path-connected. Then X is simply connected. [Hint: use van Kampen's theorem.]
(d) Conclude that $\mathbf{CP}^n$ is simply connected.

Remark. The space $\mathbf{CP}^n$ is called *complex projective space*. It is a very important space that shows up in all areas of mathematics. The space $\mathbf{CP}^1$ is called the *Riemann sphere*; it is homeomorphic to S^2 (convince yourself of this!).

Problem 2. Let X be a topological space and let x_1 and x_2 be two points in X. Given a path h between x_1 and x_2, we have seen that there is a canonical isomorphism

$$i_h : \pi_1(X, x_1) \to \pi_1(X, x_2).$$

Write $C(G)$ for the set of conjugacy classes in a group G, and let

$$\bar{i}_h : C(\pi_1(X, x_1)) \to C(\pi_1(X, x_2))$$

denote the map induced by i_h.

(a) Give an example (i.e., specify X, x_1, x_2, h and h') where $i_h \neq i_{h'}$, with proof.
(b) Show that $\bar{i}_h = \bar{i}_{h'}$ for any two paths h and h'.
(c) Assume $\pi_1(X, x_1)$ is abelian. Show that $i_h = i_{h'}$ for any h and h'.

Remark. Note the contrast between (a) and (b) — given two choices of basepoints x_1 and x_2, the sets $C(\pi_1(X, x_1))$ and $C(\pi_1(X, x_2))$ are in canonical bijection (assuming X is path connected), while the groups $\pi_1(X, x_1)$ and $\pi_1(X, x_2)$ are not.

Remark. The fact that i_h and $i_{h'}$ are not necessarily equal is why π_1 is only a functor for basepoint preserving maps.

Problem 3. Let X be a metric (and thus topological) space. Fix a basepoint x_0 in X; the word "loop" will mean "loop based at x_0" in this problem. Let ΩX denote the set of all loops in X, i.e., the set of all continuous functions $p : [0,1] \to X$ with $p(0) = p(1) = x_0$. Define a distance function on ΩX by $d(p_1, p_2) = \max_{x \in [0,1]} d(p_1(x), p_2(x))$.

(a) Show that concatentation of loops defines a continuous map $\Omega X \times \Omega X \to \Omega X$. Conclude that there is a natural map of sets $\pi_0(\Omega X) \times \pi_0(\Omega X) \to \pi_0(\Omega X)$. [Here π_0 denotes the set of path components.]
(b) Show that two loops in X are homotopic if and only if the corresponding points of ΩX are in the same path component.

(c) Construct a canonical bijection of sets $\pi_0(\Omega X) \to \pi_1(X, x_0)$. Show that this map is a homorphism, in the sense that it respects the multiplications on the two sets (the one on $\pi_0(\Omega X)$ constructed in (a) and the usual group operation on $\pi_1(X, x_0)$).

Remark. In fact, the bijection from (c) is just the first in a sequence: there are natural group isomorphisms $\pi_{n-1}(\Omega X, x_0) \to \pi_n(X, x_0)$ for all $n \geq 1$. [Here π_n denotes the nth homotopy group.]

Remark. The above theory does not at all require X to be a metric space, it just simplifies the definition of the topology on ΩX. When X is a general topological space, the appropriate topology on ΩX is the "compact open topology."

Problem 4. In this problem, we will show that every finitely presented group occurs as a fundamental group.

(a) Let G be a group, let a be an element of G and let N be the normal closure of the subgroup generated by a. [Explicitly, N is the subgroup of G generated by all conjugates of a.] Let $\mathbf{Z} \to G$ be the map defined by $1 \mapsto a$. Show that the amalgamated free product $G *_{\mathbf{Z}} 1$ is isomorphic to G/N. [Here 1 denotes the trivial group.]

(b) Let X be a topological space with base point x_0 and let $i : S^1 \to X$ be a loop based at x_0. Let X' be the topological space obtained by attaching a 2-disc to X via i; that is, X' is the quotient of $X \amalg D^2$ where an element $x \in S^1 = \partial D^2$ is identified with $i(x) \in X$. Show that $\pi_1(X', x_0)$ is the quotient of $\pi_1(X, x_0)$ by the normal subgroup generated by the class of i. [Hint: use van Kampen's theorem.]

(c) Show that every finitely presented group occurs as a fundamental groups. [Hint: let G be a finitely presented group. Pick a presentation. Start with a bouquet of circles, one for each generator. Attach a 2-disc for each relation and apply (b).]

Remark. The requirement that the group be finitely generated is completely unnecessary. The general case can be established by the same means.

Remark. There can be many very different homotopy types that have isomorphic fundamental groups; for instance, both S^1 and $S^1 \vee S^2$ have fundamental group $\mathbf{Z}$. However, given a group G there is a *unique* homotopy type with fundamental group G and contractible universal cover (or equivalently, with all other homotopy groups vanishing).

Problem 5. Let G be a topological group; thus G is simulateneously a group and a topological space, and the multiplication map $G \times G \to G$ and inversion map $G \to G$ are continuous.

(a) Show that there is a unique group structure on $\pi_0(G)$ such that the natural map $G \to \pi_0(G)$ is a group homomorphism.

(b) Show that $\pi_1(G, 1)$ is a commutative group. [Hint: if c is a loop in G based at 1 and g is an element of G then $t \mapsto gc(t)$ is a loop in G based at g. Using this you can slide one loop along another to show that they commute in π_1.]

Problem 6. Let $G = \mathrm{SL}(2, \mathbf{R})$, the group of 2×2 real matrices with determinant 1. We can naturally regard G as a closed subset of $\mathbf{R}^4$, and thus (after a few simple verifications) as a topological group. Let $B^\circ \subset G$ be the subgroup of matrices which are upper-triangular with positive entries on the diagonal. Let $K \subset G$ be the subgroup of rotations matrices. [An element of G belongs to K if and only if its two columns form an orthonormal basis of $\mathbf{R}^2$.]

(a) Show that B° is homeomorphic to $\mathbf{R}^2$, and is thus contractible.

(b) Show that K is homeomorphic to S^1.

(c) Show that the map $B^\circ \times K \to G$ sending (b, k) to bk is a homeomorphism.

(d) Conclude that G is homotopy equivalent to S^1, and thus has fundamental group $\mathbf{Z}$.

Remark. The group $\mathrm{SL}(2, \mathbf{R})$ is better than just a topological group: as a topological space it is actually a smooth manifold, and the group operations are smooth maps. Such topological groups are called *Lie groups.* They are among the most important objects in all of mathematics.