Problem Set 3

18.904 Spring 2011

Due date. Monday, May 9, 2011.

Problem 1 (5 points)**.** Let $T = S^1 \times S^1$ be the two dimensional torus and let T' be the complement in T of a small disc. Let X be the space obtained by taking two copies of T' and connecting them by a cylinder meeting the boundaries of the removed discs. Compute $H^i(X)$ for all i and describe the cup product $H^1(X) \times H^1(X) \to H^2(X)$.

Problem 2 (5 points)**.** Let X be the topological space with four points a, b, x and y such that $\{x\}$ and $\{y\}$ are closed, while the closure of $\{a\}$ is $\{a, x, y\}$ and the closure of $\{b\}$ is $\{b, x, y\}$. Compute $H_i(X)$ for all $i \geq 0$.

Remark. I believe it is true, generally, that for every compact CW complex X one can construct a finite topological space X' and a map $X \to X'$ such that the induced map $H_i(X) \to H_i(X')$ is an isomorphism. Thus finite spaces with non-Hausdorff topologies should not be ignored!

Problem 3 (5 points)**.** Let G be the cyclic group of order 3. Construct a topological space X on which G acts such that $H_2(X)$ is isomorphic to $\mathbf{Z}^2$ and the induced action of G is non-trivial.

Problem 4 (15 points)**.** Let c be a generator for $H^1(S^1) \cong \mathbf{Z}$. For two topological spaces X and Y, write $[X, Y]$ for the set of homotopy classes of maps between X and Y. Given a topological space X, we have a natural map

$$\Phi_X : [X, S^1] \to H^1(X), \qquad f \mapsto f^*(c).$$

Observe that Φ is a natural transformation of functors, that is, if $X \to Y$ is a map then there is a commutative diagram

$$\begin{array}{ccc} H^1(Y) & \longrightarrow & H^1(X) \\ {\scriptstyle \Phi_Y}\uparrow & & \uparrow{\scriptstyle \Phi_X} \\ [Y, S^1] & \longrightarrow & [X, S^1] \end{array}$$

This will be a useful observation in what follows.

The purpose of this problem is to show that Φ_X is an isomorphism whenever X is a CW complex with finitely many cells. We'll break the proof into many steps.

(a) Show that Φ_X is an isomorphism if X is a one dimensional CW complex.

We now do some basic obstruction theory. Let me remind you of a simple fact: a map $\partial D^n \to X$ extends to D^n if and only if it is null-homotopic.

(b) Let f and g be maps $D^n \to S^1$, with $n \geq 2$, and let H be a homotopy between $f|_{\partial D^n}$ and $g|_{\partial D^n}$. Show that H can be extended to a homotopy between f and g. [Hint: Interpret H as a map from a sphere and extend it to a disc!]

(c) Let X be a topological space and let Y be obtained from X be attaching a single n-cell. Show that the natural map $[Y, S^1] \to [X, S^1]$ is injective if $n \geq 2$.

(d) Notation as in (c), show that $[Y, S^1] \to [X, S^1]$ is surjective if $n \geq 3$.

For the next few steps, let X be a finite two dimensional CW complex and let Y be obtained from X be attaching a single 2-cell. Let $i : S^1 \to X$ be the attaching map. We regard X as a subspace of Y.

(e) Show that there is an exact sequence

$$0 \to H^1(Y) \to H^1(X) \xrightarrow{i^*} H^1(S^1).$$

(f) Let $f : X \to S^1$ be a given map. Show that f extends to Y if and only if $f^*(c)$ belongs to $H^1(Y)$. [Hint: Use part (e) to characterize $H^1(Y)$ as a subset of $H^1(X)$ and the fact that a map $h : S^1 \to S^1$ is null-homotopic if and only if $h^*(c) = 0$.]

(g) Suppose that Φ_X is bijective. Show that Φ_Y is as well.

By part (f) and an easy induction argument, we find that Φ_X is an isomorphism for all finite two dimensional CW complexes X. We now complete the argument. Let X be a finite CW complex and let Y be obtained from X be attaching a single n-cell, with $n > 2$.

(h) Show that the natural map $H^1(Y) \to H^1(X)$ is an isomorphism.

(i) Suppose that Φ_X is bijective. Show that Φ_Y is as well.

We now find that Φ_X is bijective for all finite CW complexes X by induction!

Remark. In fact, it is true that for any $n \geq 0$ there is a CW complex K_n such that $[X, K_n] = H^n(X)$ holds for CW complexes X. The above problem proves this for $n = 1$ and shows $K_1 = S^1$. (At least when X is finite.) It is very easy to see that $K_0 = \mathbf{Z}$. It is much less easy to see that $K_2 = \mathbf{CP}^\infty$. For $n > 2$, I do not know a nice description of K_n.

Problem 5 (10 points)**.** Let $\mathcal{C}$ be a category and let **Set** be the category of sets. There is a category $\mathcal{F} = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$ whose objects are functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ and whose morphisms are natural transformations of functors. (Recall that $\mathcal{C}^{\mathrm{op}}$ is the opposite category to $\mathcal{C}$, and that a functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is the same thing as a contravariant functor $\mathcal{C} \to \mathbf{Set}$.)

(a) Let X be an object of $\mathcal{C}$. Show that $h_X(T) = \mathrm{Hom}_{\mathcal{C}}(T, X)$ defines a functor $h_X : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$.

(b) Show that $X \mapsto h_X$ defines a functor $h : \mathcal{C} \to \mathcal{F}$.

(c) Show that the functor h above is *fully faithful*, i.e., the natural map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{F}}(h_X, h_Y)$ is a bijection.

(d) In particular, show that if h_X is isomorphic to h_Y in the category $\mathcal{F}$ then X is isomorphic to Y in the category $\mathcal{C}$.

Remark. The above result (specifically part (c)) is known as *Yoneda's lemma*. It can be very confusing when you first see it, but it is essentially tautological! Yoneda's lemma is an extremely useful organizational device: it says that an object in a category is fully determined by how other objects of the category map to it.

Remark. Let $F : \mathcal{C} \to \mathbf{Set}$ be a functor. We say that an object X of $\mathcal{C}$ *represents* F if F is isomorphic to h_X. Such an object is unique up to isomorphism by Yoneda's lemma, but need not exist. In this language, Problem 4 is just the statement that S^1 represents the functor H^1 (on the category of finite CW complex with homotopy classes of maps). Yoneda's lemma tells us that this property uniquely characterizes S^1: if another finite CW complex represents H^1 then it must be homotopy equivalent to S^1.