Problem Set 2 Solutions

18.904 Spring 2011

I don't have time at the moment to write up a complete solution set to this problem set. Everyone seemed to have most of the general ideas down, and most of the mistakes were technical. However, there is one general fact — the categorical form of the Galois correspondence — that clarifies many of the problems, and that no one employed. Perhaps we didn't cover this well enough or completely enough, but hopefully the solutions that follow will show you how to work with it.

1. Review of the Galois correspondence in categorical form

Let G be a group. Recall that a *G-set* is a set S together with an action of G on S. If S and S' are two G-sets then a *map* $f : S \to S'$ is simply a map of sets which commutes with the G-action, i.e., $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$. In this way, we have a category of G-sets.

Now let X be a path-connected topological space (with whatever other technical hypotheses are needed, e.g., locally path connected). Let x_0 be a point in X. Suppose that $p : Y \to X$ is a covering space. We claim that the fiber $p^{-1}(x_0)$ naturally has an action of the group $\pi_1(X, x_0)$. To see this, suppose that y is a point in the fiber and γ is a loop in X based at x_0. By the unique lifting property, there is a unique *path* $\widetilde{\gamma}$ in Y beginning at y which lifts γ. We define $\gamma \cdot y$ to be the endpoint of the path $\widetilde{\gamma}$. Since γ is a loop, its endpoint is x_0; thus the endpoint of $\widetilde{\gamma}$ is something which maps to x_0 under p, i.e., an element of the fiber $p^{-1}(x_0)$. Thus (after verifying details that this is well-defined and satisfies the necessary conditions), we have an action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$.

Now suppose that $p' : Y' \to X$ is another covering space and we have a map of covering spaces $f : Y \to Y'$. Recall that this means that f is a continuous map such that $p' \circ f = p$. Due to this condition, f maps the set $p^{-1}(x_0)$ into $(p')^{-1}(x_0)$. One easily checks that this map commutes with the action of $\pi_1(X, x_0)$ on each set.

The above discussion shows that associating to a covering space $Y \to X$ its fiber $p^{-1}(x_0)$ defines a functor

$$\{\text{covering spaces of } X\} \to \{\pi_1(X, x_0)\text{-sets}\}.$$

(Here we use the set notation to indicate a category.) The categorical form of the Galois correspondence states that this functor is an equivalence of categories. Thus, any purely categorical notion about covering spaces can be determined by considering the category of $\pi_1(X, x_0)$ sets.

2. Problem 2

The categorical Galois correspondence can be used to give an elegant proof of 2(b), as follows. First, we introduce a term: if $p : Y \to X$ is a covering space, a *section* is a map of covering spaces $s : X \to Y$, where X is regarded as the trivial covering space of itself. In other words, a section is a map $s : X \to Y$ such that $p \circ s = \mathrm{id}_X$. It is clear in 2(b) that giving a square root of f is the same as giving a section of $\widetilde{X} \to X$, for if we have a section s then we simply pull back $\widetilde{f}$ by s to get a square root, while if we have a square root $\widetilde{f}$ then defining $s(x) = (x, \widetilde{f}(x))$ gives a section (where here we think of $\widetilde{X}$ as a subspace of $X \times \mathbf{C}$, as all of you did).

Thus 2(b) is reduced to the following: if $\widetilde{X} \to X$ has a section then $\widetilde{X}$ is isomorphic to $X \amalg X$ as a covering space. However, this is obvious from the Galois correspondence. We know that $\widetilde{X}$ corresponds to a 2 element set S with some action of $\pi_1(X, x_0)$, while X corresponds to a 1 point set S_0 with the trivial action of $\pi_1(X, x_0)$. A section of $\widetilde{X}$ defines an inclusion $S_0 \to S$, which shows that one of the two elements of S is fixed under $\pi_1(X, x_0)$. But since there are only two elements, the other one must be fixed as well. Thus S is isomorphic to $S_0 \amalg S_0$ as a $\pi_1(X, x_0)$-set, which shows that $\widetilde{X}$ is isomorphic to $X \amalg X$ as a covering space.

This proof is perhaps not shorter than some of the ones that you all came up with, but I think its easier since you don't have to think about lifting paths. The same method does not apply verbatim to 2(c), but you can make it work. Exercise!

3. Problem 3

The categorical Galois correspondence is the easiest way to handle parts (iv) of 3(b) and 3(c). (At least if we assume X is path-connected, which the problem didn't, but which you can easily reduce to.) Since these questions are about the structure of the category of covering spaces, they can be answered by passing to an equivalent category, i.e., the category of $\pi_1(X, x_0)$ sets. In this context, the answer is clear: the category of G-sets has products and coproducts for any group G. The product of two G-sets S_1 and S_2 is the cartesian product $S_1 \times S_2$ of sets, with the diagonal G-action, i.e., $g(x, y) = (gx, gy)$. The coproduct of S_1 and S_2 is the disjoint union $S_1 \amalg S_2$ with G acting on each in the given manner.

Understanding what these operations are in terms of covering spaces is not difficult, and a useful exercise. I'll tell you the answers but not why they're the answers. Suppose $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ are two covering spaces. Then their coproduct is just the disjoint union of Y_1 and Y_2, with the obvious map down to X. Their product (in the category of covering spaces) is the *fiber product* of Y_1 and Y_2. This is denoted $Y_1 \times_X Y_2$ and is defined as the set of points (y_1, y_2) in $Y_1 \times Y_2$ such that $p_1(y_1) = p_2(y_2)$, with the subspace topology. Thus it is the union of the products of the fibers, which is why it is so named. (These descriptions of product and coproduct are valid even if X is not path-connected.)

Many of you tried to use the (non-categorical) Galois correspondence to do this problem. That's the right direction to think in, but doesn't work for two reaons: first, this bijection applies only to pointed coverings; and second (and more importantly), it applies only to connected coverings. It is clear from the previous paragraph that the coproduct of covering spaces gives a non-connected cover. It's a worthwhile exercise for you to see how the product of connected covering spaces can end up being non-connected. For instance, if Y is a non-trivial covering space of X then its self-product $Y \times_X Y$ is *always* disconnected (hint: find a map of covering spaces from Y to $Y \times_X Y$ and use this to see that $\pi_1(X, x_0)$ does not act transitively on the fiber of $Y \times_X Y$).

4. Problem 5

In this problem the use of the categorical form of the Galois correspondence makes the solution very elegant and provides much greater understanding of what's going on, at least for parts (d) and (e). Fix a point x_0 in X. By part (a) we have that $\Pi(X)$ is a covering space of X. By part (b) we know its fiber over x_0 can be identified with $\pi_1(X, x_0)$. This gives us the set that $\Pi(X)$ corresponds to under the categorical form of the correspondence. But what is the action of $\pi_1(X, x_0)$ on this set? The answer is provided in (c): $\pi_1(X, x_0)$ acts on the fiber $p^{-1}(x_0) = \pi_1(X, x_0)$ by conjugation. (The definition of this action is given by i'_h, and (c) shows that it equals i_h, which is just conjugation by h.) We thus see that $\Pi(X)$ corresponds to the $\pi_1(X, x_0)$-set $\pi_1(X, x_0)$, with action being conjugation.

Parts (d) and (e) are now very easy, when combined with some further properties of the Galois correspondence. Let's do (d) first. A covering space of X is path-connected if and only if the corresponding $\pi_1(X, x_0)$-set is transitive. The action of $\pi_1(X, x_0)$ on itself by conjugation fixes the identity element, and therefore is transitive if and only if $\pi_1(X, x_0) = 1$. Thus $\Pi(X)$ is path-connected if and only if X is simply connected.

Now for part (e). A covering space of X is trivial if and only if the corresponding $\pi_1(X, x_0)$-set is trivial, i.e., every point is fixed by the action of $\pi_1(X, x_0)$. The action of $\pi_1(X, x_0)$ on itself by conjugation fixes all elements if and only if $\pi_1(X, x_0)$ is abelian — that is the definition of abelian! This completes (e).